On The Nature Of The Fundamental Theorem Of Creation

A Logical Enquiry Into The Cross, Christ And Coexistence

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'Science, without religion, is lame … religion, without science, is blind …'

--- *Albert Einstein*

A Table of Contents can be found on pages 706-707, which will be updated as new pages are added to the essay

Some commentary on solving the field equations of general relativity for a perfect star in the presence of a dark energy singularity has been added

- *Some new material was added to the essay in this release*

May all Souls in all realms read these notes before coming here Father God. I have nothing more to give than what you have given to me, and thank you for this knowledge. Give to them now in turn.

Then, like a trumpet blast with a shower of spiraling lights, the Great Light spoke, saying, "Remember this and never forget; you save, redeem and heal yourself.

You always have. You always will. You were created with the power to do so from before the beginning of the world."

Noether's Theorem states that to every continuous symmetry in a physical situation … there corresponds a conservation law (and conversely). This deep connection requires that the *action* principle be assumed.

In this essay, the synonymous terms 'Cross and Crucifixion' should be seen as an event equivalent to the *action*. The term 'Father-Son duality' is a personalized reference to symmetries within the Godhead. The term 'Christ' refers to the person Jesus Christ who was crucified, and the term 'Holy Ghost' refers to the laws of Creation. As the essay progresses, we shed more light on these terms and definitions, and often refer to the Godhead as β-space or the β-world, throughout.

... and God created the heavens and the earth ...

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the alternating noises of dispute await their turn ... but you need not attend them …

there is nothing worse than the zeal of a convert

think radially

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By the Father and Son, and all they have done,
To give full life, to everyone …

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choose your own path, go at your own pace … intent is the Light's measure …

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`I knew with total certainty that everything was evolving exactly the way it should, and that the ultimate destiny for every living being is to return to the Source, the Light, pure Love` [Juliet Nightingale's NDE]

The Circles (we justify our perversions in the strangest of ways)

Here are 7 circles of life I think may apply. Certainly they're worth thinking about. Conquer them, if you think you can, and learn to grow into your own spiritual virtuosity.

○ hurt --> anger --> fear --> hate (*darkness*)
○ necessity --> need --> want --> greed (*desire*)
○ defiance --> resistance --> indifference --> acceptance (*denial*)
○ thought --> action --> deed --> consequence (*design*)
○ self-aggrandizement --> self-gratification --> self-righteousness --> self-importance (*delusion*)
○ knowledge --> understanding --> wisdom --> enlightenment (*discernment*)
○ joy --> peace --> courage --> love (*delight*)

The circle of circles:

○ darkness --> desire --> denial --> design --> delusion --> discernment --> delight

Note there is both an inward and outward radial component you can associate with each circle. The inward component would be the intrinsic effect felt by the person walking a particular circle and the outward component the extrinsic effect felt by others -- perhaps many others -- who were influenced by way of the ripples. Walking these circles is not a solitary experience, as our thoughts, actions, emotions, and so forth, affect not only us but many others at the same time.

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(self)  <~~~~~  | ~~~~~>  (others)
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Laws Of Creation (there are many ... you have to earn what you receive ... a law itself)

This section is intended to be a very quick overview of some of the spiritual laws of creation. It was written at a time when I knew little about them and is only intended to be a rough sketch. We learn far more about these laws and how they came to be as the essay progresses.

There are at least three fundamental spiritual laws of creation, these being free choice, forgiveness and karma. The axiom of free choice, in spite of all its usefulness, leads to many contradictions, both in mathematics and in any realm where it is applied outside the bounds of at-one-ment. One could go so far as to say that the axiom of free choice, therefore, only becomes complete and consistent once the state of at-one-ment has been reached and maintained.

The second law, used properly, can save a soul many incarnations, but beyond that I don't know all the ramifications. According to Angie Fenimore's NDE, Christ's work on the Cross forgives us of much. While I find this comforting, it is always important to remember that 'intent' is ultimately the Light's measure -- a notion of high importance within the world of spirituality.

As to the third law, much has been written about it, so I'm not going to chime in here. Suffice it to say that God is probably bound by the spiritual laws of creation which, among other things, demand that harmful debts be settled. In fact, Fenimore was told that for every harmful act, an act of suffering is required, which is a law, she says.

There is most likely a fourth spiritual law of creation which is the conservation of truth. We'll discuss that later, but for now, it is important to recognize that creation is one giant, self-correcting feedback loop with many spiritual laws. Our goal, as souls, is to live in harmony and balance with these laws by developing a reverence -- a very deep reverence -- for their power and their source.

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harmony & balance <~~~~> laws of creation

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The Second Coming and What It Really Means (it is within)

According to the Light you save, redeem and heal yourself. If so, there is no such thing as someone doing it for you. In order to accomplish this, you must learn to conquer the circles and walk away from 'all things base'. The focus has to be on the higher-self, to the exclusion of all other things. It may or may not happen in a single incarnation; odds are it will take many, but when that point arrives, you are done.

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See with the spiritual eye and not the physical, for the physical binds to physical, but the spiritual does the opposite. There is no point to the physical experience other than training the soul, though few understand this.

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Odds are, when it is all said and done, that all roads lead to Christ. There is too much NDE data to suggest otherwise. It is also probably true that Christ, meaning the ever-growing soul most favored by God, has attained complete at-one-ment with God. When this happened I do not know, but there is a chance this perfect union reached its fulfillment on the physical plane during the last incarnation. As such, there may be no distinction between Christ and God--an extremely hard thing to believe and accept, but the chances are fair to high that each is an image of the other. It is also possible that no other soul has ever attained this type of union in any realm and may never do so. Certainly the NDE data supports this conclusion, but see below for more, especially in relation to Noether's work.

If these findings are true then all religions, faiths, creeds, sects, and systems of belief are to be discounted--a notion that ties in with the work of Godel and Incompleteness. At best they become approximate truths but are inadequate nonetheless. The goal becomes one of trying to attain at-

one-ment with God, realizing that it may never be possible. The pattern, to the degree it's worth emulating, would be Christ, the ever-growing soul most favored by God. I do not feel, in my 23rd attempt, it is worth going any further.

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all roads        ~~~~~> | ~~~~~> Christ
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[11 23 11--> 1 + 10] triplet to [23,36,45] with trace 23; union with God, return to God, mastery on the physical plane, enlightenment; gap 12 which is perfection; 11th level; 23rd incarnation; Fibonacci; EDS (he) and SPD (she) ; Feb 5th/2011 see a license plate while driving 1123 KW (K is the 11th letter, W the 23rd) ... what are the odds ... April 15th/2011 see another license plate while driving with decal expiry May 19th ... what are the odds ... June 3rd/2011 see a cross in the sky made of double contrails ... June 7th/2011 someone from behind me puts their hand on my right shoulder whilst looking for bread at Superstore, but nobody is there when I turn around ... definitely felt it ...

On Aug 2nd/2011 I noticed a cross at Gyro Park, made of dry wood, standing upright and firmly embedded in the ground. It was by a tree I was sitting near , though at first I didn't see it. I had been thinking of the Cross rather intensely in the last few days, particularly after discovering that Christ does indeed have membership in the Godhead.

A year ago it was the wreath in Sutherland Park made of twisted strands of bark and wrapped in leaves. The wreath caused me to think more deeply about the sufferings of Christ back then, and no doubt sparked my interest in writing this essay. Maybe next time it will be the nails ...

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Finally, the NDEs of George Ritchie and Angie Fenimore strongly suggest a literal interpretation of this second coming is also possible. See as well Howard Pittman. And Cayce mentioned 1998 as the year of the 'birth' of Christ.

This is important to note because in Pittman's NDE, for example, Pittman says explicitly that God is building an 'army of soldiers' through whom He will work miracles of a kind never seen before. God is apparently taking this approach in preparation for the 'return' of Christ, I am presuming, in much the same way that John the Baptist was a precursor to Jesus the first time around.

As of today, such a 'voice in the wilderness' performing miracles of a kind and of a caliber that supersedes anything we've ever seen before, including the work of Moses, has not come along in my view.

Perhaps Moses was the precursor to Joshua, just as John the Baptist was to Jesus, just as 'someone' will be to Jesus again. What or who that 'someone' really is, I have no idea, but I think it's fair to

say this 'someone' will be a 'desert man' of a type and shadow, who quite possibly doesn't even know the essence of his spirit or his destiny.

When you think about it, neither Moses nor John the Baptist understood their specialness until the appointed time, but both were 'men of the desert', to be sure. In fact, only Cyrus really knew, which perhaps makes him a more special figure in history than anyone else.

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The Crucifixion (an expression of at-one-ment; hematidrosis; symmetric justification)

This has puzzled me for some time now, but on or around Sep 1/10 I had a strange thought concerning this notion. It should probably be seen more in a spiritual context than physical and most likely is symbolic of the death and resurrection of creation at large, spread throughout the entire virtual grid. A pervasive sign, for those who can see, that such a resurrection or reharmonization is inevitable. In other words, a return to unity, everywhere, as we learn to self-correct from our experiences, primarily in the physical plane.

It was the task of the Christ soul to show us, in the extreme, how we might gain mastery over the physical plane and recover at-one-ment, which I'll refer to as the pattern. The Cross, the nails, the crown of thorns, and so on, are simply remnants from this pattern, left behind for us to ponder.

By us, I mean any soul within the virtual space of creation. It (the pattern) is not specific at all to those on this planet--not by any means--and to a greater or lesser degree, all souls may have knowledge of this pattern, wherever they are.

If I am correct about this interpretation, one can see just how valuable the Christ soul is to God, for it has been charged with not only laying down the pattern, but also ensuring that each and every one of us learns to embrace it in our own unique way. I can't imagine a more important role in creation and personally, wouldn't want the job myself.

And finally, if Israel of old was ever intended to become a beacon of light for this world or any other realm, that particular design was and is woefully incomplete. You could say the Christ soul became the completion of that pattern, which is the expression of at-one-ment as opposed to the ancient notion of atonement.

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August 10th/2010 'when the student is ready the teacher will come'

An unusual dream in which an eagle divided me from a pastor I once revered. I was young in the dream, maybe in my twenties or early thirties, and handsome. The pastor was older and fat, but tried to disguise his age with a long, curly black wig. I can remember trying to talk to him through his joviality, but the eagle somehow divided us. We no longer seemed to understand one another, the pastor and I.

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The eagle represents spiritual protection, carries prayers, and brings strength, courage, wisdom, illumination of spirit, healing, creation, and a knowledge of magic. The eagle has an ability to see hidden spiritual truths, rising above the material to see the spiritual. The eagle has an ability to see the overall pattern, and the connection to spirit guides and teachers. The eagle represents great power and balance, dignity with grace, a connection with higher truths, intuition and a creative spirit grace achieved through knowledge and hard work.

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What Is At-one-ment Anyway, August 23rd/2010

Imagine two objects, A and B. Suppose each has its own individuality which we'll call the origin. If A contains B and B contains A, generating at-one-ment, then this can only happen if both A and B are infinite, otherwise, the origins would have to shift causing one to lose one's sense of individuality. Thus, at-one-ment only makes sense and can only really exist, while still preserving both origins, if A and B are infinite. If now A is God and B is any soul, this union or at-one-ment can only happen if B expands into A, since A is already infinite. Note that this blending into infinity causes one to lose one's sense of individuality because the origins of A and B would appear to converge as the expansion took place . At-one-ment, then, removes the feeling of being a single, distinct entity.

$$A \subset B \text{ and } B \subset A \quad <\sim\sim\sim\sim\sim\sim> \quad \text{at-one-ment}$$

No doubt this was the original state we as souls found ourselves in. As our thought patterns emerged that union was most likely broken and we became finite in the sense of being more aware of ourselves and those we clustered with. New realms probably emerged, according to these thought patterns, with each containing a subset of like-minded souls. This break in the symmetry or at-one-ment is what God is trying to repair, a daunting if not impossible task in my view.

As to the Christ soul, it now becomes an interesting question. Once the break occurred, was the union between this soul and God preserved or did it, the Christ soul, separate as well. It is my personal view that the union was always preserved in the higher realms, which might mean any incarnations on the part of this soul were solely for our benefit. Certainly the work of Noether supports this conclusion.

It is also highly probable that this was the only soul to retain its union with God, post-break. If so one could postulate that no other soul has managed to regain that union and may never. It is an open question for which I do not have a definitive answer, but theoretically, even an infinite

number of incarnations may not be sufficient to achieve the at-one-ment that was lost. How it is regained I do not know, assuming the question makes any sense.

But there is a corollary to all of this and it goes as follows: if the Christ soul was the only soul to retain its union with God, post-break, that was by design in all likelihood. And this can only mean that God knew the break was inevitable, but allowed it along with an embedded pattern we might emulate in order to regain or strive to regain what we 'lost' originally. How we choose to incorporate that pattern into our existence as souls is somewhat arbitrary, realizing that our intent is far more important to God than our failures. But the lifeline is there for those who need it, and it should be used because the mandate of the Christ soul is to help us along in our spiritual growth.

If I am on the right track here, the Light, which is God, was correct when it said we must learn to save, redeem and heal ourselves. It's fine to offer the pattern, but in the end, the pattern is not a saviour, redeemer or healer. We are, each and every one of us. These are my findings as of August 23rd/2010.

Porn and Sex

I wanted to say a few things about this wretched addiction. Over 50% of all evangelical preachers admit to using porn, which means ~ 80% use it on a regular or semi-regular basis. Talk about 'don't do as I do ... do as I say.' We should do our best to stay away from it, but having said that, I have not read a single NDE account where the Light mentioned the experiencer's use of porn, if applicable. The Light, in general, doesn't seem too interested in our sexual preferences, orientation or habits, choosing instead to focus on motive and intent, cause and effect. My little goal in life is to make God's job here a little simpler by becoming a soul he doesn't have to worry about too much. I think if I can do that I'll be satisfied when it's time to go, which hopefully won't be very much longer.

To me, anyway, the world is an overcrowded insane asylum for the spiritually bankrupt and I feel very uncomfortable being here. At the same time I realize how critically important the physical plane is for growth and understanding, but nonetheless would prefer to let the dead have sex with the dead. It isn't for me anymore.

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According to the literature, there are over 500 forms of paraphillia. Even if we reduce it to the 80 most common, which you can read about on Wiki, and assume a normally distributed population regarding each of these forms in particular, there is roughly a 93% chance that any person in the population will be plagued by at least one type of paraphillia or another. In other words, restricted to sexually distorted behavior alone, most of the planet is dysfunctional even though any member seems normal.

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Open Questions

Has the Christ soul always been in perfect union with the Light in the higher realms ? My hunch is 'yes'.

Has the Christ soul always been in perfect union with the Light on the physical plane with each incarnation ? I don't know but my hunch is 'yes' in the last incarnation we know as Jesus [cf Susan Blackmore's OBE].

Has any other soul always been in perfect union with the Light in the higher realms ? My hunch is 'no'.

Has any other soul always been in perfect union with the Light on the physical plane with each incarnation ? Most likely 'no' regardless of how many incarnations that soul has had.

Conclusions

The goal is at-one-ment with God, irrespective of realm, region, remnant or reality. The pattern is the Christ soul to the degree it's worth emulating. You go at your own pace and you choose your own path. Intent is the Light's measure.

A Prayer To God (which I hope he reads and runs with)

Please Dear Father God, begin to tear down the structures on this earth that are harming us the most. There are many souls, from Sodom and Gomorrah of old, who are here today and would seek to damage again and again all that you are trying to repair. You know who they are, what they do and the designs of their hearts. If you will not intervene and put these souls in their rightful place things will only get worse, for they continue to spread their errors, their confusion and their culture, taking many with them. They should be made to forfeit their own lives in all fairness.

At the same time you must enlighten each and every one of us who desires the truth despite our many failures. That enlightenment goes well beyond the teachings of any religion but is available to all who seek it. As our spiritual understanding grows the world will change, but it needs to be instilled in those who are able to receive. These are the ones that love truth, which is to say, you Father God.

So you need to do both--tear down and build up--and that is what I would want the most from you.

The Focus Of My Prayers

Since God probably isn't interested in 'begging prayers' I thought I'd focus mine in five different areas, and try to make them as fluent as possible -- adoration, contrition, gratitude, understanding, separation. We'll see where it takes me, if anywhere.

A Note On The Hierarchy and Physics Of Realms (just my ideas)

No doubt, each usable or emerging planetary system has an associated set of higher and lower realms. Beings working in the light-filled realms would be functioning in some beneficial capacity tied to the planetary system, and beings in the lower realms would be there because of entanglement issues. The realms are most likely infinite in nature, and every usable planetary system in every galaxy in every universe would have such a set of local realms.

Galaxies and universes may also be associated with light-filled realms, in the same way as planetary systems, but may or may not have lower realms. The complete distribution of light and dark-filled realms throughout the virtual grid of creation is unclear to me.

The management of this hierarchy would be top-down I should think, from the universal to the galactic to the planetary realms. One can see just how insignificant we really are as a lonely planet on the outer edges of an ordinary galaxy, amongst an almost infinite number of galaxies in our universe, which is one of an almost infinite number of universes.

In between physical universes is what I call the pre-big bang consciousness of God, for lack of a better term. It may be part of the physical plane, but most certainly could not be detected by any human or man-made device. Whatever this energy field is, it is something we as humans do not know, but it is the medium through which information and force flow from one universe to another. As such, when designing a theory of the cosmos, say Relativity Theory or whatever, one cannot look only at this universe, but must look at all universes together, as each has an influence on all the others.

No doubt universes orbit universes, just as stars orbit stars, and something like gravitational pull would exist at the inter-universal level, just as it does at the inter-galactic level. So when talking about the bending of a universe, one cannot look at only the material inside a universe but must also look at external influences carried through the pre-big bang energy field. Whatever curvature tensor or momentum tensor is used to explain something like Relativity, it has to be designed to accommodate all objects in the physical plane and the forces that exist inside and outside any particular universe. This would mean understanding the pre-big bang energy field, among other things.

No doubt the pre-big bang energy field does not contain the dimensions of time and space as we know them, which means a whole new physics would have to accommodate the flow of information and force between universes. Once that information or force reaches the boundary of any universe in particular, and penetrates it, the physics we have today could probably be used to describe things. As such, physics as we now know it is but a tiny subset of a larger physics hidden in dimensions which could be described as more spiritual than physical. It is on the boundary of any universe that the Larger Physics quite possibly morphs into the Smaller Physics, with the former based on an almost infinite number of spiritual laws and the latter based on a smaller set of disorganized laws that we have before us at present.

It is unlikely we will ever unite the Larger Physics with the Smaller Physics as this is akin to uniting the Spiritual with the Physical, and we as humans in particular, are a long, long way off from ever becoming spiritual beings first and physical beings second. Until we change, stamp out

the corrupt ego that dwells within each of us and find at-one-ment with the Father God, we are doomed to live out our days in a state of utter spiritual ignorance. How sad.

A Note On The Human Shell and How It Came To Be (necessary foolishness for the foolish, necessarily)

I don't regard the shell we currently live in here on earth to be very important, as any vehicle would suffice if we could learn sufficiently well through it. Having said that, the most likely explanation is the one offered by Cayce, which goes something like this:

Sometime in the distant past, maybe 2 million years ago or less, a first wave of souls decided to descend into whatever shells were available on the earth at the time and deemed usable. These would have been ape-like creatures of the Homo erectus type, in all probability. But the shells really were not sufficient for learning in, and upon leaving them, these souls were not able to find their way back to their origins and instead, became entangled in lower realms, where perhaps some or maybe many still linger to this day.

A second wave of more enlightened souls would have come along sometime later, not so much to descend into similar shells, but rather, to guide the course of future evolution by modifying DNA in these shells, so as to produce a suitable vehicle we now know as the Homo sapien or modern human being. The reason for doing this was to assist the first wave of souls (now in an entangled state) in freeing themselves from entanglement by allowing them to descend into the new and improved shells, with the hope that lessons learned would return them back to the Light upon death.

These newer shells were also designed so that many other souls who have never had the opportunity to descend could now do so, thus giving them lessons they really couldn't obtain in any other way. The earth, therefore, becomes a school for souls, designed to instruct and correct, so that ultimately, after a sufficient number of incarnations, the need to return to this planet goes away. It is my view that such a process will continue for millions of years into the future, if not billions.

I also believe an evolutionary intervention could have taken place in any other usable planetary system in any galaxy in any universe for similar reasons, but have no way of really knowing.

A Note On Noether's Theorem and The Father-Son Duality (around Sept/2010)

In what follows, the words image, symmetry, parity, connection and duality will be used interchangeably. Here we go:

`for every symmetry there is a conservation law and for every conservation law there is a symmetry` according to the work of mathematician Emmy Noether, regarded as one of the most influential thinkers in the last century.

I listened to a one-hour youtube talk on Emmy Noether and her work in algebra which ties together symmetries and conservation laws. The theorems are used to this day by physicists in many disciplines.

At about the same time as I listened to this talk I remember finding a wreath in Sutherland Park, my favorite summer place to go. I also remember watching Mel Gibson's The Passion of the Christ, so maybe the stage was being set for what follows ...

I postulated a while back that Christ and God may be duals, meaning symmetric. That is, looking at one is like looking at the other. Is it possible that the conservation law which follows is really the law of conservation of truth? If this is so then either could speak on behalf of the other, for they are symmetric duals. There is a reference to this idea in Angie Fenimore's NDE which might be a hint, so let's keep going.

One could also ask the reverse: what symmetry results when speaking of the law of conservation of truth? A symmetry of some kind must exist, but what is it? I'm thinking it is some mirror image or dual of God, and the only image that makes sense to me at this point is the Christ soul, based on NDE research.

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      Noether's Theorem
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duality <--> action <--> truth
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The law of conservation of truth is most likely an axiomatic spiritual law upon which all of creation rests, just like the axioms of free choice, forgiveness and karma. It would be the fourth law I've become aware of. What is amazing about this law is that it necessarily implies the existence of a symmetry in the Godhead, which might be a duality, triality, or something even beyond this. At a minimum it suggests a duality, which is likely the Father-Son connection through at-one-ment that I've been looking for. If the symmetry is greater than this, it could ultimately extend to all souls, which may be the symmetric state of creation that existed before the almost spontaneous break which led us to where we are now.

In other words, we were all in union with God, wholly and completely, before we decided not to be, but the law of conservation of truth could not have been violated, which means that even 'post-break', a symmetry had to exist and that symmetry most likely was the Father-Son connection through at-one-ment.

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If I am correct, the Christ soul never lost its at-one-ment connection to God in any realm, which would answer some outstanding questions I have had for a very, very long time. In fact, the Father-Son connection through at-one-ment is what upholds the law of conservation of truth, and this is perhaps the most profound thought I have ever had in my 55 years of living on this earth in what is probably my 23rd incarnation. May all souls in all realms reading this fully appreciate what is being said here.

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In short, the Father-Son connection, through at-one-ment, is both a necessary and sufficient condition for the law of conservation of truth, which can neither be created or destroyed. Truth, then, in God is both complete and consistent, and you could, therefore, say truth is the embodiment of the spiritual laws of creation.

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In the vision or dream I had where two perfectly symmetric, extremely powerful, beautiful male beings appeared, I got the feeling that we were tied together through the numbers on the 3 solid-core orbs. Perhaps at a deeper level this 3-fold symmetry, as exemplified in a set of triplets, is also present in the Godhead [Father, Son, Spirit]. Something to think about.

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Think of a chessboard with red and black squares. Let the red squares emit a truth, defined by the arrangement of those squares on the board. Now paint the red squares black and the black squares red, and let the red squares emit the truth again. Since the board is as it was initially, the same truth is emitted from what were originally the black squares. Thus, because of this symmetry the black and red squares have emitted exactly the same truth, so we would say truth is conserved under this symmetry.

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Noether's Theorem Revisited (truth ... the embodiment of the spiritual laws of creation)

I am going to dig more deeply into this theorem, for it states that 'differentiable symmetries via the action of a system lead to conservation laws'. If here the system really is the [primordial] Godhead, and the conservation law is the Law of Conservation of Truth, then some action exists within the primordial Godhead for which there is a differentiable symmetry that upholds this universal conservation law.

Furthermore, if each law contained within Truth is conserved, then a corresponding symmetry tied to some action within the Godhead upholds that law. There may be many 'discrete symmetries' that exist as a result, but in the aggregate they would lead back to the more universal notion mentioned above.

What does this mean ? It probably means that God is the union of a set of (perhaps infinite) laws generated through corresponding symmetries and that this union of symmetries itself is a symmetry that implies the Law of Conservation of Truth. The set of symmetries and their union could well be one and the same thing.

What the underlying action of this system we will call the Godhead really is, I don't know just yet, but such a notion probably exists and may be 'characterizable' through some of the math used to prove Noether's Theorem. I am going to have a go at this one, depending on how much brainpower I've got left.

As an example, consider the law of 'harm and suffering' noted by Fenimore. Every act of harm must be met with an act of suffering, according to AF. If this law is conserved, a corresponding symmetry in the Godhead exists which upholds it. That symmetry is most likely the Father-Son

duality through the at-one-ment connection, via the action of Christ's sufferings, which culminated in his work on the Cross. Since there is no time in the β-world, these sufferings always were, and are both necessary and sufficient for the removal of any and all harmful acts throughout the virtual space of creation, in my view. In essence, this is the law of 'harm and suffering' as dictated to Fenimore, from the perspective of the Godhead.

It should also be said that for every spiritual law which is conserved, the (symmetry, action, conservation law) triplet might have been formed in the β-world 'long before' the physical plane 'existed', and perhaps long before souls were brought into being. In the case of the 'harm and suffering' law above, we note the action, in part, is manifested in the physical plane. This might best be explained by noting that harm has also manifested itself in the physical plane, or it may be God showing us a glimpse of his profoundly forgiving nature. Whether action always manifests itself physically for every conservation law, I do not know, but like its two counterparts, symmetry and invariance, action always was.

Finally, I am of the belief that (symmetry, action, conservation law) are all equivalencies, taken two at a time. Thus, in the example above, the action of Christ's sufferings, which culminated in his work on the Cross, is equivalent to the Father-Son duality through the at-one-ment connection, and both are equivalent to the law of conservation of 'harm and suffering'. Seen in this light, Christ not only becomes a symmetry in the Godhead, but also a suffering Christ whose work on the Cross is both a necessary and sufficient condition for the removal of any and all acts of harm.

In a nutshell, this 3-fold equivalence, which always was, is God's gift to mankind, for which I am deeply grateful. And it is the only such 3-fold equivalence I am aware of that has manifested itself physically, thus providing any observer on earth some deeper insight into the nature of the Godhead. Indeed, it may be the way back home.

Jesus said "I am the life, the way, and the truth ... (symmetry, action, conservation law). It seems the Bible was correct after all, as these words are materially no different than the 3-fold equivalencies discussed here.

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duality <--> action <--> law of h&s <--> truth
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A Virtual Creation With and Without The Law Of Harm and Suffering

Imagine a virtual creation where this law did not exist. There could be no harm, for there would be no way to 'cover' the harm through suffering. Thus, any soul would, in all likelihood, be confined to the Godhead in a state of at-one-ment, even if it were an embryonic utopia for that soul. In short, there might be little to do.

With the law, it is a given that harm will exist, though now, necessarily, it will be covered by suffering to keep the books balanced, as they say. But also, any soul is now allowed to make as

many mistakes as it needs to in an effort to grow, self-correct and learn throughout its seemingly endless cosmic journey.

There is no longer at-one-ment with the Godhead--by design--but in return, the soul is presented with a virtual creation in which there are an infinity of choices as it inches ever closer to perfection by self-correcting at each step along the way.

As such, at-one-ment, like infinity itself, may be illusory. The aim of the soul is not to be in a state of at-one-ment with the Godhead, but rather, to strive to attain this virtually unattainable state. Creation and its laws would have it no other way--particularly the law of harm and suffering.

So was there ever a 'fall from grace' ? I say there was not. We may have all lived in an 'embryonic utopia' initially, just as an unborn child makes the womb its home for a while, but eventually it would have been time to move on and begin the grand journey.

The words the Light spoke to Mellen Thomas Benedict could never be more true -- you do indeed save, redeem and heal yourself. We always have ... we always will.

The Equivalency Of The Spiritual Laws Of Creation (ignorance is no excuse for the law)

We are arriving at a point where we can start to think more sensibly about these laws and whether or not they are, in fact, equivalent. By equivalent, I mean they imply each other, taken two at a time.

Let's start with the law of free choice, and assume throughout, that all spiritual laws of creation discussed here are conserved. In an open creation, meaning non-embryonic-utopian, the law of free choice will inevitably lead to acts of harm, implying the existence or necessity of the law of harm and suffering in order to 'cover' these acts.

Conversely, in an open creation, the law of harm and suffering can only exist if acts of harm exist that must be covered. But acts of harm can only exist if they are voluntary or involuntary and not all such acts will be involuntary, particularly in the physical plane. Thus, the law of harm and suffering implies the existence or necessity of the law of free choice.

So this argument seems to be telling us that the law of harm and suffering and the law of free choice are indeed equivalent, even though they seem unique. It is also telling us that in a closed creation, meaning embryonic-utopian, neither law exists, and indeed, within the Godhead alone [closed creation] there may be no laws at all. They would only actively exist in an open [non-embryonic-utopian] creation.

Now we know that the Father-Son duality through at-one-ment (symmetry) and Christ's sufferings (action) are equivalent to each other, and both are equivalent to the law of harm and suffering (conservation law). Replacing the law of harm and suffering with the law of free choice leads us to the conclusion that the same symmetry and action which upholds the former also upholds the latter.

One might expand on this train of thought and conclude that any two spiritual laws of creation are indeed equivalent, and that the same action and symmetry which upholds one also upholds the other.

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In other words, there is only one symmetry responsible for each and every spiritual law of creation, and there is only one action responsible for each and every spiritual law of creation. The symmetry is the Father-Son duality through the at-one-ment connection, and the 'action' is Christ's sufferings which culminated in his work on the Cross. Somehow, the completion of the physical component of this action upholds [is equivalent to] both the symmetry in the Godhead and all spiritual laws of creation simultaneously, for it (the action) binds these two pieces together.

It's almost as though Christ's work on the Cross unified, completed, activated, solidified and made equivalent all spiritual laws in creation simultaneously, and justified [upheld] his symmetry in the Godhead at the same time. Truly a remarkable offer and accomplishment for mankind to puzzle over, for as they say ignorance is no excuse for the law.

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If truth (necessarily) is now defined as the union of the spiritual laws of creation, truth becomes the union of a set of laws where each law is equivalent to any other, and thus the same symmetry and action which upholds any one of these laws also upholds truth. This was our starting point when we first discussed Noether without regard to action and also laid down a heuristic definition for truth.

We have, therefore, come full circle and located the 'action' which binds the symmetry in the Godhead [Father-Son duality] to this more general law; namely, the law of conservation of truth. Whether there are other equivalent actions and symmetries is something we're going to look at a little later.

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duality <~~> action <~~> law of truth
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                |
              Cross
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The Embryonic Utopia, Action Revisited and All That Jazz

The embryonic utopia, which you could also refer to as the womb of God, is perhaps the birthing place for all souls. Whether birthing is a continuous process I do not know, but it is probably safe to say that those souls who linger in this 'womb' are typically more reluctant to move forward with their cosmic journey once they leave it.

That is because the Godhead [embryonic utopia] is really a state of perfect equilibrium in which no laws exist that are active. The spiritual laws of creation really only exist in an active sense of the word outside this womb, in what we call the non-embryonic-utopia, or open creation.

Many of these reluctant types might never have incarnated even once on the physical plane, and it wouldn't surprise me that those who do are more prone to suicide, depression and other related phenomena.

On the other hand, braver souls that long ago left the womb and have since incarnated many times, would have gained the courage necessary to deal with challenges in any physical realm, and could well function as mentors for the 'younger' ones. So goes the cycle of helping and being helped within this loving, linked community we call God's children.

It may also be true that a soul decides at some point it no longer wants to participate in creation. In such a case, that soul might be able to re-enter the womb under dissolution and never return to the open creation again. Its essence, however, would be gone if dissolution were allowed.

Some evidence for this notion can be found in Sylvia Browne's writings, but regardless, it probably doesn't happen very often, if at all.

Earth, The Planet Of Sorrows (Feb 18th/2011)

It occurred to me recently that it is no coincidence that this planet [earth] is also known throughout the virtual space of creation as 'the planet of sorrows'. Why might this be ?

Because I suspect there is no other planet like it, anywhere in any universe, and if I am right, it is the only planet a soul can come to where the learning experience is optimized.

If I am correct, it is then no coincidence that the action which binds the symmetry in the Godhead to the Law of Conservation of Truth was also manifested here on earth. That 'action' [Christ's sufferings and the Cross] was, no doubt, established in 'blueprint form' within the β-world [closed creation] long before it was mapped into the open creation--indeed, long before there may have even been an open creation.

The mapping needed an endpoint in the physical plane in order to create legitimate suffering and illumination. It appears as though earth was the best candidate, given its instructional mandate, and given the fact that it is uniquely defined by its sorrows. I suspect there really was no other choice, which, among other things, would serve as a testament to the importance of this world and its continuation well into the future.

What's In That Godhead, Anyway ?

I was thinking about the Godhead lately, and realizing that it is a perfect equilibrium which contains no active laws, thought it might contain everything there is to know about the open creation, in blueprint form. One could even speculate, for example, that all things in the Godhead, in blueprint form, are mutually symmetrical or equivalent.

An analogy might be the two-dimensional plane with a unit disc centered at the origin. Every point in that disc has a counterpart outside it, extending to infinity, with the center of the disc representing infinity in the open plane. One could think of the center of the disc as some kind of singularity, not unlike a black hole found at the center of a galaxy, for example.

What's interesting about this analogy is that a law surfaces, namely the law of conservation of information, because every point of information inside the disc has a corresponding point outside the disc. Nothing is lost in going from the closed creation to the open creation, and everything that ever was, is and will be is contained within that disc--the closed creation.

We could call this disc the Godhead; home to the symmetry that gives rise to the Law of Conservation of Truth by way of an action that was mapped to the open creation and onto the physical plane according to the system of things.

The mapping was necessary in order to induce suffering to bring into being, or complete, if you will, the law of harm and suffering. By doing this, all laws were somehow unified, completed, solidified and made equivalent to one another in the open creation, thus forming the basis for the Law of Conservation of Truth. It should also be said that such a mapping would be independent of time and space, dimensions we are all too familiar with here on earth.

One might say, then, that Truth was 'infused' into the open creation by way of this (action, symmetry) pair, independent of any dimensions associated with the open creation. What we observed 2000 years ago or so, is simply what the dimensions of time and space let us see back then, but in reality, the infusion always was.

* * *

If you are really bold, and equate the open creation with Truth, itself a union of the spiritual laws, you could say the same mapping that infused Truth actually brought forth the open creation. Or said another way, the 'action' that binds the closed creation to the open creation is indeed the Cross, and all three of these--closed creation, open creation, and the Cross--are really equivalent. There may be no other action that does this though admittedly, a huge leap of faith is required to accept what we are saying here.

The Cross, then, becomes a spiritual portal if you will, a gateway through which an observer in the open creation can peer into the closed creation and examine any symmetry that exists there. It is the gateway to the Father-Son duality, for example--something I wouldn't have guessed originally. Similarly, the Father-Son duality through the Cross can look back at an observer in the open creation and examine things accordingly.

Perhaps the proper analogy would be the boundary of the unit disc that separates the two parts of the plane, much like the event horizon of a black hole. This spiritual boundary, it turns out, is both necessary and sufficient for the existence of each creation--the open and the closed--and is, from what we've said, equivalent to the Cross itself. Admittedly, though, other actions may exist.

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The parallel in mathematics would be the Jordan Curve Theorem, if you want to get technical, and may be worth looking into further. Maybe in the end, God really is 'mathematics personified' ...

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A More General Look At The Virtual Space Of Creation

Well I wasn't going to write on this subject at all, but feel pressed to do so. I'm going to start by dividing up creation into the three pieces we've been talking about and offer a little detail on each. Since I really don't know the physics of it all, we'll have to be content with generalizations.

First we've got the closed creation, which I'll refer to as the Godhead. This is the place where there are no active laws, but contains all that ever was, is and will be. One could argue it is a fusion of symmetries and equivalences bathing in a perfect equilibrium of nothingness -- no dimensions, no space, no time, no anything you might be familiar with through your senses here on earth. Among other things, it is home to the Father-Son duality that we've talked about in the past, but there may be many more symmetries or equivalencies in this creation--even infinitely many. The best analogy I can come up with at this point is the black hole from physics and cosmology.

Then we've got the open creation, which exists beyond the boundary of the closed. In the open creation are realms, regions, remnants and realities endowed with their own unique dimensions and infused with the active spiritual laws of creation. Here you will find, among other things, the physical plane; its dimensions of space and time, however illusory, and souls functioning through shells on different planetary systems in different galaxies and universes. The best analogy would be everything outside the black hole, in both the physical and non-physical planes.

And finally, we've got the boundary separating the closed creation from the open creation. This is a most peculiar region, because it binds together the open with the closed without explicitly being one or the other. Here you'll find the action we've talked about that sits between them, and is equivalent to, the Father-Son duality and Truth. That 'action', as we've said, is the Cross--an event portrayed to us in the physical plane some 2000 years ago. The best analogy concerning this boundary would be the event horizon of a black hole, a kind of scary, eerie concept for most ordinary folk.

Now the boundary may contain or emit many 'actions', perhaps infinitely many, which may all be equivalent to one another. Any one of them could serve as a binding between the closed and open creations, not just the Cross that we've studied already.

It's quite possible the physical plane intersected this boundary [event horizon] at a single point in space-time from our perspective, manifesting the event we call the Crucifixion and were witness to on this planet almost 2000 years ago. Other observers on other planetary systems might also have experienced a similar intersection at some point in space-time, but seen the 'action' differently, even though any two actions on the boundary would be equivalent. Whether that observer's space-time point was in our past or our future is irrelevant; only its existence matters.

So in the most general sense, we've got the closed and open creations, and the boundary which joins them together. All three are equivalent to one another, in so much as you can't have one without the other two, and theoretically anyway, the boundary could contain or emit an almost infinite number of equivalent actions, any one of which could serve as a binding. That binding, in turn, would lead to some equivalent symmetry in the Godhead which could then be tied back to Truth.

Why is this important to us ? Because it generalizes creation to the point where souls in different parts of the physical or non-physical planes, for example, can peer into the Godhead by way of some 'action' [portal] located on this event horizon and identify with some [symmetrical] aspect or facet inside it. It just so happens that on earth, a planet uniquely defined by its sorrows, that action is the Cross, allowing us to ponder more deeply the Father-Son duality which itself is equivalent to Truth. Other observers could well see it differently.

Thus, there may be many paths and portals to God, depending on your frame of reference within the open creation. In the end, however, the Light is not going to care about your specific set of beliefs; rather, your intent. It always has ... it always will ... and necessarily must ...

Noether's Theorem ... The Rosetta Stone Of Creation ... Maybe (April 23rd/11)

We're going to extend the notion above by way of an example on the physical plane in some other part of the space-time grid. The aim is to discover other symmetries in the Godhead and corresponding actions that bind the former to the law of conservation of truth, where truth is defined as the union of the spiritual laws of creation. I should point out initially that this note is somewhat hypothetical, and has no basis in fact when studying the near-death research. Nonetheless, it is worth considering, even if it's nothing more than a theoretical curiosity.

So imagine, if you will, a civilization on some other planet in some other galaxy in some other universe. In this world, there is no concept of offspring or reproduction, nor do shells really age. New shells are simply brought into being by some cloning process that allows for distinguishing features, and thus individuality. We'll assume male and female, and fix the age of each at about 30 years, from our perspective.

Additionally, we'll assume this is a very advanced civilization that has learned to live in harmony and balance with the spiritual laws of creation. There is no evil here, no dark side ... just the opposite of what you see on earth, for example.

Now let the boundary of the Godhead, where actions are stored, intersect the space-time grid at exactly the point where this particular world is located. The question could then be asked 'how best would the Godhead show itself to this civilization ?'

To answer that question we have to take the view that Noether's Theorem is like a Rosetta Stone of sorts, tying together in a universal way the notions of (symmetry, action, conservation law). We have already conceded that many actions and symmetries might well exist that ultimately lead to the law of conservation of truth, and that there is an overarching equivalency among and between these three notions.

It is highly unlikely the Godhead would present itself as a 'Father and Son' symmetry since there are no offspring on this planet, nor would it make sense to offer up the notion of a 'suffering Son' as some kind of 'action' binding this symmetry to truth. Indeed, that action has already been mapped to the physical plane elsewhere.

Instead, it is my view that a male-female symmetry would make more sense with the law of unconditional love being the focus of the third component of Noether's Rosetta Stone. What

remains is some action sitting between this symmetry and this law which binds the two together--a very real, physical action that this particular civilization would have been witness to in a very otherworldly way. Such an event -- a visitation, cloud formation or configuration of the stars, for example, could easily do the trick, particularly if it was deemed to be both mystical and extraordinary.

And thus, this world, through Noether's Rosetta Stone, sees the equivalency between male-female duality in the Godhead and the law of unconditional love by way of an otherworldly action that joins the two together. Quite the opposite of the Father-Son duality and its equivalent relationship to the law of harm and suffering via the action of the Cross that we see here on earth, and indeed, quite comforting as a matter of fact.

Nonetheless, both triplets are equivalent to one another, and both are equally valid paths back to the Godhead if we are to adopt Noether's Rosetta Stone as the standard through which these equivalencies arise when looking at the generic notion of (symmetry, action, conservation law).

And finally, we have to concede as well that Noether's Rosetta Stone may always be at work in some unique way, no matter where we are within the virtual space of creation. No two observers will necessarily agree on what they see or feel, but to be sure, nothing stops the Godhead from continuously manifesting some (symmetrical) aspect of itself through some very real action which might, in turn, bind this symmetry to a spiritual law of creation and ultimately the larger law of truth.

One cannot, therefore, underestimate the importance of what is going on here, for it means, above all other things, that we must learn to tolerate and accept those who have a different perspective, realizing that at best our view is only part of a much larger picture--a view that is both incomplete and inconsistent according to Godel.

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many dualities <~~> many actions <~~> truth
       o                o              o
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       |                |              |
        --- Noether's Rosetta Stone ---
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Noether's Rosetta Stone Revisited, A CounterArgument

While the concept of Noether's Rosetta Stone is interesting, it probably goes against the grain of how nature actually works and how the Godhead might work as well. Nature is continuously minimizing the energy it expends for maximum gain, and I expect the Godhead to be no different.

Thus, there really is no need for more than a single (action, symmetry) pair within the Godhead that gives rise to the spiritual laws of creation, and so other equivalent actions or symmetries would seem redundant. One cannot, quite simply, avoid the variational principles at work here if one is to view the Godhead as perfectly efficient.

There is also no hard-core near-death evidence to support other (action, symmetry) pairs, and so one is almost forced into a corner whereby the Father-Son duality by way of the Cross leads to truth, where truth is defined as the union of the [equivalent] spiritual laws of creation. It is almost irrefutable--this conclusion--whether we follow the ancient texts, near-death research currently available to us, or an extension of Noether's Theorem.

Indeed, if one were to find such an equivalent action, it would have to be akin to the horror and the anguish associated with the Crucifixion, for example, and retain the same level of mysticism associated with the Resurrection. In addition, the corresponding symmetry in the Godhead would have to offer up the 'godlike' evidence Christ produced by way of his miracles and his sayings, leading to a very specific law such as the law of harm and suffering.

In other words, one would want evidence for such a (symmetry, action, conservation law) triplet which goes beyond folklore, and to be honest, I haven't seen it, anywhere. Indeed, the harder I look, the less likely it seems ...

Having said that, Noether's Rosetta Stone has both theoretical and contemplative value in so much as it allows us to consider other (action, symmetry) pairs even if they don't really exist. In that sense, it helps us to develop compassion, understanding and tolerance for others who are on their own road to discovering truth, and in my view anyway, there can be no greater lesson to be learned from all of this.

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one duality <~~> one action <~~> truth
   o                o              o
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   |                |              |
    --- Noether's Rosetta Stone ---
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The Crucifixion Revisited (weaving together those laws through a seemingly latent action)

On April 25th/2011 I had another strange thought concerning the Crucifixion--so strange, in fact, that I thought I'd write it down.

We've learned that Noether's Theorem tells us the Father-Son duality exists by way of the Cross, and leads, in turn, to truth, where truth is defined as the union of the (equivalent) spiritual laws of creation. This is the de facto (symmetry, action, conservation law) triplet upon which creation hangs, literally everywhere.

Indeed, without this 'action' (the Cross), you really don't have the closed and open creations discussed elsewhere in this essay, nor do you have the boundary separating them. In fact, all you may have is the undifferentiated energy of God everywhere and nothing more. Those differentiable symmetries (personalities) within the Godhead exist because of this (latent) action and lead to the (active) spiritual laws, which is to say, the open creation.

So one has to ask the question 'what really was going on, from the perspective of the Godhead, where there is no time and space, during the Crucifixion ?' What did Christ really mean when he said 'It is finished' whilst hanging on that Cross ?

To answer the questions we have to remember what the higher-ordered beings told Howard Storm during his NDE. The Bible, while true, is commentary and should not be interpreted literally. It is some kind of encoding which tells us how creation was woven together absent the dimensions we live in--that is to say, from God's perspective.

My guess is that during the time Christ was suffering on the Cross (about six hours), the open and closed creations were being formed, along with the boundary that would separate them. The open creation would have been completed by infusing into it the spiritual laws of creation that heretofore only existed in 'blueprint form'. Once this was done, a natural boundary would separate the closed creation [Godhead] from the open--a boundary which, from our perspective, anyway, is none other than the Cross.

No doubt the open creation, at this time, was lit up with an almost infinite number of 'big bangs' as universes came into being on the physical plane and started to expand into their current forms. Other realms on other planes might have come into being as well, and once this grand infusion of light, laws and energy into the open creation was complete, souls would have started to leave the Godhead to begin the great journey. The magnitude of such an event cannot really be measured by the human mind, leaving one to only imagine what must have happened in God's 'now' according to the Noether connection.

Genesis speaks of 'six days of creation' and Cayce mentions the number six as the Christ-force in nature, in addition to the former, and wouldn't you know it ... Christ spends six hours on the Cross. Almost too coincidental, in my view, that we should see this type of alignment without there being a deeper meaning.

What we saw 2000 years ago is simply what the dimensions of space and time let us see, and we are fooled into believing there is nothing more. In reality, we were very likely witnesses to the unfolding and authenticity of creation itself, which concluded with the words of Christ 'It is finished'.

In this sense, Christ's time on the Cross is the physical expression of an equivalent latent action stored on the boundary that divides the open and closed creations. From God's perspective there is no material difference between the two--they are both equal and simultaneous actions that cannot be separated. It is only from our perspective in the physical plane that things become confusing because we have bought into the notion of time and space--a masterful trick if there ever was one.

Without that suffering in the physical plane, you don't have completion of the law of harm and suffering, and without this law, by equivalency, you don't have any active spiritual laws of creation, meaning you don't have an open creation. In turn, this means you've lost both the open creation and the closed creation that would contain any symmetries, and by implication, the action itself. Effectively, all you are left with is the undifferentiated energy of God, everywhere, as we said before. This might well have been the state of the primordial Godhead prior to everything unfolding into what is now deemed life throughout the virtual cosmos.

Conversely, if there is no action, then clearly there is nothing, and in particular, there is no Cross. Thus, one could argue the Crucifixion really is equivalent to the latent action (boundary or membrane) that brought forth both the symmetries in the Godhead and the spiritual laws of creation, simultaneously. This is a profound notion, to say the least.

'In the beginning, God created heaven and earth' really means the undifferentiated energy of God that existed everywhere, initially, collapsed into a subspace of the virtual cosmos, bringing into being the open and closed creations, and the boundary that divides them. At this point differentiable symmetries in the Godhead would have emerged along with the (latent) action and (active) spiritual laws of creation--the ultimate big bang if there ever was one you might say. All three notions should be seen as equivalent to one another according to Noether, even though it may be hard for the mind to do so. We'll explore the idea a little more in the next section.

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primordial fabric <--> Noether <--> differentiable mosaic
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                          |
                          o
            duality <~~> action <~~> law of truth
                          o
                          |
                          |
                          o
                        Cross
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Alas, a tiny fragment of God's 'now' paid us a visit some 2000 years ago, and 2000 years later there is no one who seems to understand what really happened or why. If Noether's Theorem does apply, it is probably the best explanation we are ever going to get in terms of an answer, for it seems to unify things in a way that no mere mortal ever could, moving beyond the familiar dimensions of reality we are used to and up into the rarified space where only gods dwell.

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Christ never did die to 'save anyone's soul' and 'dying for the sins of mankind' doesn't come close to describing what really happened and why. Without that remarkably heroic act we call the Crucifixion it's very likely there wouldn't be a creation, for there would have been nothing to sustain it according to Noether.

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A More Detailed Look At That Thing We Call The 'Action' (May 14th/2011)

Well, we are arriving at a point where we can think much more deeply about the different components associated with Noether's Theorem and how they relate to the building of creation, spiritually speaking.

Richard Feynman, the great quantum physicist, believed very much in using the action principle to resolve the mysteries of a system even though he didn't understand why it worked--only that it did. So we'll go with good old Rich as we try to unravel a similar mystery concerning the unfolding of creation, even though I'm no physicist. Effectively, we're using Noether's Theorem with a little more detail to try and do the unthinkable.

The action is a sum of parts, an integration if you will, leading to a single value. From it, various symmetries can be deduced that lead to conservation laws. In my mind, this means the action is an encoding of sorts which lives in one dimension below those dimensions that contain these symmetries and these laws. Decoding the action leads to symmetries and laws and conversely, these symmetries and laws can be encoded rather compactly back into an action.

So if we now think of a unit disc centered at the origin it only makes sense to locate this action on the boundary of the disc and the corresponding symmetries within it. The active conservation laws can be thought of as living in the region outside the disc and all three should be seen as equivalent. In other words, if you lose one piece, you lose the whole thing.

We stated before it was most likely the collapse of the undifferentiated energy of God, everywhere, that brought into being the closed and open creations, and the boundary separating them. We now should feel reasonably certain the (latent) action is indeed that boundary and contains the encoded energy needed to derive both the differentiable symmetries within the Godhead and the (active) spiritual laws of creation outside it. The crucial thing to remember here is that none of this could have ever happened had there been no collapse to begin with--a collapse which changed both the energy and the geometry of the virtual cosmos. It is also crucial to understand that there is no material difference between these symmetries, the (latent) action itself and the (active) spiritual laws of creation. All three are equivalent and therefore, indistinguishable.

So if you are wondering who God really is, God is not one thing, even though He may present himself to you that way. God is simultaneously the differentiable symmetries resulting from the collapse; the union of the spiritual laws of creation resulting from these symmetries, and the encoded energy which glues these two pieces together. All three are equivalent and inseparable, and so quite naturally this leads us to the next question 'who is Christ' ?

Well, in all too familiar a way, we are used to seeing Christ as a person hanging on a Cross, dying for the sins of mankind. This is the typical message from Christian doctrine, for example, but it must be seen in a very different light.

In reality, Christ is probably one of those symmetries within the Godhead, and the same encoded energy which gave rise to this symmetry gave rise to all the differentiable symmetries within the Godhead. This encoded energy or latent action, if you will, may have been the same energy that made a descent into the body we refer to as Jesus Christ, who then went on to complete the law of harm and suffering, and by equivalency, all the spiritual laws of creation. Had this not happened, as we now know, there would have been no creation, essentially.

When Christ said 'If you have seen me you have seen the Father', he was basically saying what we are saying here. He may indeed be a symmetry (personality) within the Godhead but at the same time a full encoding of the energy which gives rise to both these symmetries and the resulting

spiritual laws of creation. Christ is probably all three, and you could, therefore, say that up to personality, anyway, Christ is God.

* * *

Think of the unit disc in the plane and divide up its interior into 12 equal sectors, say, like slices of a pie. Alternate the colors of these slices using black and white. Note that if you paint the white sectors black and the black sectors white you have the same geometric layout. In terms of (symmetry, action) you could think of the black sectors along with the boundary as being one component of the Godhead, and the white sectors along with the boundary as being the other. Black might represent one personality and white another.

Now reflect these white and black slices along the boundary and add two additional slices between neighboring reflections in order to begin the process of tiling the disc's exterior. Make sure to change color as you do the reflections or add sectors between them.

What you'll find is that the disc's exterior will be tiled with an infinite number of black and white slices in a beautifully symmetrical arrangement, and you could think of this arrangement as representing the spiritual laws of creation, which are not only conserved but equivalent by way of this symmetry.

You can even get more creative and add additional coloring to these exterior sectors all the while preserving a more complex symmetry. Each colored area would represent a law, say, and any two colored areas extended to infinity would be symmetric. Taken as a whole, the colored areas would tile the disc's exterior.

In this way, you have a geometric layout that portrays the (symmetry, action, conservation law) triplet associated with the Godhead in very simplistic terms. Adding more color to the interior sectors also produces more symmetries, but unfortunately there is no way to know just how many symmetries actually exist in the Godhead, nor can we be sure how many spiritual laws of creation there really are. I suspect there are many more laws than symmetries, but this is just a hunch on my part.

Interestingly, we've used coloring throughout this example to generate symmetries and spiritual laws, and loosely associated color with personality. In the same way that personality can be associated with symmetries in the Godhead, it can also be associated with the spiritual laws of creation. It seems as though these laws, while being equivalent to one another, are each endowed with their own unique identity.

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The Principle Of Least Action and The Uniqueness Of Creation

I've wondered for some time now just how unique this approach to creation by way of Noether really is. In the tiling example above, nothing stops us from dividing the interior and exterior arrangements with a boundary shaped in the form of an ellipse, for example. However, one should always remember that God is going to be perfectly efficient when it comes to encoding and expending the energy needed to build the symmetries and the laws and so, would choose a least

action leading to the most compact arrangement of symmetries and laws reflected in some corresponding geometry.

This is where the circle comes into play, because it is probably the most efficient corresponding geometry for least action if we arbitrarily tile the interior and exterior with sectors. By bending the circle into an ellipse, the interior and exterior tiling structure would become warped leading to asymmetries, and I highly doubt God would do such a thing.

You could also bend the circle into a rectangle, say, and replace sectors with squares when doing the tiling, but intuitively I would surmise that this would not lead to least action. Indeed, if you look up into the nighttime sky, or view an eclipse or examine soap bubbles, what do you see ? You see objects that are spherical in shape because they are built based on variational principles. God probably took a similar approach to building the virtual cosmos.

Thus, least action tells us there is only one unique action from which the symmetries and laws can be decoded, and conversely, the symmetries and laws, when encoded, lead back to this same unique action. Hence, there is only one unique physical expression of this (latent) action which completes the spiritual laws of creation and this is most certainly the Cross. Least action is telling is there is no other choice.

Similarly, least action is also telling us there is a unique arrangement of symmetries within the Godhead just as there is a unique (symmetrical) arrangement of the spiritual laws of creation. I suspect only the minimum number of symmetries and laws were built in order to make everything work.

* * *

In one shot, using least action with physical expression, the undifferentiated energy of God that existed everywhere collapsed into a remarkably beautiful but unique geometry, according to Noether, generating the symmetries and laws that make up the virtual cosmos. This, in a nutshell, is probably how it was all put together.

* * *

Are Souls A Necessary Part Of Creation ?

I was driving around town the evening of June 8th/11 and pondered the idea of souls, within the context of this essay. My question was pretty simple -- could you have a creation without souls using Noether's principle as your guidepost ?

Well, suppose no souls were ever made, leaving you with just the symmetries and the laws that make up the virtual cosmos by way of an action that itself is equivalent to the Cross. Since there are no souls there could be no Crucifixion, which means, in turn, the law of harm and suffering could not have been completed. Indeed, there would be no need for such a law and so, by equivalency, the spiritual laws of creation vanish along with the action and the symmetries. Once again we are back to the undifferentiated energy of God that existed everywhere before creation came into being.

This tells us the mosaic of creation necessarily implies the existence of souls and conversely, the existence of souls necessarily implies the mosaic. In other words, the mosaic of creation and the notion of souls are equivalent to one another -- if you lose one you lose the other.

We can, therefore, extend our understanding of things by saying that creation or God, if you will, is the union of four distinct, yet equivalent, pieces. These are souls, symmetries, least action and spiritual laws. Losing any one piece means losing all four, or put another way, everything everywhere is intricately connected.

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In God's 'now', souls came into being when everything else came into being -- symmetries, least action and laws. They most likely resided in the Godhead until it was time to leave and explore the virtual cosmos which we have before us today. Souls are as fundamental to creation as are any symmetries within the Godhead, the encoded energy from which these symmetries arise, and the corresponding spiritual laws. All four pieces are equivalent to one another and creation, quite simply, would not exist if even one of these pieces were to disappear.

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  |      fundamental components      |
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duality <~~> action <~~> laws <~~> souls
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Why Does Evil Exist and The Law Of Purification Of The Void (June 27th/2011)

The timeless question of why evil exists and why God does not destroy evil can best be answered now within the context of Noether.

We know there are at least four equivalent components to creation, these being souls, symmetries, least action and spiritual laws. Losing one component means losing all four, and thus a reversion to the undifferentiated energy of God that existed everywhere before creation came into being. A cosmic vacuum, if you will.

Suppose now that all souls ever created are malevolent, thus occupying the void. If God destroys these souls, thereby destroying the void, souls no longer exist anywhere. This implies, by equivalency, that creation ceases to exist and so leaves God in the position of not being able to eradicate this evil after all. Connectedness and equivalency prohibit such an action, unfortunately.

God has no choice but to purify the void, even if it is in small steps over eons of time, according to our notion of time. In other words, the existence of creation necessarily implies purification of the void, and conversely, purification of the void implies the existence of creation. You could, therefore, argue that purification of the void is a law, equivalent to all the other spiritual laws of creation and without it, creation ceases to exist, just like it would cease to exist if any of the spiritual laws were to vanish.

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God is bound by the (equivalent) spiritual laws of creation, just like any soul is. These laws were designed to allow us to self-correct as we move through a seemingly endless cosmic journey, even to the point where extricating ourselves from the void is possible.

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|    God ⊂ Spiritual Laws ⊂ God   |
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The law of free choice, for example, guarantees us complete freedom of thought and discovery, but is also met with the very powerful law of harm and suffering, designed to balance and cover our poor choices. The more we know about these laws, the better our chances of succeeding as souls no matter where we are within the virtual cosmos.

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It may be that souls are also connected to one another forming a larger whole. If indeed there is a dependency (equivalency) amongst souls God would be prevented from destroying even one of them, for to destroy one would mean destroying the whole, and this is not possible according to our findings above. In addition, any destruction of souls would contradict the Law of Unconditional Love and the Law of Principled Indifference, which we'll look at later.

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Comparison Of The Empty Fabric and The Differentiable Mosaic

Well, it's June 30th/2011, and I hope I'm getting to the end of this rather interesting essay. In this section we are going to talk about the empty fabric and compare it to the differentiable mosaic we now see before us.

It struck me in the last few days that the empty fabric -- also referred to as the undifferentiated energy of God -- is indeed a closed system that doesn't leak or suffer dissipation, and so, Noether's Theorem almost surely applies in terms of describing how it morphed into the remarkably beautiful mosaic we see today.

The empty fabric is the state of things prior to the existence of anything, where there is no differentiation and no discernable structure -- all things cease to be things, implying an

undifferentiated, unified 'whole' of all that is, whatever this finally means. Such a fabric still contains the information necessary to build the mosaic, and indeed, that information would be conserved in moving to the latter, albeit rearranged, just like rearranging a bitmap, for example.

At some point, a 'movement' or 'shift' in the empty fabric caused least action to come into being, giving rise to symmetries in the Godhead and hence the larger law of conservation of truth, as well as (perhaps through a quantum effect) souls and other fundamental components of creation. Almost surely this happened, because the empty fabric, seen as a closed system, would not have had an action of any kind without such a shift or perturbation, however subtle.

The empty fabric and differentiable mosaic could also be seen as equivalent to one another, although it is highly improbable we would ever see the mosaic dissolve into emptiness. Statistical considerations alone would prevent an infinitely long, irregular bitmap from ever becoming ordered, and this is roughly what would need to happen if we were to see the differentiable mosaic disappear into nothingness, so to speak.

The empty (or primordial) fabric is, however, probably the purest definition of God you are going to get, and from God's perspective, no doubt, this fabric and the differentiable mosaic are one and the same thing. As such, the notions of separateness and uniqueness in the differentiable mosaic are simply illusions created by our experiences in the latter, which perhaps, is a good thing, for they lend credence to the viability of such a mosaic, so long as you follow the laws. The price you pay for your sense of individuality, one might say, even though in reality all things are connected and unified everywhere.

* * *

Ask yourself again, according to Noether, what it took to bring into being the law of harm and suffering, and thus the remarkably beautiful creation we have before us today ? It would seem we carried little, if any, of the burden, and indeed, could not have if we, as souls, are not a symmetry in the Godhead. Within the differentiable mosaic, at least, this seems to be the case.

We can, however, appreciate what was given to us, and indeed, the only expectation God would have is that we follow the laws as best we can and hopefully, whilst here on the physical plane, put our spiritual interests ahead of our physical interests. For those who do, it will have made the trip worthwhile, to say the least.

* * *

Searching For The Universal Least Action (July 3rd/2011)

We noted in the last section the empty fabric is a closed system that suffers no leakage or dissipation. As such, it is my belief Noether's Theorem applies in characterizing this system as it morphs into the mosaic comprised of symmetries, least action, laws and other fundamental components of creation such as souls. In other words, we don't just have a universal action, but rather, a unique universal least action from which everything else flows.

At the apex of this hierarchy are the symmetries within the Godhead and the larger law of conservation of truth, where truth is defined as the union of the (equivalent) spiritual laws of

creation. Each contains the other, so to speak, and by way of some connection to this universal least action all other fundamental components of creation, such as souls, emerge.

All of these pieces -- symmetries, least action, laws and souls -- should be seen as equivalent to one another in so much as losing one piece means losing the whole. All things, at the apex of creation, are inextricably bound together and unified into a whole even though the illusion of separateness and individuality exists in the mosaic.

This should not come as a surprise, especially when you think about what the empty fabric really is and how the differentiable mosaic came into being via Noether. Quite frankly, separateness is not possible forcing us instead to accept the notions of equivalency, connectedness and unification.

We also know, from near-death research in particular, that all physical things flow from spiritual things, and thus we may conclude that the physical laws of creation are derivative forms of their spiritual counterparts. Since the spiritual laws are indeed connected to the whole, it stands to reason that the physical laws are also connected to the whole by way of equivalency and unification.

This is an important observation, for it means the unique universal least action, were it ever to be found, would be sufficient to describe the physical laws of nature and indeed, that description would contain the equivalency and unification we have talked about. The physical laws of creation, no matter how you analyze creation, really are connected, equivalent and unified into a whole even though here on earth we cannot see it.

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In physics today, there is a great deal of activity regarding grand unified theories and the like, but it is my view that no such theory will ever come about until (a) we become spiritual enough to recognize the oneness of all things and (b) develop the necessary mathematical tools to describe this oneness. Even then, Godel may prohibit us from obtaining a complete and consistent picture of things.

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duality <~~> action <~~> law of truth
   o            o            o
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   |            |            |
   |            o            o
   o <~~> least action <~~> physical laws
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An Interesting Corollary (July 14th/2011)

Because Noether can be used to characterize the 'morphing' of the empty fabric into the differentiable mosaic, we have to accept that some symmetry within the Godhead gave rise to all the (equivalent) spiritual laws of creation simultaneously, by way of an action, and not just some of them. Indeed, we have defined the union of these laws to be the law of conservation of truth, which

is essentially an all or nothing proposition. Either this overarching law exists in some active sense of the word, fully and completely, and by way of a (symmetry, action) pair, or it doesn't exist at all.

In particular, this tells us the law of harm and suffering would have been established in its entirety when the differentiable mosaic was established, meaning the Crucifixion -- itself an event equivalent to the unique universal least action -- took place as the empty fabric unfolded into the remarkably beautiful mosaic we have before us today.

Such a conclusion can only be reached if we remove the dimensions of space and time from our analysis and come to an understanding that from God's perspective, anyway, there is no time and space as we perceive these dimensions. The past, present, future and space itself are all compressed into what we would call 'God's now', and it is from this perspective that we must examine both the empty fabric and the virtual cosmos we live in today. Any other approach will not work, in my view.

* * *

Noether not only forces us into a position where all things in creation are unified, connected and equivalent, but also, tells us that dimensions such as time and space don't really exist after all.

We have learned, through Noether, that a symmetric duality in the Godhead gave rise to the larger law of conservation of truth by way of a unique action equivalent to the Cross, and that truth is indeed the union of the (equivalent) spiritual laws of creation. These laws, while equivalent, may have stronger or weaker personalities, but it is both fair and correct to say that personalities within the Godhead are equally balanced because of symmetry.

Finally, Noether also tells us there are at least four equivalent fundamental components of creation, these being symmetries, least action, laws and souls. If we were to lose even one component creation would cease to exist, according to Noether. All things, it seems, are unified into a connected whole through equivalency.

* * *

Some 2000 years ago a piece of God's now 'tore open' the space-time continuum we seem to have 'fallen asleep in'. For those who have ears to hear, and eyes to see, wherever they are in the virtual cosmos, let them hear and see ...

* * *

A Note On Einstein, Godel and Noether

Throughout this essay I have used principles of Einstein Relativity Theory, Godel Incompleteness and Noether Invariance to justify my understanding of things. These three people were probably some of the brightest minds of the last century and it is doubtful we will ever see an intellectual confluence of this kind again for a long, long time.

Einstein lived between 1879 and 1955, Godel between 1906 and 1978, and Noether between 1882 and 1935. So there was a period of time when all three were alive and, no doubt, knew about each

other. To me, at least, this is not a coincidence, nor is it a coincidence that I have taken advantage of what they said within the context of this paper.

If we believe in Noether's Theorem extended to spirituality, and Einstein's notion of Relativity, which in essence allows for the concept of God's 'now', and Godel's Incompleteness which implies consistency and completeness only within the Godhead, then the conclusion reached above in 'The Crucifixion Revisited' is reasonable.

I'll be pondering this for some time to come, to say the least ...

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|            Three Pillars           |
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|  Einstein <~~> Noether <~~> Godel  |
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Where Are We With Those Spiritual Laws ???

It's time for a recap on the spiritual laws of creation. We've learned a few more, so here's a quick summary:

a) the law of free choice
b) the law of forgiveness
c) the law of harm and suffering
d) the law of earning what you receive
e) the law of conservation of information
f) the law of conservation of truth
g) the law of unconditional love
h) the law of purification of the void
i) the law of leaving the Godhead
j) the law of principled indifference

It is my view that all these laws are equivalent, not just some of them, and that Truth is indeed the union of the spiritual laws of creation. Indeed, Truth very likely is creation.

A Note On The Law Of Earning What You Receive

Well, it's July 11th/2011 and I keep saying I'm done writing, but I thought it important for those who read this essay to talk a bit about this law. We know from Fenimore's NDE account that such a law exists and that it is definitely tied to our spiritual progress. In fact, one could argue that our spiritual progress ought to precede any gain or acquisition we might aspire to in the physical plane, including, but not limited to, bodily healing.

For whatever reason, I went back and read the account of Samson (man of the sun) on Wiki, probably because I had been thinking about his story in recent times. It struck me that his great strength was lost when his hair was cut, which led to a breaking of the Nazirite bond he had formed

with God. As the hair regrew, Samson's strength returned, most likely because he became reacquainted with his own sense of spirituality that was lost when he flirted with forces or powers that were only too willing to take it from him.

The lesson here is quite simple. Your spirituality can be developed or destroyed by your own thoughts and actions, and thus it is up to each one of us to guard our progress by living in harmony and balance with the spiritual laws of creation. The more we know about these laws the better our chances.

If we do this other things such as physical healing will flow more freely and naturally from God. In Samson's case, he was healed physically by God once he first healed himself spiritually. Unfortunately, Samson's spiritual healing was a long and drawn out process that was both painful and embarrassing for him, but eventually he got it right and the rest is history, as they say.

The story of Samson may be a good example of the law of earning what you receive, for it details the account of a man endowed with great supernatural power and strength initially, only to lose and then regain it as he recovered his own sense of spirituality. It was never about Samson's hair per se, or any Nazirite vow, but rather, where Samson was with respect to his spiritual self. A poignant lesson for the rest of us who, in particular, aspire to be healed physically -- heal thyself spiritually first.

A Note On Spiritual Ratios (July 11th/2011)

For the last few days I've been thinking about reincarnation and how our reincarnations are tied to spiritual progress. It struck me that we must actually look at two things -- the number of reincarnations we've had as a soul and the level of progress achieved to date.

I'll call this quotient of 'reincarnations to level of progress' the spiritual ratio, and make the obvious remark that the lower this number, the more advanced a soul really is, or has become. It does us virtually no good to talk about the number of reincarnations solely by itself, especially if there is no progress with each descent. Indeed, in some cases the experience of descending into the physical plane can actually be regressive, undoing any progress made in previous incarnations -- something to keep in mind for the overly ambitious in the higher realms may I say.

In the world of Cayce's numbers there appear to be twelve levels of understanding, with the last (or twelfth) being perfection. I doubt any soul will ever reach the final level, irrespective of the number of incarnations, but who knows.

In my case there is sufficient evidence to support the conclusion that I've reached the eleventh level in 23 descents, yielding a spiritual ratio of about 2 using the Cayce numbers. I recall as well in my dream (vision) the other two beings I saw had 36 and 45 emblazoned on their solid-core orbs, suggesting that (a) these were the number of incarnations each has had and (b) their spiritual ratios are probably no lower than 3 and 4, respectively, if we assume neither has ever reached the twelfth level.

As perfect and beautiful as these beings were, it seems to me, ironically, that their spiritual ratios are actually larger than mine. Since they were identical twins and since we are tied together by the

orbs and their corresponding numbers (23, 36, 45), it is my belief that (a) we are identical triplets and (b) I may actually be more spiritually advanced than both of them.

This comes as a surprise to me, for I have thought for a long time now that I was the 'youngest' of the three in terms of spiritual progress. That may, in fact, not be the case after all.

A Note On Spiritualizing The Virtual Cosmos (July 12th/2011)

As I was sitting in the park today, it finally dawned on me that it was time to summarize all that I've learned through this essay and ask the fundamental question 'how does God go about transforming the virtual cosmos ?'

We know Noether's principle can be used to explain how creation likely came into being, but what indeed is the point of this knowledge if it can't be used to enlighten others ? Knowledge, on its own, isn't worth much unless it leads to understanding, wisdom, and finally enlightenment -- the circle of discernment, if you will.

Noether tells us the Father-Son duality gives rise to the larger law of conservation of truth by way of a unique action, itself equivalent to the Cross. Our poor choices, in essence, are covered by the law of harm and suffering which was completed via the Crucifixion some 2000 years ago, and indeed, the differentiable mosaic exists because of this remarkable event, though few, if any, understand the import of what is being said here.

For whatever reason, our planet was chosen to stage the Crucifixion according to the system and timing of things, meaning no other planet anywhere, at any time, can or ever will bear witness to what we have seen. Earth is indeed a highly special place to God, for souls who come here to learn have a golden opportunity to discover these mysteries if they really want to. Sadly, most never do no matter how many times they make the descent.

But for those who do gain the enlightenment, they can, upon returning to the higher realms, pass on their knowledge to others and too, sow the seeds of understanding on other planets in other galaxies in other universes through incarnation.

In M's NDE, for example, we learn of a Creator-level mandate recently issued to bring an end to the 'Luciferian Rebellion'. To me, this simply means a mandate has been issued to spiritualize the virtual cosmos in the most efficient way possible. A needed thing, in my view, since by my reckoning about one-third of creation has been lost to darkness already and that percentage isn't likely to fall unless something of a drastic nature reverses the decline.

M's NDE speaks of Leaders of Light from everywhere throughout creation convening in the Library of Light to discuss the matter and find a resolution. These beings, from what I gather, are immensely powerful, capable entities -- almost on the order of God himself -- and so, it appears we are in good hands after all.

Perhaps the best approach involves sharing the secrets Noether has given us -- secrets that can help unlock and decode the mysteries of the Godhead, thereby leading to a fuller understanding of the

spiritual laws of creation, and finally creation itself. In the end, isn't this what it's all about -- living in harmony and balance with these laws and knowing how and why they came into being ?

Einstein, Noether, and God's Now (The Godhead) -- Digging A Little Deeper (July 15th, 2011)

It seems every time I want to put the pen down another thought pops into my head, and sure enough, that happened today as I was sitting in the park and enjoying my tea.

It wasn't long ago I found a paper on the Internet written by Nina Byers from UCLA in 1998. It was a paper devoted to the celebration of Noether and a discussion of the deep connection between symmetries and conservation laws in particular. What I learned from this paper is that Noether's Theorem actually led to the discovery of the laws of General Relativity Theory, thereby strongly supporting the idea, in my view, that we can indeed use Noether to explain how creation actually came into being. In other words, if Noether was good enough for Einstein, perhaps she is good enough for creation at large.

Adopting whatever tools the First Lady of Mathematics gave us, we have already come to the conclusion that from the perspective of the Godhead, anyway, there really is no time and space. The Godhead, you may recall, is the closed creation that contains the symmetries responsible for the law of conservation of truth via some action.

Let us now turn to Einstein, who along with others, came to the realization long ago that in our physical plane, time and space are woven together -- there is, in fact, no way to separate these dimensions. We also know that light has no clock, and so, time seems to stand still for observers travelling at this speed. In turn, this would seem to indicate space vanishes as well for these observers because it is interwoven with time.

Since Noether implies the existence of God's now [ie the Godhead], neither space nor time can occupy the β-world, anywhere. On the other hand, Einstein tell us this can only happen if the Godhead is filled with light everywhere, extending out to a perceived infinity which omits the familiar dimensions we live in.

In other words, were you to cross over into the β-world by passing through the membrane (action) that divides the closed and open creations, any rod or clock used to measure distance or time would not work. Indeed, you might find your rod or clock dissolving into the very fabric of the Godhead, and thus becoming light itself. Taking it one step further, you might find yourself completely melding with the Godhead for the same reason.

The Godhead, were you able to see it by being in it, would probably look like a colorful array of symmetries generated solely by the purest forms of light one can imagine. No matter where you were within the Godhead, these symmetries would persist, seemingly out to an infinity not measurable through the familiar dimensions that exist here in the physical plane.

And because the notion of dimensionality would cease to exist, you would most likely become one with the Godhead, knowing all that ever was, is and will be. Some NDErs come back saying this has in fact happened to them. Perhaps we now know why ...

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|     The Godhead     |
|                     |
| symmetries of light |
|                     |
|    no dimensions    |
|                     |
|   all things known  |
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A Trip Through The Realms Outside The Godhead (July 16th/2011)

I'm going to try and characterize the continuum of realms outside the Godhead, taking account of rods, clocks and light, and see if we can learn anything. At the outset I should like to say emphatically that Godel applies here, meaning my view of things is most likely incomplete and inconsistent, but nonetheless, interesting.

We now know from Noether and Einstein that within the Godhead no time or space exists, and so any observer will find himself blending completely with other observers here and with the purest forms of light imaginable -- an experience which would seem eternal and infinite but not measurable using familiar norms. In the Godhead, there is no sense of individuality at all -- everything is meshed together into an infinite whole you might say and the concept of rods and clocks vanishes altogether.

Let us now move out and beyond the membrane (action) dividing the closed and open creations, and into a realm 'relatively close' to the Godhead. In such a realm, beings would nearly have the same awareness they had in the Godhead, but at the same time retain a very weak impression of 'clock ticks' and 'spatial events'.

They would, for the most part, feel completely blended together with each other and with the light, in what would seem like an infinite expanse that could not be measured with rods or clocks. Indeed, rods and clocks would seem strange to them, but not altogether, meaning these beings would feel complete at-one-ment with each another on the one hand, and a limited sense of individuality on the other. The light in such a realm, while very pure, would not be as pure as the light in the Godhead because to a very small degree it would be moving through trace elements of time and space. In that sense, time and space, however you wish to define them, diminish the purity of light and thus the notion of at-one-ment.

Continuing our journey, we arrive at the physical plane some distance from the Godhead where we find the notions of time, space and light to be roughly the same in terms of sensation or awareness. Clocks tick, rulers measure and light beams down from the sun. It all seems rather natural and well-ordered from the perspective of a being who lives in this plane, however there is quite a bit of drag here because light must move through or 'compete with' the density of time and space, thereby diminishing its purity even more.

Our sense of individuality is heightened and the notion of at-one-ment we might have felt in one of the higher realms, or even the Godhead, is lost. We no longer blend with the light and each other the way we did before because time and space have both distorted and impeded the sensation of this phenomenon. Indeed, time and space have virtually removed this sensation altogether.

Moving down further into one of the lower realms, typically a void-like region where anguish reigns, we find that time and space begin to overshadow light uniformly. The presence of light is severely diminished or vanishes completely depending on how low the realm actually is, leaving an observer with the sensation of extreme individuality -- that is to say, complete arrogance and ignorance.

Because time and space begin to overshadow the light, an observer perceives continuous misery in a more 'spatially aware' sense of the word. It seems to be everywhere around him with no letup. It is in this sense that time and space finally get the better of this observer, and so they should, because our thoughts and actions ultimately form the basis for our spiritual destiny.

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Even in the physical plane it is quite possible to put yourself in a void-like region by way of your thoughts and actions. If you do, it is more than likely you will pass over into such a realm upon death, but now your perception of that misery will be heightened by orders of magnitude. Conversely, thoughts and actions that are light-filled will do just the opposite -- they will be magnified by orders of magnitude upon leaving the earth plane.

The more we know about the spiritual laws of creation, and creation itself, the better our chances of succeeding as souls no matter where we are within the virtual cosmos. For those that have ears to hear and eyes to see ...

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 ------------------>---  time & space intensify  --->--------------------
|            |                                                            |
|  Godhead   | light-filled realms ~~> physical plane ~~> darker realms   |
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Is The Physical Plane A Fundamental Component Of Creation ? (July 20th/2011)

Well, we've done our trip through the realms and come to the realization that time, space and light, in varying degrees of 'awareness', determine the 'quality and purity' of any particular realm. But the question could be asked 'how fundamental are realms to creation and what would happen if they didn't exist ?'

Well, if no realms existed anywhere souls would be confined to the Godhead indefinitely, resting in a state of perfect equilibrium with the symmetries of light that dwell there. The active spiritual laws of creation, such as the law of harm and suffering, do not apply in this domain and so Noether tells us the differentiable mosaic vanishes -- we are back to the empty fabric that existed everywhere before this mosaic was brought into being. Conversely, if there is no differentiable

mosaic, then clearly there are no realms, and this in turn means realms beyond the Godhead are indeed a fundamental component of creation.

In the case of the physical plane it should be clear that if it didn't exist, nor would the Cross. Conversely, if there were no Cross, you would have nothing, since the Cross is an event equivalent to the action that binds together the open and closed creations. Thus, one could argue the physical plane in particular is a fundamental component of creation -- if you lose it, by implication you lose everything.

God, then, has a choice. Either he elects to have a physical plane and induce suffering through it in order to bring about the law of harm and suffering, or he finds another way to complete the law which avoids this plane altogether.

If God chooses the latter, he must find a realm above or below what would have been the physical plane and stage an event accordingly. Choosing a higher realm would make no sense, and so he is left with considering a lower realm made up of discarnate souls who largely suffer ongoing mental anguish, amplified by an awareness of their selves. Physical pain, in these lower realms, really doesn't exist.

God opted for a combination of physical and mental anguish in the extreme, which was probably the basis for bringing into being the physical plane originally. Without maximum suffering God determined that the law of harm and suffering, which forms the underpinning to creation, could not be completed. 'Extreme suffering in exchange for extreme harm', you might say.

* * *

In a sense, it appears as though the entire physical plane - universes, galaxies, stars, planets and the like - was built by God to pay homage to the Crucifixion. A grand, cosmic tribute to the one who completed the law of harm and suffering and thus sustained (became the raison d'etre for) the very creation we see before us today.

And Jesus said “I tell you,” he replied, “if they keep quiet, the stones will cry out.”

* * *

It is no coincidence the physical plane exists. While it provides souls the best opportunity to grow spiritually at rapid rates, it is also the plane that hosted the Crucifixion and bore witness to maximal pain and suffering, lest we forget. The physical plane is a fundamental component of creation.

Realms are also a fundamental component of creation. Necessarily they exist outside the Godhead and most likely form a continuum. The existence of realms also lends credence to the idea that souls were never meant to remain in the Godhead; rather, they were meant to discover and develop the virtual cosmos in accordance with the spiritual laws of creation. Indeed, if you think about it more deeply, souls in all likelihood can't stay in the Godhead, which could be seen as a law.

* * *

To wit, if all souls in the differentiable mosaic remained in the Godhead indefinitely, the active spiritual laws of creation cease to exist, which means you are back to the empty fabric containing no discernable structure. On the other hand, if souls leave the Godhead then necessarily the differentiable mosaic exists and so the notion of souls leaving the Godhead becomes a spiritual law of creation equivalent to all the other laws -- there never was a 'fall from grace'.

We are all very likely subject to the laws according to the notion of 'principled indifference', discussed below. In particular, the law of leaving the Godhead would work this way, and so it is doubtful there would be any exceptions -- we would all need to go at some point.

***

We have now extended our knowledge of creation to six fundamental components which are dependent on one another. These are symmetries, least action, laws, souls, the physical plane, and realms. All six components form a unified whole, connected together through equivalency. Losing any one piece means losing the whole.

```
    --------------------------- creation ---------------------------
   |                                                                |
   |                                                                |
   |                     fundamental components                     |
   |                                                                |
   |                                                                |
 duality <~~> action <~~> laws <~~> souls <~~> realms <~~> physical plane
   |                                                                |
   |                                                                |
   |                                                                |
    ----------------------------------------------------------------
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The Law Of Principled Indifference -- A Few Notes (July 29th/2011)

In order to better understand laws such as 'purification of the void' or 'leaving the Godhead', we need to know how all the laws work at a deeper level. In other words, is there a 'traffic cop' law of sorts that determines the ethical behavior of each and every one of them ?

So let's start with Godel where we learn that in any region of the open creation incompleteness and inconsistency will exist, thereby making it impossible to 'grasp the big picture' in its entirety. But Godel's theory itself would also be inconsistent and incomplete, and so there is no reason not to believe a region of the virtual cosmos exists where Godel's principles do indeed break down. This region or 'singularity' we have termed the Godhead or God's now.

Turning to Einstein and Noether, we have learned the Godhead is a region of infinite expanse filled with symmetries of light absent the dimensions of time and space. All that ever was, is and will be is known here and you could, therefore, say God's now is indeed both wholly consistent and complete.

In a peculiar kind of way Einstein, Godel and Noether reinforce one other when seen in a more spiritual light -- a perspective we must always adopt if we are to attain a deeper understanding of things.

Noether, in particular, tells us there is an intimate relationship between these symmetries of light in the Godhead and the law of conservation of truth which exists beyond the membrane (action) that divides the two. However, both are equivalent to one another and in fact, you could say each is a reflection of the other via the membrane.

If this is so, then any characteristics or qualities associated with these symmetries would quite naturally be inherited by the spiritual laws of creation. Noether, taken one step further, tells us that not only do these laws exist by way of the symmetries, but also, inheritance factors are very likely preserved because of the reflection.

The spiritual laws of creation, then, not only exist but form as well a complete and consistent unified whole. In short, they must operate in a perfectly ethical manner showing no bias or prejudice when carrying out their tasks. We call this the law of 'principled indifference' and use it throughout this essay to discover other laws and various practical applications.

```
Law A  |  ~~~~>
Law B  |  ~~~~> show no bias ...
Law C  |  ~~~~>
       .
       .
       .
```

Does The Differentiable Mosaic Rotate About An Axis ? (July 30th/2011)

Lately, despite my own personal woes, I've been thinking more about this idea, especially when considering how realms (rings) might be laid out in concentric fashion about some central axis.

We have a lot of evidence to support the idea of a Godhead, but if the entire virtual cosmos 'rotated' it is almost a certainty that at the center of it all you would find a singularity of some kind where things break down. Time and space cease to exist here and light, because it is no longer impeded or distorted by these dimensions, is of an unimaginable purity. In fact, there would be no fair way to characterize this kind of light based on our experiences in the physical plane and indeed, it may be incorrect to simplify things in this manner.

Rotation, too, would lend credence to the idea of rings (realms) at ever increasing 'distances' from this center, with time and space 'intensifying' the 'further away' you were. Indeed, at some point these dimensions, however you wish to define them, would overshadow light to the extent that a realm (ring) would seem dark and a soul would become far more aware of itself at the expense of others -- the arrogance and ignorance syndrome, if you will. Such realms have been referred to as 'the void'.

In the physical ring we are aware of the law of conservation of angular momentum and know too that all physical laws are derived from higher spiritual counterparts. Thus it is not unreasonable to suppose a similar law concerning momentum exists in the differentiable mosaic, and so, can only bolster our view that the virtual cosmos does indeed 'rotate', however you wish to define this phenomenon.

Rotation, then, could play a fundamental role in (a) how realms form and (b) how time, space and light interact with one another in each of these individual rings that make up some kind of continuum. For example, the closer we get to the Godhead (singularity), the 'faster' a ring might spin, thereby leading to purer forms of light at the expense of time and space. A soul in such a realm would be more aware of blending with the light and less aware of its own individuality -- just the opposite of what you'd expect to see in a distant ring rotating more 'slowly'.

Noether also validates our ideas here, for her theory tells us angular momentum is conserved under rotation by way of an action. Within the virtual cosmos that action can be seen as a membrane of sorts, dividing the Godhead from the open creation in which we live and operate, and containing all the encoded information and energy needed to build and sustain the differentiable mosaic.

Because angular momentum is conserved, rings closer to the Godhead would carry less 'mass' but spin more quickly, and conversely for the more distant ones. We can't know within this context what spin and mass really mean, but no doubt a soul in a less 'massive' ring feels a 'lightness' of sorts, free of the usual things that seem to burden us down in the physical plane, for example. In a darker, more massive ring it would be the opposite -- not only does misery seem to be everywhere with no letup, but a soul now feels pulled in and heavily bound to such a realm from which there appears to be no escape.

* * *

Rotation should be seen as that effect which endows each realm (ring) with something we'll call resonance. The closer a ring is to the Godhead the purer the light because it is less impeded or distorted by dimensions such as time and space. Souls here easily blend with all that is, and are less aware of their own individuality.

A soul finds its way into such a realm because the harmonics of both soul and realm are compatible.

* * *

The Membrane and The Cross - A Comparison (July 20th/2011)

I thought it would be interesting to say a few things about these two notions, in light of all we've learned. We know the membrane is a spiritual structure of sorts that divides the closed and open creations and contains, in some encoded format, all that is needed (in terms of information and energy) to bring everything into being. We often refer to this membrane as least action and have determined it is both unique and equivalent to the event we call the Crucifixion.

Indeed, if the spiritual laws of creation omitted the law of harm and suffering because it wasn't needed, the event we call the Crucifixion would have never been staged and all we'd be left with is the membrane. However, there would be no way to characterize this membrane beyond saying that it exists, and so, the Cross should be an indication to us that it (the Cross) is special and warrants further investigation.

In this sense, then, the Cross should be seen as an event equivalent to (or emitted by) the action, but not necessarily the action itself. The two, however, are inseparable and wholly dependent on one another -- losing one means losing the other, most assuredly.

Passing through the membrane from the outside means entering the Godhead. There a soul will find it has reached a state of perfect at-one-ment with the symmetries of light that make up this region of infinite expanse, absent the dimensions of time and space. A complete loss of one's sense of individuality, you might say, in exchange for total immersion with all that exists here.

Since the membrane is equivalent to the Crucifixion, one could argue that 'bonding with the Cross' is akin to penetrating this structure. We can never know as souls experiencing in the physical plane what it must be like to pass through the membrane directly, but we might, in some religio-philosophical sense, recognize the Cross as a gateway of sorts leading to the Godhead.

Indeed, it may be all that God is asking of us as we strive to attain at-one-ment no matter where we are in the virtual cosmos. For those who do, one could argue they are on a path to both a higher and more complete understanding of things.

* * *

Jesus said "No one comes to the Father if not through me." Perhaps we are saying the same thing ...

* * *

```
Godhead <~~> membrane <~~> cosmos
   o            o            o
   |            |            |
   |            |            |
   o            o            o
duality <~~> action <~~> law of truth <~~> union of laws
   o            o            o
   |            |            |
   |            |            |
   o            o            o
               Cross
```

The Geometric Distribution Of Realms and Sectors (July 23rd/2011)

At the center of everything is the Godhead or God's now, followed by the membrane (action) that divides the β-world from the open creation containing all realms and the physical plane. For lack of a better term we could think of these realms as annuli in two dimensions or more, all surrounding the Godhead at ever-increasing 'distances' and forming a continuum. The further out the ring (annulus) the darker the realm.

In M's NDE there is a reference to both 'Leaders of Light' and 'sectors' throughout the virtual cosmos that they manage. It strikes me now that these rings are probably chopped up into pieces determined by radial boundaries emanating from the center of the Godhead, not unlike the spokes of a wheel. Such a 'carving up' of things would lead to sectors within any annulus, including the physical ring (plane) that contains universes, galaxies, planets, and so on.

No doubt there are local realms as well, for we read in near-death research about the 'earth-bound' plane -- a void of sorts for those who never could detach from their physical addiction to this world while invested with a body. However, in this section we are primarily concerned with the bigger picture.

Near-death research also speaks of 'tunnels' transporting a soul to a realm, and it is my view that these tunnels probably move a soul from some point in the physical ring to another point in another realm (ring) and sector commensurate with that particular soul's resonance. Some mechanism has to exist and the tunnel seems like a reasonable choice.

In the same way that you have 'Leaders of Light' and 'Libraries of Light' mentioned in M's NDE, it wouldn't surprise me if the opposite were true in the outermost rings where darkness reigns supreme. Sectors here would probably be managed by recalcitrant souls and may, in fact, be highly organized. There is some evidence for this kind of thing in Pittman's NDE as a matter of fact, though hard-core evidence is lacking in my view.

The physical ring is perhaps the strangest of all annuli in so much as it divides the higher and lower realms, and serves as the final destination for any incarnation where, it is hoped, lessons learned will return the soul to a more enlightened plane of understanding upon death. This ring is peppered with universes, galaxies, planetary systems and the like, giving any being in this plane the overwhelming impression of an endless, eternal physical expanse as measured by the familiar dimensions of time, space and light. Truly a remarkable oddity, and yet a necessary one in so much

as it was designed to host the Crucifixion -- an event upon which everything, everywhere hangs, as we now know.

Most souls are pretty well-behaved, and so it is my guess that a 'median realm' might exist somewhere between the Godhead and the physical ring. Here souls would experience life where the feeling of at-one-ment with others would persist in a light-filled region of infinite expanse, but at the same time, each would be well aware of their own individuality, creative talents and uniqueness. Even so, this awareness would always be overshadowed by the blending together of everything.

Why Did The Crucifixion Happen 2000 Years Ago ? (July 27th/2011)

We probably have enough information now to answer this question, which is a rather deep mystery in itself. Noether tells us symmetries of light within the Godhead give rise to the spiritual laws of creation by way of an action equivalent to the Cross. Furthermore, we've learned there are at least six fundamental components of creation, among them the physical ring which hosted this remarkable event we call the Crucifixion about 2000 years ago according to our notion of clocks, rods and light.

So imagine, if you will, a physical ring which has persisted throughout the eons of time, dotted with universes that come and go in a seemingly infinite expanse that is literally beyond human comprehension. In this ring there is no absolute way to measure time meaningfully except in your own local frame of reference, and so, when speaking of the Crucifixion, we must realize the only valid perspective is that of the Godhead. Here, as we know, the familiar concepts of time and space vanish completely.

The point at which the Crucifixion took place in the physical ring, from the perspective of the Godhead, could be seen as a 'phase transition' of some kind -- a critical point in creation at which it was determined that 'imparting the Cross' to a particular planet in a particular universe was deemed appropriate. Whatever the balances and imbalances are throughout the various realms, including the physical plane, they have now reached a point where physical expression of the universal least action is necessary, along with the knowledge it imparts.

To be sure, the law of harm and suffering had to be completed at some point, and by doing so, we as souls on the planet that hosted the Crucifixion have been given unique knowledge of what is probably the only event in the physical ring that offers us any insight into the inner workings of the Godhead. We have, you might say, been allowed to pass through the membrane that divides the Godhead from the open creation in which we live and operate, by way of the Cross.

This knowledge and the power behind it could ultimately be used to rebalance the virtual cosmos, using our planet as a base or springboard, if you will, from which to carry out an almost impossible task. That is to say, sowing the seeds of understanding throughout the virtual cosmos, according to Noether, thereby leading to a deeper impression of things.

Thus, decoding the mysteries of the Cross is akin to decoding the mysteries of creation. Perhaps this is what God is trying to tell us after all …

So Who Really Is The One We Call The Christ ? (July 28th/2011)

The ageless question of who Christ really is can now be answered affirmatively. We recently learned of the Law of Leaving the Godhead and know through Noether that symmetries of light within the Godhead uphold the law of conservation of truth by way of an action equivalent to the Cross. This much is irrefutable, based on our knowledge of things.

If Christ is a soul like the rest of us, he cannot stay in the β-world; indeed, he must vacate at some point otherwise the law of leaving the Godhead would be violated. But if he leaves, he is not a symmetry within the Godhead, and thus has no part in upholding the law of conservation of truth. Christ, like us, becomes an observer in creation but has no special membership in God's now, which is to say, β-space.

But by definition, Christ was the one who was crucified, thereby completing the law of harm and suffering and hence the larger law of conservation of truth by way of equivalency. Thus, we are forced into accepting the fact that he really is a symmetry within the Godhead after all, and not a soul like the rest of us. Refusing to accept this conclusion only leads to a contradiction.

As a corollary, it follows that no soul can atone for its acts of harm. That, indeed, has already been done, and you could, therefore, say that any effort to atone before God would seem foolish. Our job is to recognize how, why and by whom these spiritual laws of creation were brought into being and then, of course, to live in harmony and balance with each and every one of them. To be sure, if there ever was a 'sacrificial lamb' in all of this, it is Christ.

* * *

And Jesus said, "But that you may know that the son of man has power on earth to forgive sins ..."

* * *

Christ is indeed a symmetry (personality) within the Godhead and you could say that up to personality, anyway, Christ is God. There really is no other viable option.

If Christ does have 'soul-like' qualities, they are unlike anything we would be familiar with -- they would be an extension of the characteristics and qualities associated with his essence as a symmetry in the Godhead and therefore, not something we could easily comprehend no matter where we were in creation. There is evidence for this idea in David Oakford's NDE, for example, and other NDEs as well, that speak of the purity of Christ's light.

* * *

I met a man -- very meek and mild -- almost 40 years ago, in a park who asked me where I was going that day. I didn't answer him with much clarity but I remember him telling me to go to church. I disagreed and he never spoke to me or looked at me again, that day.

Later on in life I realized this man was an exact copy of the image of the person you see in the Shroud of Turin -- the burial cloth of Christ. Maybe I should have done it differently, looking back, especially in light of what I now know.

What Did It Take To Get To This Point ? (August 7th/2011)

In some ways, the major theme of this essay was to find out, if we could, who Christ was. To me, at least, it has been a lifelong passion and so I'm glad I've had a chance to learn of things from a new perspective. Indeed, it is the only perspective that makes any sense from my standpoint.

Initially, through Noether, we felt there was a duality of sorts within the Godhead that gave rise to the law of conservation of truth by way of an unseen action. Nothing more was known at the time and even these ideas weren't defined very well.

```
duality <--> action <--> law of truth
```

We then learned, quite concretely, of the law of harm and suffering through Angie Fenimore's NDE, and realized the Cross might be seen as an event that was somehow connected to the action. Indeed, this law is one of many contained by truth, so at this point there seems to be a link of sorts between symmetries in the Godhead, the action and the Cross, and various laws.

```
duality <--> action <--> law of h&s <--> truth
                |              |
                |              |
             Cross ----------o
```

Things are now beginning to heat up a bit as we discover all laws are equivalent to one another, thus making it simpler to recast the bigger picture by saying symmetries in the Godhead lead to the larger law of conservation of truth by way of an action that is somehow connected to the Cross. It's looking more and more like Christ is involved in all of this, but we really can't be sure just yet.

```
duality <~~> action <~~> law of truth
                |
                |
             Cross
```

At about the same time we learn Noether can indeed be used to characterize the unfolding of the primordial fabric into the differentiable mosaic we have before us today. Both the fabric and mosaic are important for our understanding of things to come.

```
primordial fabric <--> Noether <--> differentiable mosaic
```

Six fundamental components of creation are discovered through Noether, these being symmetries, action, laws, souls, realms (rings) and the physical plane. Remarkably, all are proven to be equivalent to one another and it is also shown that the Crucifixion is an event equivalent to the action. A minor breakthrough, you might say.

```
duality <~~> action <~~> law of truth
                o
                |
                |
                o
              Cross
```

Two new spiritual laws of creation are also discovered, these being the law of leaving the Godhead and the law of principled indifference. Without them, we really aren't any closer to understanding who Christ is.

Either Christ is an ordinary soul or he isn't. If so, he must leave the Godhead at some point and thus has nothing to do with upholding the law of conservation of truth, thereby contradicting his involvement with the law of harm and suffering. Christ, it turns out, is a symmetry within the Godhead after all, which we refer to as the Father-Son duality.

```
          duality <~~> action <~~> law of truth
             o            o
             |            |
             |            |
             o            o
     Father & Son <~~> Cross  <~~> law of truth
```

In summary, though it took a lot of work to get to this point, one can say the Father-Son duality within the Godhead upholds the law of conservation of truth by way of an action equivalent to the Cross. Christ is that unique Son.

Moses, Joshua, John The Baptist and Jesus

On or around Sep 12th/10 I had a thought on these remarkable figures in history. It's not much, but it may serve to justify the Noether connection above.

Joshua superseded Moses in so much as he led the Israelites into Canaan. Jesus is the English translation of the Greek Yehoshua and ditto for Joshua and the Hebrew Yehoshua--the name ultimately given to Joshua by good ol' Moses. So we can probably conclude that the Joshua soul superseded Moses in some sense, using both the leadership and naming factors as evidence.

The transfiguration of Jesus also demonstrates 'soul' supersedence' relative to Moses and Elijah, the latter being equivalent to John the Baptist, who saw Jesus as a superior, perhaps even divine, incarnation.

One possible conclusion then is that the Joshua and Jesus souls are equivalent, and that this soul is somehow 'above' the souls of Moses and John the Baptist. Certainly Cayce would agree with such a conclusion and here's a little graph:

```
  Moses       John the Baptist  (precursors)
    |                |
    |                |
 Joshua            Jesus        (equals)
```

Incompleteness, Godel and At-one-ment

I wasn't going to write on this subject, but think it may be appropriate to say a few things after all. As I understand it, according to Godel, there is no way to construct a mathematical system which is complete and consistent. To me this means that any mathematical system you build will never be able to discover, much less prove, all truths it was intended to capture, and within that system, many inconsistencies will arise.

This notion can be extended to the construction of almost any other system outside the bounds of at-one-ment. Quite simply, no matter how you build a foundation for that system, it will eventually run aground if built outside this boundary.

By outside the bounds of at-one-ment, I simply mean a state in which the soul-God duality has been broken. Such a state exists here on earth and most likely in all other realms as well. This means, among other things, that our thought patterns cannot be shaped to fully capture the notion of a complete and consistent set of truths which I will simply refer to as 'truth in God'. Perhaps this is a blessing in disguise, especially when you think more deeply about the law of harm and suffering as it pertains to an open versus a closed view of creation discussed earlier.

Creation really was meant to be an open versus a closed design, complete with all the laws that would be required in order to allow us to grow, learn, make mistakes, self-correct, forgive and inch ever closer to an illusory state referred to as at-one-ment. It is indeed incomplete and inconsistent from our perspective as souls, but both complete and consistent from the perspective of the Godhead. We are not meant to return to the Godhead, but are meant to discover and develop creation in such a way that our actions are in harmony and balance with the spiritual laws that brought it into being.

The Notion Of Infinity and Counting

I wasn't going to write on this subject either, but around March 10th/2011 I watched a youtube series on four rather tragic figures in math and science; namely, Cantor, Boltzmann, Godel and Turing. Each man was well ahead of his time in terms of his thinking, but died a rather tragic death--a death, no doubt, tied to the futile search for truth.

How does such a search come about ? It seems to me it starts with the basic notion of counting, which leads to numbers and eventually various infinities. The problem with this approach is that it isn't long before your ship runs aground and truth eludes you once again.

For example, the continuum hypothesis is undecidable, which means given our current understanding of things, there is no way to know if an infinity of sorts exists between the integers and the reals. I suspect most things are like this in any system we construct for ourselves; namely, undecidable, assuming you can even uncover them.

So what do we do ? To me, at least, it seems we need a new way to count and a new way to look at infinity. The thought I had involved regions divided by boundaries containing information. Either regions are infinite or finite in scope and contain an infinite or finite amount of information. A boundary would always divide regions and be of lower dimension. Seen this way, infinity

becomes less of a mystery in so much as there is now connectedness between the finite and the infinite through a boundary, which itself is of lower dimension and may contain information.

In the current school of thought, you 'never get to infinity' through some counting process; you simply let the counting process go on indefinitely and derive some result accordingly. But in reality, there is no such thing as 'reaching infinity' or 'crossing over into infinity' because the counting process is a linear notion that never stops.

As long as our thinking remains enumerative or 'unwoven' instead of 'woven', this is what you're left with--an illusory view of infinity [nature] brought on by a rather childish notion of counting that ultimately leads to inconsistency and incompleteness.

With 'woven' logic, where [mathematical] objects are now designed in such a way that each would have an influence on the other, it may be possible to recast ancient notions like counting and infinity in a new light and design a 'higher language' through which we begin to understand this immensely beautiful and mysterious creation the Father God has given to us. I only wish I knew where to go from here.

***

Let's take a simple example, based on an idea I had on May 17th/2011. I was talking to a chap I know up at Starbucks and we got to chatting about Noether's Theorem and the (symmetry, action, conservation law) connection. Among other things we discussed famous equations in mathematics including the only known equation that ties together pi with other numbers. This equation is

$$e \wedge (i * \pi) + 1 = 0$$

Here is what is interesting about this. If you think of the left-hand side as being the law and the right-hand side as being the action, then Noether tells us there is indeed a symmetry amongst numbers (presumably) from which the law emerges by way of an action whose value just happens to be zero in this case.

Written out in a slightly fuller form, then, we would have

$$<|> \;==\; 0 \;==\; e \wedge (i * \pi) + 1$$

The == symbol simply means 'equivalent to' and the <|> simply means 'symmetry'. Whatever the symmetry is, the law on the right-hand side comes from this symmetry, which in turn is generated from an action that can be associated with a value of 0. There may be many, possibly infinitely many, such laws that are generated from symmetries, which in turn are derived from actions whose encoded values are 0, 1, 2, ... and so on.

I suspect, then, that all such useful 'laws' in mathematics can be derived from symmetries, and that these symmetries are woven together in a much larger (geometric) mosaic based on actions. If so, such a mosaic would lead to many beautiful relationships in number theory, mathematics and physics that heretofore have gone undiscovered. Something to think about for the future, for it may turn out to be the mother of all Rosetta Stones.

Where Do We Go Now ?

When presented with the idea that Christ and God are symmetric duals, you have a choice. You can accept the notion as a child would, and believe it to be so, or you can reconcile that duality in another way, if such were possible. I have chosen to reason it out by way of the writings above, but in order to do so, I've had to consider axioms, conservation laws, symmetries, actions, Godel, Noether, Einstein, Jordan, NDEs, etc., and attempt to unravel the mystery over a very long period of time. I wonder now if it would have been better to be that child.

In the end, if you are lucky or marked by favor, I suppose you will probably come to this same very basic conclusion, which is to say:

Father-Son duality <==> Christ <==> Truth <==> Perfection <==> Completeness

You might also be willing to accept the fact that many other equivalent actions and symmetries exist, all of which lead back to Truth. As such, we are obligated to tolerate and encourage one another, wherever we are in the virtual space of creation.

About Me, A Self-Analysis (it's not what you achieve ... it's what you overcome ...)

You, my friend, are afflicted with Ehlers Danlos Syndrome and most likely Schizoid Personality Disorder. Your earthly father was probably the carrier for EDS and your mother the carrier for various mental disorders. Whether you willingly took this shell or not is unclear, but there it is--a defective, deficient body from the skin to its core which isn't getting any better as the years pass.

Even worse, you may be from a suicide pool if the psychic in Washington DC was correct. If this is so, your experience with God in this, your 23rd incarnation, is probably non-negotiable. People who suffer far less stand to gain far more in such a case, because they wouldn't be from such a pool.

Having said these things, you are probably more of a 'mathematical spiritualist' than a 'maniacal religionist' and so wouldn't fit in very well with any form of organized worship. You simply aren't one of them, and never will be.

All you can do is try to improve in small steps and not put too much pressure on yourself with the time remaining. Whether they did or didn't do the right thing a long time ago is not very important now. What's important is learning from this document you've written and apply what you can in a practical, useful way. If you can do that, your life in this, your 23rd incarnation, will have served its earthly purpose.

* * *

1 Thus says Yahweh to his anointed one, to Cyrus whom, he says, I have grasped by his right hand, to make the nations bow before him and to disarm kings, to open gateways before him so that their gates be closed no more:

2 I myself shall go before you, I shall level the heights, I shall shatter the bronze gateways, I shall smash the iron bars.

3 I shall give you secret treasures and hidden hoards of wealth, so that you will know that I am Yahweh, who calls you by your name, the God of Israel.

... was it really for me ? ...

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Addenda To The Essay

The following addenda add insight into this work, both from a philosophical and mathematical perspective

That Strange License Plate 1123 KW

What are the odds that I cross paths with the license plate 1123 KW, remembering that (a) 11 and 23 are two numbers that have followed me continuously throughout life and (b) K and W are the 11th and 23rd letters of the alphabet, and (c) Kevin Williams, the founder and maintainer of the website www.near-death.com, has these letters as initials--a website I've been studying rather intensely for the last two years … …

List of reasonable assumptions: (^ means raised to the power of and * means multiply by):

- plates of form 3 numbers 3 letters or 2 letters 4 numbers or 4 numbers 2 letters :
  10^3 * 26^3 + 2 * 26^2 * 10^4 which equals 31,096,000 combinations

- number of most common cities and towns in BC ~ 100

- odds of us both being on the road at same time are 1/4

- odds of both of us being in the same 30 * 30 square foot area given a 5 mile * 5 mile driving area in Kelowna are 1 / 774,400

- odds that I look up and see the plate, if we are within the same 900 square foot area at same time are 1/10

Odds that I cross paths with this license plate :

$$\frac{1}{31{,}096{,}000 * 100 * 4 * 774{,}400 * 10} \sim 1 / (10 ^\wedge 17) \quad \ldots$$

Fundamental Theorem of Creation and Melchizedek

Symmetries in the Godhead uphold the Law of Conservation of Truth … by way of an *action* equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
          |                  |                  |
          |                  |                  |
          |                  |                  |
          -------------------- Cross --------------------
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The ancient texts refer to Melchizedek as a priest-king in the time of Abraham, functioning as a precursor of sorts to Jesus Christ. In the essay, we identify Christ as a symmetry in the Godhead, which can be associated with kingship, and also as an essence in the open creation whose job it was to complete the law of harm and suffering. In other words, a high priest of sorts willing to make the ultimate sacrifice, and intercede on our behalf where appropriate.

This hybrid characterization of Christ, using Melchizedek as the example, lends even more credence to the Fundamental Theorem of Creation mentioned above, because it allows for both the duality within the Godhead and the priestly role outside the Godhead simultaneously.

How this simultaneity is accomplished via the action or 'membrane' that divides the Godhead from the open creation is a mystery, to be sure, but even so, its existence and uniqueness is virtually irrefutable.

We can also see an analogy to the Fundamental Theorem when looking at the inner and outer parts of the sanctuary, divided by a veil. The book of Hebrews speaks of this notion, but of course, cannot establish the Noether connection. Here, the inner sanctuary would represent the Godhead and its symmetries, the outer sanctuary the open creation and its active spiritual laws inherited from these symmetries, and the veil a symbolic representation of the membrane (action) binding these two pieces together. Since the Cross and the action are equivalent to one another, the Cross then becomes a replacement for the veil, and a gateway of sorts into the Godhead. It should come as no surprise, therefore, that this veil was torn in half when Christ's work on the Cross was complete, for his job was to unify the spiritual laws of creation by completing the law of harm and suffering, in particular.

And this he did, with *admirable* courage, giving any soul anywhere an opportunity to bind with the Godhead permanently, by embracing what is probably the highest form of truth available to us -- the Cross, along with all its magic, mystery and redemptive powers …

Godel's Theorem, Green Tao Theorem and The Godhead

It struck me recently that Godel's Incompleteness Theorem is probably equivalent to the recently discovered Green Tao Theorem. The latter tells us the prime numbers contain arithmetic sequences of arbitrary length, and the former tells us, loosely speaking, no system can ever discover (much less prove) all truths it was intended to capture.

Godel's Theorem ≡ Green Tao Theorem

If the two are equivalent, then it stands to reason it is impossible to design a perfectly random system -- one which has no regions of predictable regularity, anywhere. For if such a perfectly random system existed somewhere in the open creation, Godel's Theorem itself would suffer a contradiction, and we have seen already that such a contradiction can only exist in the form of a singularity, within the Godhead itself.

This tells us, in turn, the Godhead is not only made up of perfectly balanced symmetries, but also, that there are no regions of predictable regularity, anywhere, within this singularity. No matter 'where you are' within the Godhead it would never be possible to identify a region or subspace of predictable structure -- you would, in fact, be dwelling within a closed creation that was (a) perfectly random and (b) perfectly symmetric.

Although the two ideas seem at odds with one another, they probably are not, particularly when you think of something like a fractal, which is self-similar but never identical, as you move from one region of the structure to another. Quite possibly, then, the Godhead too is fractal in nature, but well beyond anything the human mind could ever comprehend …

An interesting consequence of this rather odd characterization of the Godhead is what I term 'perfect uncertainty' -- an observer dwelling within the closed creation (or without) can never determine the Godhead's decision-making process, for there are no arbitrarily long sequences of 'yeses' or 'noes' that will ever be found anywhere within this singular region.

It seems, therefore, as ordinary souls operating within the open creation, we can never know how the Godhead really thinks or communicates with its constituent parts; that is to say, the echoes that are exchanged between the various symmetries here. Indeed, 'perfect uncertainty' might make it impossible to decode the mysteries of the Godhead altogether, unfortunately.

Another interesting consequence involves the spiritual (and hence physical) laws of creation themselves. We know from the essay the spiritual laws of creation are equivalent to symmetries in the Godhead, and that these laws inherit their properties from these symmetries. As such, the laws may also be 'perfectly uncertain', meaning there is no way to characterize them with any degree of predictability even though they form a fully consistent and complete framework.

perfect uncertainty | symmetries Godhead | spiritual laws

In the same way that it is impossible to decode the Godhead, 'perfect uncertainty' makes it equally impossible to decode the spiritual (and hence physical) laws of creation, suggesting, in turn, that

they too are fractal in nature. To wit, perfectly random structures which seem to go on forever -- were we able to see them ...

```
      perfectly uncertain                          perfectly uncertain
                |                                            |
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
                |                        |                   |
                |                        |                   |
                |                        |                   |
                -------------------- Cross --------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

Green Tao Theorem and Some Consequences

Green Tao Theorem

The sequence of prime numbers contains infinitely many, arbitrarily long arithmetic (regular) sub-sequences. Prime numbers cannot be generated using any functional algorithm, and thus can be considered a random sequence. Arithmetic sequences, though, are always predictable since they can be generated by simple algorithms.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29 … the first few primes

3, 7, 11, 15, 19, 23, 27, 31 … an arithmetic sequence with period 4

Early Intuitive Consequences

There is no such thing as a perfectly random system, for if so, it has no regularity (predictability) anywhere. However, *any* random system is 'roughly' equivalent to the primes, but Green Tao tells us even the primes contain arbitrarily long, regular sub-sequences throughout.

There is no such thing as a perfectly random encryption scheme, for if so, it would not only be random, but perfectly random everywhere, and this is not possible according to Green Tao. Any encryption scheme will always contain arbitrarily long sequences of predictable information.

The complete number line is a continuum, which is random, since there is no algorithm that can predict each and every element. But by Green Tao it is not perfectly random and so contains sub-sequences of arbitrary length which can be generated algorithmically, like the integers and fractions.

Any continuum will always contain regions of regularity since no continuum is perfectly random everywhere. This may explain why universes, embedded within a larger virtual space, exist and are endowed with physical laws.

Tossing a coin infinitely many times is a random system, but it is not perfectly random according to Green Tao. Thus it contains arbitrarily long, regular sub-sequences of just heads and just tails, necessarily.

Free will, meaning the ability to choose freely, infinitely often, is a random system, but not perfectly random according to Green Tao. Thus it contains sub-sequences of arbitrary length which are predictable. There is no such thing as perfect free will even though you might be fooled into believing there is ...

Noether's Theorem, Commutativity and The Godhead

Consider an encoding, which we'll define as 45 and written as follows:

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 <~~> 45

By reflecting the left-hand side nothing changes, but a law is *automatically* introduced which is known as the Law of Commutativity Under Addition in Mathematics:

```
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9
                                    <~~>  45  <~~>  Law of Commutativity
9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1
```

It is the *very act* of reflection *itself* which brings this law into being, for without such a symmetry no such law could exist. This, in a nutshell, is Noether's Theorem -- symmetries leading to laws and vice-versa by way of an action that encodes both.

In terms of the Fundamental Theorem of Creation the following diagram applies:

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
          |                  |                  |
          |                  |                  |
          |                  |                  |
          -------------------- Cross --------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

Two important corollaries follow which we state here:

> Creation would have collapsed had Christ not completed his work on the Cross. Although this is self-evident from the diagram above, it is a profound notion which is nearly impossible for the human mind to grasp.
>
> You cannot atone for your own acts of harm. Reaching a state of at-one-ment with the Godhead means passing through the Cross. Only Christ can justify this union between any soul and the Father according to the Fundamental Theorem of Creation.

We can expand a bit on the second corollary by noting that at-one-ment is equivalent to zeroing out all acts of harm for any soul. Such an equivalency seems reasonable, especially when you consider

that such a union with the Godhead, by definition, requires the harmonics of a soul to be perfect. At the same time, at-one-ment is equivalent to passing through the Cross, according to the Fundamental Theorem, implying, in turn, that the Cross itself is equivalent to zeroing out all acts of harm.

```
at-one-ment with the Godhead  <~~> zeroing out all acts of harm
                |                    |                   |
                |                    |                   |
                |                    |                   |
                ------------------ Cross ------------------
                                     |
                                     |
                                 law of h&s
```

And Jesus said, "I have come not to abolish the Law but to complete it ..."

This brings us back to the second corollary -- no soul can or ever will be able to atone for its own acts of harm. It is only granted uniquely through Christ's work on the Cross and least action principles.

The law of harm and suffering (h&s) is a difficult one to interpret, partly because of its discretionary nature and partly because it forms the underpinning to all of Creation itself. However, if we think of it in contractual terms, using the notions of *offer* and *acceptance*, the Cross becomes a touchpoint of sorts where the meeting of two minds takes place.

On the one hand there is the offer of at-one-ment with the Godhead through Christ's work on the Cross, and on the other, there is acceptance of this offer by the soul formed through intent and understanding. No contract or covenant, if you will, can ever materialize unless there is a meeting of the two minds, meaning that the soul, in turn, must reach a point where such an understanding is possible.

In its broadest scope the law of harm and suffering is both necessary and sufficient to remove any and all acts of harm throughout the virtual space of Creation, and at the same time ... ... rebalance and reharmonize Creation through the notion of at-one-ment. However, the law is also discretionary in nature and so cannot go any faster than the 'slower moving parts'. In this sense, it is up to each of us to reach a point of understanding and acceptance by embracing the Cross, fully and completely, and realizing that it truly is the gateway to the Godhead ...

Noether's Theorem and The Gospel of John

I have just finished reading the Gospel of John, and was struck by some of the parallels you find there vis-a-vis the Fundamental Theorem of Creation. Let us start by recalling this theorem:

```
               life                      way                    truth
                |                         |                       |
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
                |                         |                       |
                |                         |                       |
                |                         |                       |
                ------------------- Cross -------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

And Jesus said, "I am the way, the truth and the life. No one can come to the Father except through me." (John 14:6)

In the Gospel of John, and only here, we read that Christ says 'I am the way, the truth and the life'. It is a profound statement which I'm sure has puzzled readers down through the ages, but seen within the context of the Fundamental Theorem, has a very deep meaning indeed.

When Christ said this he was really referring to the Fundamental Theorem above. We can interpret *way* as the action, binding together the symmetries in the Godhead [*life*] with the spiritual laws of creation [*truth*]. Christ was saying he is all three, and indeed, according to the Fundamental Theorem of Creation, all three are equivalent to one another in so much as losing one piece means losing the whole.

```
Symmetries Godhead  <~~>  Action  <~~>  Union of Spiritual Laws
              |                      |                          |
              |                      |                          |
            life                   way                        truth
```

John the Apostle could not have known what he was writing at the time because he didn't have the Noether connection leading to the Fundamental Theorem of Creation. As such, it is more than likely the statement really was made by Christ, suggesting, in turn, that even Christ himself was trying to tell us almost 2000 years ago how he and his Father brought everything into being. Evidently, no one was listening ...

Also in the Gospel of John, we read what is probably the most popular scripture in the entire Bible:

> *For God so loved the world that he gave his only begotten Son, that whosoever believeth in him should not perish but have everlasting life.*

There are some interesting things that can be gleaned from this statement, the first of which is the existence and uniqueness of the Son [*only begotten*] versus the Father. Within the context of the Fundamental Theorem, it is a personalized reference to symmetries in the Godhead [*Father and Son*] that give rise to the spiritual laws of creation -- among them, the law of *harm* and *suffering*, which was completed through Christ's work on the Cross [*gave his Son*].

```
symmetries Godhead  <~~>  Cross
                            |
                            |
                       law of h&s
```

The remainder of the statement focuses on the notion of *offer* and *acceptance*, where the Cross serves as a touchpoint of sorts. It is, in reality, commentary on the law of harm and suffering without sharing many of the details. More specifically, however, we can say this law functions according to the following diagram:

```
at-one-ment with the Godhead  <~~> zeroing out all acts of harm
              |                  |                  |
              |                  |                  |
              |                  |                  |
               ----------------- Cross -----------------
                                 |
                                 |
                            law of h&s
```

John 3:16, in all its mystery, is an attempt to describe and interpret the Fundamental Theorem of Creation. It actually does a pretty good job and you could say the statement was well ahead of its time. No doubt, too, it is authentic, for John himself would not have had the insight to make these inferences based on his own understanding of things back then.

Taken together, these two distinct statements in the Gospel of John (John 3:16 and John 14:6) provide a complete framework for the Fundamental Theorem of Creation outlined above. Naturally the statements are opaque and so, on their own, anyway, could never establish the connection. But they do serve as a nice confirmation of the theorem, looking back in time, just as mathematics, physics and current near-death research do, looking forward in time.

The Fundamental Theorem of Creation stands on its own, whether we look to the past or to the future for a better understanding of how and why things came into being. When it's all said and done, however, we have to believe, necessarily, that all roads lead to Christ ...

The Axiom of Free Choice and The Law of Harm and Suffering

We learned in a previous article titled *Noether's Theorem, Commutativity and the Godhead* that no soul can or ever will be able to atone for its own acts of harm. Let us begin by revisiting the diagram that illustrates the law of h&s in some detail --

```
at-one-ment with the Godhead  <~~> zeroing out all acts of harm
               |                    |                   |
               |                    |                   |
               |                    |                   |
               ---------------- Cross ----------------
                                    |
                                    |
                               law of h&s
```

every act of harm must be covered by an act of suffering

We can now ask the question 'why is this so' ? Why was creation designed in such a way that souls have neither the rights nor the privileges to self-atone ?

Our starting point is the *Fundamental Theorem of Creation,* which represents both a framework and a design that reflects the principles of least action --

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
           |                   |                   |
           |                   |                   |
           |                   |                   |
           ------------------- Cross -------------------
```

Using both diagrams we can see that the singular *action* of the Cross is both necessary and sufficient for the removal of any and all acts of harm throughout the virtual space of Creation, and that it (the Cross) is, in fact, a *least action* (optimal).

If we now allow any soul the right to self-atone, it's rather like granting that soul a dual form of free will -- how to make choices and how to atone for the poor ones. Such a design is doomed to fail because the axiom of free choice itself, outside the bounds of at-one-ment ..... leads to many inconsistencies which cannot be resolved. By granting a soul the right to self-atone, you inevitably run the risk of making things worse, unfortunately, generating even more harm than you had initially.

```
axiom of free choice ---› acts of harm
         |                      |
         |                      |
         |                      |
acts of harm  ‹---  self-atone free choice
```

This vicious loop can only be broken by removing the notion of self-atonement altogether from the axiom of free choice, and cancelling the harm in some other way. That way, according to the Fundamental Theorem of Creation, is the Cross -- a single, yet compelling act of profound suffering which erases this harm everywhere throughout the virtual cosmos ...

Energy Wells and The Law of Harm and Suffering

We have reached a point where we can probably answer the question as to why every act of *harm* must be covered by an act of *suffering*. Angie Fenimore's NDE tells us this is an axiomatic truth, but unfortunately does not tell us why.

Our starting point is the following diagram --

```
at-one-ment with the Godhead  <~~> zeroing out all acts of harm
              |                  |                  |
              |                  |                  |
              |                  |                  |
              ---------------- Cross ----------------
                                 |
                                 |
                             law of h&s
```

every act of harm must be covered by an act of suffering

If we think of an act of harm as a deformation within the fabric of the virtual cosmos, caused by the presence of negative energy, for example, it stands to reason that at some point this deformation will need to be 'smoothed out' by the corresponding presence of a positive energy so that the two do indeed cancel one another.

```
act of harm  ------› deformation in the cosmos
      |                          |
      |                          |
      |                          |
act of suffering    ‹-------    smoothed out
```

This positive energy finds its source in the Cross [Christ], according to the Fundamental Theorem of Creation, but still leaves us with the question as to why suffering was actually required to cancel the harm.

The answer seems to be tied to one's *frame of reference*, for within the Godhead itself *no* degree of suffering is required and there is nothing to be gained by using this notion to cancel out opposing energies. However, outside the Godhead [virtual cosmos], where souls live and operate, tying this cancellation of energies back to a compelling act of suffering makes eminent sense, for without such an act it is doubtful the typical soul could ever appreciate its offences and self-correct accordingly.

The law of harm and suffering, then, implies not only a *least action* [optimal], but also an action with *physical expression* [Cross] designed to teach and instruct so that the soul will indeed learn to self-correct at every step of the way in its cosmic journey. To be sure, these acts of harm are zeroed out, but at a very deep cost -- something we must always remember as we learn to grow into our own spiritual virtuosity …

```
removal of negative energy    ‹~~›    zoroing out acts of harm
            |                                   |
            |                                   |
            |                                   |
   Symmetries Godhead         ‹~~›          Law of H&S
            |                                   |
            |                                   |
            |                                   |
            ------------------ Cross ------------------
                                 |
                                 |
                             atonement
                      complete and consistent
```

Energy Wells and The Law of Harm and Suffering, Part II

In Part I, we attempted to answer the question as to why every act of *harm* must be covered by an act of *suffering* -- something we know is true according to Angie Fenimore's NDE, and indeed, is a law she says. In this follow-up note, we expand on the notion, somewhat, by offering the following diagram --

```
                      generalized law of h&s
                               |
                               |
        energy-  ‹~~  ------------------------------  ~~›  act of harm
           |                   |                              |
           |                   |                              |
energy+ ‹~~ membrane  ‹~~  ------------------------------  ~~›  Cross ~~› suffering
           |                   |                              |
           |                   |                              |
          ‹~~  ----------  Symmetries Godhead  ----------  ~~›
                               |
                               |
                           atonement
```

every negative energy caused by an act of harm must be
cancelled by a positive energy caused by an act of suffering

Rewriting the law of h&s as above, we can see that this law, acting as an agent on behalf of symmetries in the Godhead, draws upon a positive energy source [induced by suffering] to cancel various negative energies in the open creation associated with corresponding acts of harm. This positive source of energy, according to the diagram above, seems to originate within the action [Cross] itself -- a boundary of sorts, separating the Godhead and its symmetries from the spiritual laws, and containing all the information needed to bring Creation into being, according to Noether.

But because of equivalency, it really doesn't matter what path you traverse -- the end result is the same. By going up the right-hand side, for example, we see that symmetries in the Godhead cancel the harm by passing through the Cross. And by going up the left-hand side we see that symmetries in the Godhead cancel the negative energy by passing through the membrane. Either way, the law of harm and suffering [law of h&s], operating as an agent on behalf of these symmetries, is actively at work, maintaining all the checks and balances that are required in Creation by drawing on a positive source to do its job.

Whether we see this positive source as being associated with the Cross or the membrane [*action*] is irrelevant, since both are equivalent to one another according to the Fundamental Theorem of Creation. Without the *action* [membrane] there is no Cross and without the Cross there is no *action*--each upholds the other and losing either piece means, quite simply, losing all of Creation itself.

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
        |                   |                 |
        |                   |                 |
        |                   |                 |
        -------------------- Cross --------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

The deeper mystery of the Crucifixion, and why it was needed, can now be explained by using a two-fold approach, and keeping in mind the generalized version of the law of h&s. On the one hand, suffering on the Cross was done to make the soul aware of its offences and self-correct accordingly; we know as much from earlier writings, where we refer to this notion as a *frame of reference* problem.

But on the other hand, suffering through the Cross, by way of a symmetry in the Godhead, *quite literally* generated *all* of the positive energy that would ever be required ... to cancel *all* of the negative energy produced, by each and every eventual act of harm, throughout the virtual cosmos....

```
                    Spiritual Laws
                          |
                          |
positive energy   <~~>  Cross  <~~>   act of suffering
                          |
                          |
                  Symmetries Godhead
```

This positive energy had to come from somewhere and, it seems, the dual action of the Cross was chosen as the optimal device through which to get the job done. No doubt this additional energy was stored in the *action* [membrane], where it could be used as needed by the law of h&s to balance the books, as they say. Doing so would not only make sense, but also and quite naturally, explain both the equivalency and the intimacy shared between the Cross and the membrane that we see glimpses of in the Fundamental Theorem of Creation, for example.

According to the ancient writings, Christ only spent about six hours on the Cross -- roughly one-twelfth of the usual amount of time required to cause death using crucifixion as the punishment. It seems as though this is all it took in order to generate and store all of the positive energy that would eventually be needed to rebalance Creation at large. And a good thing too, for by my reckoning, anyway, we probably weren't worth the price of admission into his kingdom to begin with -- a price that, by any measure, cannot be comprehended ...

The Fundamental Theorem of Creation and Completeness

The Fundamental Theorem of Creation should be seen as a compact description for the grand design itself , which leads us to the question of just how complete and consistent this theorem really is--

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
        |                    |                   |
        |                    |                   |
        |                    |                   |
         -------------------- Cross --------------------
                             |
                             |
                         atonement
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

Godel's Theorem tells us that no matter where we are in the open creation, beyond the Godhead, our view of things will be both incomplete and inconsistent. No matter how you try to build a framework for knowledge, say, eventually your ship will run aground according to the theory.

However, the Fundamental Theorem of Creation itself is an *all-encompassing* prescription, linking the Godhead to other constituent parts through *equivalency*. Since the Godhead is both fully consistent and complete, it stands to reason that these other parts must be as well, and so we may conclude that the Fundamental Theorem of Creation is *not only* a design which is both *fully consistent and complete*; but also, one which is *optimal* because of least action principles.

This means, for example, that the spiritual laws of creation form a fully consistent and complete framework which is optimal; no other approach could, or ever would, result in a better configuration replacing what we have now. And, the same can be said for atonement -- it too operates in an optimal fashion, without any *conflict or inconsistency,* because it is a divine process [of the Godhead] equivalent to the Cross [law of h&s].

Self-atonement, on the other hand, through the axiom of free choice, leads to many conflicts and inconsistencies, and so, is not an option within the grand scheme of things. Indeed, you could say it has been removed from this axiom altogether, leaving the Godhead, in turn, with *no choice* but to atone on our behalf. But this is a good thing, for it guarantees every soul fair treatment according to their actions and intent, showing no bias one way or the other. Creation, it seems, would have it no other way ...

Degenerate Actions and The Primordial Fabric

In this note we are going to discuss, in a little more detail, the primordial fabric and various *actions* that may have existed there before everything morphed into the beautiful mosaic we see before us today. Our starting point, in this case, is the Fundamental Theorem of Creation, shown below --

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
          |                  |                  |
          |                  |                  |
          |                  |                  |
          -------------------- Cross --------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

`High above he pitched a tent for the sun, who comes forth from his pavilion like a bridegroom, delights like a champion in the course to be run ...` [19th Psalm]

The words above from the 19th Psalm are, no doubt, a reflection in the heavenly realms of Christ's zeal for his eventual mission here on earth, and in particular, his work on the Cross which would ultimately unify and complete the spiritual laws of creation.

We now know that Christ was indeed successful, and should be seen as both a hero and a champion for not only running the course, but finishing it with the admirable courage and dignity he retained throughout. But we could also ask the question -- what would have happened had he failed ?

According to the Fundamental Theorem of Creation depicted above, creation would have collapsed, leaving us with no symmetries, no laws, and by implication, no *viable action*. Indeed, the action, seen as a membrane of sorts, would have (a) necessarily dissolved into the primordial fabric itself or (b) collapsed into a singularity somewhere within the fabric or out at infinity. Either way, however, the encoded energy and information associated with this *degenerate action* would have been preserved, and could be used again to rebuild creation if that was an option.

Without the completion of Christ's work on the Cross there would be no creation, for there would be nothing to hang it on, according to the Fundamental Theorem. Truly a profound ... and almost incomprehensible result, but one we must accept if Noether and *least action* principles apply. It seems, with all we know, that indeed they do ...

```
primordial fabric ‹~~› Noether ‹~~› differentiable mosaic
                          |
                          |
          duality ‹~~› action ‹~~› law of truth
                          |
                          |
                        Cross
```

†††   †††

On The Equivalency of At-one-ment and Atonement

The Fundamental Theorem of Creation is an all-encompassing prescription for the grand design itself and, as we now know, one which is not only optimal, but also fully consistent and complete --

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
          |                  |                  |
          |                  |                  |
          |                  |                  |
          -------------------- Cross --------------------
                             |
                             |
                      ~~~~~~~~~~~›  atonement
     at-one-ment   ‹~~~~~~~~~~~
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

If we move left to right along the diagram above, we can see that symmetries in the Godhead pass through the Cross and into the open creation, where the spiritual laws were unified and completed via the law of harm and suffering [law of h&s] -- something referred to as *atonement* in earlier writings. Conversely, a soul moving right to left in the diagram above must pass through the Cross in order to enter the Godhead, and achieve a state of *at-one-ment*, for however long it can be maintained.

Both notions, namely at-one-ment and atonement, are not only equivalent to one another, because each is equivalent to the Cross; but also, *divine processes* under the control of the Godhead and beyond the axiom of free choice. As such, no soul can or ever will be able to atone for its own acts of harm, and by equivalency, no soul can, by any means possible, reach a state of at-one-ment under its own power -- both are only granted uniquely through Christ's work on the Cross and least action principles, according to the Fundamental Theorem of Creation.

```
                 Spiritual Laws
                       |
                       |
 at-one-ment   ‹~~›  Cross  ‹~~›   atonement
                       |
                       |
               Symmetries Godhead
```

The centerpiece in the grand design, from our vantage point, really is the Cross, for it binds together symmetries in the Godhead with the spiritual laws of creation, by way of an *action* [membrane] that gives rise to the equivalent notions of *at-one-ment* and *atonement*, simultaneously.

Each soul, therefore, must ultimately reach a point of understanding and acceptance by embracing the Cross, fully and completely, and realizing that it truly is both the essence and the substance from which all things in Creation are made, and woven together into a unified whole ...

On The Equivalency of At-one-ment and Atonement, Part II

In an earlier note titled Noether's Theorem, Commutativity and the Godhead, we established an equivalency between the Cross and zeroing out all acts of harm through the transitive notion of *at-one-ment*. Here we are going to clarify things a bit more by starting with the Fundamental Theorem of Creation, a fully consistent and complete description, which is shown below –

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
         |                   |                 |
         |                   |                 |
         |                   |                 |
         -------------------- Cross --------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

Clearly at-one-ment is equivalent to the Cross, according to the diagram above, and it should also be clear that in such a state the soul becomes *one* with the Godhead, implying (necessarily) that all acts of harm have been removed. The soul's harmonics are perfect, so to speak.

```
at-one-ment  ==> zeroing out all acts of harm
     |
     |
   Cross
```

In turn, this means the Cross implies zeroing out all acts of harm, and since, by the Fundamental Theorem, zeroing out any and all acts of harm necessarily implies the Cross [law of h&s], the two notions must be equivalent.

```
Cross  ‹~~›  zeroing out all acts of harm      (*)
```

This distinct equivalency between the Cross, and removing all acts of harm for any soul, which we refer to as *atonement*, now makes both at-one-ment and atonement equivalent to each other, as shown in the following diagram --

```
               Spiritual Laws
                     |
                     |
at-one-ment  ‹~~›  Cross  ‹~~›  atonement
                     |
                     |
             Symmetries Godhead
```

Such an event -- the *simultaneous* experience of at-one-ment and atonement -- must be rare in the higher realms and almost certainly would never occur in the physical ring, but it could be seen as a lofty ideal to which each and every soul might aspire, as they move along in their cosmic journeys. Indeed, according to Thomas Sawyer's NDE, choosing to merge *completely* with the light of God means never returning to physical life again [under our own will], implying that the dual experience of at-one-ment and atonement really is the end of the reincarnation cycle for any being in this rather fortunate position.

More generally, the law of h&s [Cross] operates in a discretionary manner, in all likelihood, cancelling some acts of harm on behalf of the soul, and retaining others. Forgiveness, unconditional love, reincarnation and other expressions of learning can all be used to mitigate things somewhat, and will eventually lead to a higher path for the soul, but ultimately the decision to cancel these offences rests solely with the Cross, meaning Christ himself.

To see this more clearly, consider the case where a soul has one and only one act of harm associated with its cosmic history. If that soul were allowed to cancel this harm, thereby zeroing out all acts of harm, we would be forced to conclude that self-atonement is indeed equivalent to the Cross, which is an absurdity (*). Now assume *n* acts of harm, tied to this soul's history, have been cancelled through the Cross, and that an *(n+1)-th* violation occurs sometime later. By what we have just said, the soul cannot self-atone, even in the case of a single, outstanding offence, regardless of where it occurs along the cosmic timeline, and so we are forced to conclude that the *(n+1)-th* act of harm really is erased by the Cross after all.

```
                    zeroing out one act of harm
                                 |
n acts of harm zeroed out   ==>  n + 1 acts of harm zeroed out
           |                     |                     |
           |                     |                     |
           -------------------------------------------
                                 |
                               Cross
```

Thus, by induction we see that no soul can or ever will be able to atone for its own acts of harm. It is only granted uniquely through Christ's work on the Cross and least action principles, according to the Fundamental Theorem of Creation. And rightly so, for if self-atonement really was part of the axiom of free choice, it would inevitably cause more problems than it would fix, and alas, become the very disease it was intended to cure ...

A New Heaven and a New Earth

From what we have learned so far, the Fundamental Theorem tells us that ultimately Creation will be rebalanced and reharmonized by way of the Cross. We can see this more clearly by noting that souls, too, are a fundamental component of Creation, and so, by virtue of equivalency, have *already* reached a fully consistent and complete state in God's simultaneous now.

```
              souls     <~~>     physical ring   <~~>   realms
                |                      |                    |
                |                      |                    |
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
                |                      |                    |
                |                      |                    |
                |                      |                    |
                -------------------- Cross --------------------
                                       |
                        consistent, complete, optimal
```

This means, among other things, that every soul will eventually return to the Light of God, and attain a parallel state of *complete* at-one-ment and atonement, according to the diagram below --

```
                    Spiritual Laws
                          |
                          |
at-one-ment   ‹~~›  Cross  ‹~~›   atonement
                          |
                          |
                 Symmetries Godhead
```

* * * * *

> `I knew with total certainty that everything was evolving exactly the way it should and that the ultimate destiny for every living being is to return to the Source, the Light, pure Love` [Juliet Nightingale's NDE]
>
> `For look, I am going to create new heavens and a new earth, and the past will not be remembered and will come no more to mind` [Isaiah 65:17]
>
> `Then I saw a new heaven and a new earth; the first heaven and the first earth had disappeared now, and there was no longer any sea` [Rev 21:1]

* * * * *

When we combine all three statements above, it is not too hard to infer that at some point, however far into the distant future, Creation is going to morph into a new kind of mosaic where (a) all souls

will be in a state of complete at-one-ment with the Godhead and (b) all rings outside the Godhead will merge with the Light.

In this sense, the differentiable mosaic, as we see it today, will no longer exist. A new heaven [higher and lower realms] and a new earth [physical ring] will likely emerge, with characteristics and qualities similar to those of the Godhead. This new mosaic will, no doubt, be more *homogeneous* in nature, and infused with a uniform resonance that permeates all realms and rings everywhere, thus giving any soul anywhere the feeling of being *at one* with both its Creator and all other living things, simultaneously.

Just how long it's going to take to reach this endpoint is unclear, as many souls today still languish in the void and many others still have a desire to experience in the physical plane, for better or worse. An estimate is provided in a later note titled *How Large Is The Physical Ring*, and there the number is roughly 10^{30} earth years, an incomprehensible span no matter the measure.

In Old Testament writings we read of a long-suffering God, but clearly this is an understatement in light of everything we now know. Still, out of love for his Creation, God is willing to wait for the day when all is well again. As such, it is up to each and every one of us to do our part, assisting where we can and evolving as we must. There really is no other option, when you think about it ...

> `Then Yahweh passed before him and called out, 'Yahweh, Yahweh,
> God of tenderness, and compassion, slow to anger, rich in faithful
> love and constancy ...` [Exodus, 34th chapter]

Love In The Primordial Fabric

According to all that we know, the Fundamental Theorem of Creation tells us there was a primordial fabric of sorts before everything was brought into being. This fabric could not have had any discernible structure, and so, should be seen as more of a framework, if you will, containing the undifferentiated energy of God everywhere.

At some point, though, a shift took place [within the fabric], causing symmetries and laws to come into being, along with other components, as shown in the diagram below --

```
                 souls     <~~>    physical ring   <~~>   realms
                   |                     |                   |
                   |                     |                   |
  Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
                   |                     |                   |
                   |                     |                   |
                   |                     |                   |
                   -------------------- Cross --------------------
                                         |
                              consistent, complete, optimal
```

But exactly what caused this shift seems to be a mystery, for there is no reason it ought to appear unless, of course, the undifferentiated energy of God [everywhere] *expressed* a desire for change driven by some kind of emotion.

That emotion [or feeling], might well have been *divine love* for a creation [child] that didn't exist yet, but could be brought forth under pain and suffering [Cross] if the will to do so was there.

```
Primordial Fabric  ~~›  Creation
                     |
                     |
              Love and Suffering
```

Evidently the will existed, for what we have today, according to the Fundamental Theorem, is a beautiful mosaic which could be seen as God's *offspring* -- a creation fully formed, complete and consistent within the simultaneous *now*, one might say.

> For God so loved the world that he gave his only begotten
> Son, that whosoever believeth in him should not perish but
> have everlasting life [John 3:16]

When we read the words of John above, it actually is a rather remarkable statement, for not only does it apply within the context of the differentiable mosaic, but also, the domain of the primordial fabric itself. Even before there was a mosaic there was a fabric, and even here there was *love* before there was anything else; a love so strong, evidently, that it was willing to suffer grievously for the child it would eventually give birth to ...

On The Holy Ghost and The Fundamental Theorem of Creation

Up until now, we have discussed the Fundamental Theorem of Creation through the familiar terms shown in the diagram below --

```
         Father & Son            membrane          Holy Spirit
               |                     |                   |
               |                     |                   |
  Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
               |                     |                   |
               |                     |                   |
               |                     |                   |
               ------------------- Cross --------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

In Christian doctrine we read of a Father, Son and Holy Ghost forming a Trinity of sorts, but up until now have not considered the third member of this Trinity; namely, the Holy Spirit. The First Epistle of John says 'it is the Spirit that bears witness, for the Spirit is Truth', and so we can infer from this that the Holy Ghost itself becomes a moral compass, if you will, acting as an agent on behalf of the Godhead.

If our characterization of things is correct here, it would only make sense to equate the Holy Spirit with Truth in the diagram above, meaning the spiritual laws of Creation as a whole. Under this scenario, the Father & Son duality and the Holy Ghost become reflections of one another via the membrane, and so can be thought of as a trio of equivalencies [Trinity], according to the Fundamental Theorem.

The Fundamental Theorem of Creation, in its broadest sense, is both fully consistent and complete, so if indeed the Holy Spirit does exist as a separate entity, it must exist, perforce, somewhere within this description. In James T's NDE, for example, the Holy Ghost is seen as the *mind* of God and made up of *antimatter* that coexists with *matter* everywhere. As such we can, if we wish, think of the spiritual laws as being *infused* into the open creation and operating in a perfectly ethical fashion at all times [without bias], even though they couldn't be detected by any ordinary means possible.

```
Spiritual Laws  ~~~›  Creation
       |                  |
       |                  |
  Holy Ghost    ~~~›  antimatter
```

Whether the Holy Spirit really does exist as a third component within the Christian Trinity is somewhat up for grabs in my view. Almost nothing is said about it in the NDE research and no

one, to my knowledge, has ever seen such a distinct essence. As well, the Fundamental Theorem doesn't require anything more than a duality within the Godhead to make everything work -- a notion that is only reinforced when you consider least action principles and equivalency among and between all of the spiritual laws of Creation. In the final analysis, it may be nothing more than terminology, as they say ...

Relativity Theory and The Spiritual Laws of Creation

Recently I was thinking a little more deeply about the spiritual laws of creation and wondered if there might be a law tying together the very basic ideas of curvature and stress in a more universal way. If I take a piece of paper and apply pressure to the edges with my hands, the paper will flex; similarly a wind blowing through the trees in a park will cause the branches to bend, and oddly enough, if I'm *stressed out*, my fingers or lips may curl up.

More generally, then, we might say there is a fundamental duality between curvature and stress originating in the spiritual domain, and with a physical counterpart, which goes something like this:

> *For any curvature tensor C there exists a stress tensor T such that T induces C, and conversely. Properties of C are inherited by T, and vice versa, because of duality.*

Let us now apply this law to general relativity by noting first [from the Bianchi identities] that we always have, for any smooth, simply-connected manifold,

$$\mathrm{cov}[R(u,v) - 1/2 * R * g(u,v)] = 0 \quad \text{.... (1)}$$

where cov should be interpreted as the covariant derivative in u or v. There is never a point on the manifold for which this is not true, and so because the bracketed expression is a curvature tensor [which we'll define as C(u,v)], a stress tensor T(u,v) *must* exist which induces C(u,v) and inherits any of its properties. In other words,

$$\mathrm{cov}[C(u,v)] = 0 = \mathrm{cov}[T(u,v)] \qquad \text{..... (2)}$$

for all u,v and all points in the manifold. Thus, by the Fundamental Theorem of Calculus, we may now infer that

$$C(u,v) = k * T(u,v) + K \qquad \text{..... (3)}$$

where a numerical scaling constant k has been added and cov(K) = 0. In a nutshell, this is general relativity as seen through the eyes of the fundamental curvature-stress duality.

When you look at (2) above, you can also see hints and glimmers of Noether, in so much as symmetries associated with C(u,v) lead to conservation laws associated with T(u,v) through an action that just happens to have a value of zero. More generally, I suspect any unified field theory in physics, if it is ever found, will always have a curvature tensor C for which cov[C(u,v,w,x, ...)] is 0, in order to preserve the conservation laws associated with T that follow ever so naturally.

Just what C or T really are I cannot say, but to be sure, the world of physics today is fraught with attempts of one kind or another to build a theory of everything. However, it is my view that until we look at things in a more spiritual light, and take our inspiration directly from the Godhead, no such theory will ever materialize. Even then, Godel's work may prohibit a fully consistent and complete description of Creation outside the bounds of at-one-ment, just as it should ...

The Continuum Hypothesis and Unified Theories in Physics

The Continuum Hypothesis tells us we can never decide if there is or isn't a class of numbers S sitting between the integers and the reals, given our current understanding of axiomatic set theory. Typically, mathematicians assign a cardinal number of **N** to the integers and **c** to the reals, and tell us we cannot know if there is a cardinal number **r** corresponding to S such that

$$\mathbf{N} < \mathbf{r} < \mathbf{c}$$

Turning to physics, we know there are two competing theories which have never been reconciled -- Quantum Mechanics on the one hand, and General Relativity on the other. The first theory is largely a *state-driven* probability model over the integers, while the latter should be seen as a *continuously smooth* model over the reals. Intuitively, then, and without loss of generality, one can assign a cardinal value of **N** to Quantum Mechanics, and a cardinal value of **c** to General Relativity, giving us the following diagram :

Quantum Mechanics	bridging theory	General Relativity
N	**r**	**c**
	undecidable	

In doing so, the question now becomes one of whether or not there really is a *bridging theory* that joins the other two pieces together; in other words, can we move from the tiny world of Quantum Mechanics over **N** to the larger world of General Relativity over **c** by way of an intermediate theory over **r** which glues everything together ?

According to the Continuum Hypothesis this question has no answer, meaning if you pose the query today or a thousand years from now the response will not change -- quite simply, it cannot be decided. Philosophically, then, we are left with *no* choice but to conclude a *unified* theory of physics spanning Quantum Mechanics to General Relativity really doesn't exist after all, for if it did, any bridging theory sitting in between the two would have to emerge at some point, thereby contradicting the notion of *undecidability*.

Although there is a great deal of activity in physics today devoted to the idea of a theory of everything, it is highly unlikely such a theory will ever be found. Godel's work alone tells us such a quest is practically futile, and now the Continuum Hypothesis seems to be saying the same thing. Until we see things in a more spiritual light and take our inspiration directly from the Godhead, unified theories in physics will come and go like the seasons, leaving behind a legacy of confusion on the one hand, but a glimmer of hope on the other -- a hope that someday, just maybe, a complete description of Creation will finally appear on the horizon ...

Godel's Theorem and The Green Tao Theorem Revisited

In an earlier note we postulated that Godel's Theorem and the Green Tao Theorem were equivalent notions. Here, using a rather heuristic approach, we are going to offer a little more evidence for this idea, and then look at some generalizations.

Godel's Theorem tells us [roughly] no system S can ever discover (much less prove) all truths it was intended to capture, and that within S many inconsistencies will arise. Green Tao, on the other hand, tells us the prime numbers P contain arithmetic sequences of arbitrary length, literally everywhere. A random system R, almost by definition, ought to be just that -- random throughout, but surprisingly, this turns out not to be the case at all, at least with respect to the primes.

If we think of these regular sequences in P as exceptions [ripples or aberrations which shouldn't be there], it becomes rather evident a striking parallel exists between the inconsistencies which surface in S and those that appear in P. In other words, the *anomalies* in S become, if you will, the *unexpected* regions of *regularity* in P, leading to a correspondence of sorts as shown in the diagram below.

```
      S ---------------------------- P
      |                              |
    Godel                        Green Tao
      |                              |
      |                              |
  anomalies      ‹~~~›       regular regions
      |                              |
```

Since ~[Godel] is, by definition, an attribute of the Godhead, it is not much of a leap to suppose ~[Green Tao] is as well. And, because all attributes in the Godhead are equivalent by virtue of symmetry, we may conclude

```
    ~[Godel]     ≡    ~[Green Tao]
        |                  |
        |                  |
 fully consistent    no regularity in
   & complete          randomness
```

thus proving [at least intuitively] the equivalency of these two theorems in our reality, by taking their complements in the expression above.

Understanding just what ~[Green Tao] means is not an easy task; however, by inference it should be clear that *any* random system R containing predictable regions of arbitrary size is probably not a candidate for the Godhead. In particular, this rules out the primes and, at the same time, strongly suggests the only kind of system you could ever have [within the Godhead] is one which is perfectly random. That is, a random system containing *no* regions of predictable regularity [of arbitrary size], anywhere, you might say.

```
Godhead      ‹~~›     Symmetries
   |                      |
   |                      |
   ------------------------
              |
              |
      perfectly uncertain
```

Whether our conclusions are true is hard to tell; however, the Godhead is not going to give up its secrets easily, and if we insist on a fully consistent and complete design in accordance with the Fundamental Theorem of Creation, perfect uncertainty seems like a good fit here. Indeed, in the final analysis Godel may demand it ...

On The Equivalency of The Spiritual Laws of Creation

In the *essay* we gave a rather heuristic argument for the equivalency of the spiritual laws of creation. Here we are going to tighten up that proof a little bit, using the Fundamental Theorem as our starting point --

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
        |                   |                 |
        |                   |                 |
        |                   |                 |
        ------------------- Cross -------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

Suppose Truth is defined as $U\{L(i)\}$, with $L(1)$ being the law of h&s, always. If we now lose the Cross then by equivalency we lose the *action*, and so

$$U\{L(i)\} = 0, \quad i >= 2$$

Thus, $L(1) = 0 \implies L(i) = 0$ for all $i >= 1$, meaning all laws vanish. Assume now $\{L(1), ..., L(n)\}$ is a set of laws all equivalent to one another, and that $L(n+1)$ is also a law which is not equivalent to any element in the previous set. If $L(1) = 0$ then $L(n+1) = 0$, so by assumption [equivalency], $L(i) = 0 \implies L(n+1) = 0$ for $i <= n$. In other words, we have the relationship

$$\sim L(i) \implies \sim L(n+1), \quad i <= n$$

meaning $L(n+1) \implies L(i)$, for any $i <= n$. Also by assumption, no $L(i) \implies L(n+1)$, for if even one did, they all would, making $L(n+1)$ equivalent to each of them. In particular, $L(1)$ does not imply $L(n+1)$, but $L(1)$ is equivalent to the *action*, according to the Fundamental Theorem, and thus we have a contradiction.

By induction, then, we see that *all* the spiritual laws of creation are indeed equivalent to one another -- losing any one of them means losing the whole, most assuredly. In particular, this applies to $L(1)$, the law of h&s, and so we may conclude that Creation really is an *all-or-nothing* proposition after all -- either you have a fully consistent and complete set of laws to begin with, or you have nothing at all. As such, we must always see things within the *simultaneous now* when looking at the Fundamental Theorem, and move from one frame of reference to another when it seems convenient to do so. Creation, it appears, would have it no other way ...

* * * * *

You'll notice in the proof above L(1) functioned as an anchor of sorts, being equivalent to the Cross on the one hand, and a spiritual law itself on the other. In fact, it is because L(1) ~ Cross *holds* that we were able to show *all* of the spiritual laws of Creation [Truth] are equivalent to one another, thereby allowing us to frame the Fundamental Theorem the way we have. Indeed, in the same way that breaking L(1) ~ Cross causes the whole of Creation to collapse, forming this equivalency does just the opposite -- it causes the whole of Creation to come into being !

When Christ said "I have come not to abolish the Law, but to complete it ...", he was very likely talking about completing the law of h&s on the Cross; to wit, a necessary *physical* expression in the *ordered now* which would ultimately unify all of the spiritual laws through equivalency in the *simultaneous now*, and thus Creation itself. As a result, and among other things, only a simple duality [Father & Son] would ever be needed within the Godhead to uphold these laws, thereby paving the way for the many discoveries the Fundamental Theorem has led to.

simultaneous now | *ordered now*

all the laws ‹~~› membrane ‹~~› action ‹~~› Cross ‹~~› law of h&s

spiritual frame | *physical frame*

In a peculiar sense, when you think about it more deeply, Christ might have been trying to tell us a little more about the nature of these symmetries in the Godhead when he made this comment, but just like today, it is highly unlikely anyone was really listening ... after all, why would they ?

* * * * *

Perfect Uncertainty and The Spiritual Laws of Creation

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
          |                   |                  |
          |                   |                  |
          |                   |                  |
          ------------------- Cross -------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

The Fundamental Theorem of Creation is both fully consistent and complete, as we know, and so by equivalency it should be clear that ~[Godel] is an attribute of T … … where T is defined to be U{L(i)}, the set of laws. Since

~[Godel] ≡ ~[Green Tao, R]

where R is any random system, we can also say ~[Green Tao, R] is an attribute of T and thus conclude T is indeed *perfectly uncertain*.

```
                 Light
                   |
      --------------------------------
     |                                |
     |                                |
  Godhead      ‹~~›      Symmetries     ‹~~›     Laws
     |                                |                |
     |                                |                |
      --------------------------------        perfectly uncertain
                   |
           perfectly uncertain
```

Among other things this means the laws are *unknowable*, just like the corresponding symmetries, and so any search for a fully consistent and complete description of these laws, outside the bounds of at-one-ment in the *current* mosaic, will likely be futile.

The Continuum Hypothesis tells us we can never know if there is or isn't a *bridging theory* which joins together the worlds of Quantum Mechanics and General Relativity, for example, and so prevents us from ever finding a *unified* theory of physics which combines the two. This shouldn't come as a surprise now, especially once we understand that perfect uncertainty, by definition, means unified theories in general are [unfortunately] nothing more than empty promises in disguise.

In the *unified* mosaic, however, life will be different. Everything will be in a state of complete at-one-ment with the Godhead in those days, but alas, by the time this happens the symmetries *and* laws that currently hold everything together will have *disappeared*, according to the Fundamental Theorem of Creation. A rather ironic end, to what must have seemed like a promising start, for those who believed originally, and still believe today, in the notion of absolute knowledge beyond the membrane ...

The Cross, The Crown and The Gold Medal

This particular note wasn't supposed to be part of the *Essay* folder, but recently I was looking at the gold medal I won back in 1978, on my birthday, and realized there was an odd similarity between the engraving on this medal and the geometric layout of the Fundamental Theorem of Creation --

```
Symmetries Godhead  <~~>  Action  <~~>   Law of Truth
         |                   |                  |
       wings                owl               wings
         |                   |                  |
         |                   |                  |
         -------------------- Cross --------------------
                             |
                           crown
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

On one side of the medal we see an owl with its wings outstretched, and hovering over a *crown*. The *wings*, to me at least, represent *symmetries* in the Godhead, along with the corresponding spiritual *laws* of creation, while the owl itself corresponds to *action*, a structure equivalent to the Cross [crown].

The owl, according to folklore, is a spirit animal emblematic of a deep connection with wisdom and intuitive knowledge, and those who have this totem, it is said, are likely to see what's usually hidden to most. With regard to the diagram above, we can see an immediate comparison to Christ, where now Christ becomes this magnificent creature with its wings outstretched, and hovering over its prey [Cross] -- the ultimate prize in all of Creation, you might say.

From our perspective in the physical ring, the Crucifixion does appear to be an event which is both shameful and humiliating; but in reality and from a spiritual vantage point, the Cross actually becomes God's *crowning* achievement in Creation, for without it everything would have collapsed, as we now know. As such, we must always take the spiritual view when considering these mysteries, looking beyond what seems familiar to us and moving up into those rarefied spaces where only the enlightened ones dwell.

> `High above he pitched a tent for the sun, who comes forth from his pavilion like a bridegroom, delights like a champion in the course to be run ...` [19th Psalm]

On The Equivalency of Noether's Theorem and Closed Systems

Up until now, we have tacitly assumed Noether's Theorem could be used to describe the morphing of the primordial fabric into the differentiable mosaic, which we see before us today --

primordial fabric ‹~~› Noether ‹~~› differentiable mosaic
|
|
duality ‹~~› action ‹~~› law of truth
|
|
Cross

Now we'd like to tighten things up a little bit by offering a reasonable proof that Noether may have been the *only* available mechanism by which this transition occurred, assuming the geometry of the primordial fabric *was* [and still *is*] both simply-connected and closed. The first of these two properties is always presumed to be true throughout, in what follows ...

Let us begin by noting that if a Noether solution N exists *everywhere* on M, then M must be simply-connected and closed, for if not, M would *roughly* be equivalent to S(n) + holes, according to Poincare, where S(n) is a sphere of dimension n. But if a singularity exists somewhere on S(n), topologically N would cover S(n) almost *nowhere* [think of poking a hole in the sphere S(2), say, and shrinking the resulting infinite plane down to a single point], and so we may conclude M cannot contain any singularities after all. It is, in mathematical parlance, homeomorphic to the n-sphere, for some integer n.

On the other hand, if M is a simply-connected and closed manifold, we know by Poincare M ~ S(n), and here we can always find a Noether solution, everywhere, for the latter. Thus we have the following rather deep equivalency --

> If M is any simply-connected and closed manifold, there is always a Noether solution which covers M, and conversely. In other words, for the simply-connected case …

$$N \supset M \quad <\sim\sim> \quad M \sim S(n)$$

This result is somewhat startling because it tells us, at least geometrically, the primordial Godhead may have had *no* choice but to differentiate itself by way of Noether, assuming the manifold was equivalent to S(n) initially. In all likelihood such was the case, implying, *a fortiori*, a rearrangement of the indiscernible primordial energy leading to symmetries and laws by way of an action equivalent to the Cross.

We identify with these symmetries, on a more personal level, as the *Father* and the *Son*, according to Christian tradition, but any pair of labels would work just as well when thinking back to Noether's Rosetta Stone. Indeed, according to Ken Vincent, in his book *God Is With Us*, the post-conscious [or post-conventional] state of moral development is what we all ought to be aiming for

anyway, where the usual norms and systems of belief are tossed in favor of something that is more universally true. In my view, nothing says it better than the Fundamental Theorem of Creation -- a fully consistent and complete description of the grand design itself, touching on virtually every aspect within the system of things, or so it would seem ...

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
          |                  |                 |
          |                  |                 |
          |                  |                 |
          --------------------- Cross ---------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

Covering Theorems Over Manifolds In Creation

In the last note, we discussed Noether's Theorem over a simply-connected manifold M, and saw that

$$N \supset M \quad <\equiv> \quad M \sim S(n) \qquad (*)$$

where N is any Noether covering for M. Here we want to develop the theory just a bit more, and look at *all* theorems {T} for which (*) might be true on a manifold spanning the *whole* of creation.

So let's begin by stating the following rather self-evident postulate --

> ′ If T is any fully consistent and complete theorem over M, then necessarily T is perfectly uncertain, and conversely. If E(T) is the energy associated with T, there will always be one such T for which E(T) is minimal, implying symmetry ′

In other words,

$$\exists \;\; T \supset M \;\; \P \;\; \sim[\text{Godel}, T] \;\; <\equiv> \;\; \sim[\text{Green Tao}, T] \;\cap\; E(T) <= E(T') \;\; \forall \; T,T' \; \varepsilon \; \{T\} \qquad (\dagger)$$

Philosophically, then, this means no matter what theorem T we come up with, spanning creation, it must always be fully consistent and complete on the one hand, and perfectly uncertain on the other, in so much as these two attributes go hand in hand because of equivalency. Furthermore, the energy associated with T must be minimal, in accordance with variational principles, and so the theorem itself would have to function according to some *least action* principium, whatever this finally is.

Either T is a Noether covering for M or it isn't. If not, then T is most likely a set of laws over M, absent a viable action [or boundary] containing any symmetries. However, because of (†), one could argue the laws are *symmetrically* equivalent in this case, and if so, T becomes a *degenerate* Noether covering for M, where the symmetries and laws now become *one* and the *same* thing, so to speak. Thus, we may conclude --

$$N \supset M \quad <\equiv> \quad T \quad <\equiv> \quad M \sim S(n)$$

In other words, the theorem we are searching for *is*, in all probability, Noetherian over the n-sphere, where symmetries imply laws, and vice-versa, by way of a *minimal* action linking the two together.

Such a conclusion makes eminent sense, for nothing could be simpler, more compact or more efficient than this description of things, and so we can say, with some confidence, that the Fundamental Theorem of Creation is not an arbitrary result -- in all likelihood it emerged, necessarily, by way of variational principles associated with both the energy and geometry of the original primordial fabric ...

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
        |                   |                  |
        |                   |                  |
        |                   |                  |
        ---------------------- Cross ---------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

LOOKING FORWARD

Later on, when we study *coexistence*, we'll learn that for a local observer in β-space, the energy of β is always 0. Thus, if T is any theorem here, we must have

$$0 \leq E(T) \leq E_\beta$$

which implies E(T) = 0. Hence, T is always a *minimal* energy theorem in β-space, implying it must be Noetherian.

Sayings and Musings

The following are a list of sayings collected over a period of time, while all this research was going on. No attempt is made to interpret any of them; rather, the intent is simply to ponder their meaning ...

- just as gold is tested in the fire, so are the chosen tested in the furnace of humiliation ...
- I didn't realize the juxtaposition of black and white could be so beautiful ...
- he is charged with finding the being who will complete his healing ...
- your law, my God, is deep within my heart (holy spirit) ...
- he traps the crafty in the snare of their own cunning ...
- the Cross ... God's crowning achievement in Creation ...
- learn to distinguish between the precious and the base ...
- man will prey on man until man will pray for man ...
- they are what they need and need what they are ...
- silence ... the highest form of enlightenment ...
- we bind the books ... and then they bind us ...
- spiritual freedom ... a gift from the Son ...
- Taoism ... silence and effortlessness ...
- I didn't give you permission to sin ...
- bend my heart to your instructions ...
- his suffering justifies the quest ...
- you cannot inherit moral maturity ...

Hints and glimmers of the Unified Mosaic ...

- when everything has been subjected to him, then the Son himself will be subjected to the One who has subjected everything to him, so that God may be all in all ... [Cor, 15th chapter]

- Jesus, whom heaven must keep till the universal restoration comes ... [Acts, 3rd chapter]

Perfect Uncertainty and The Cross

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
          |                    |                  |
          |                    |                  |
          |                    |                  |
          ------------------- Cross -------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

The Fundamental Theorem of Creation is both fully consistent and complete, as we know, and so by equivalency it should be clear that ~[Godel] is an attribute of the Cross. Since

$$\sim[\text{Godel}] \equiv \sim[\text{Green Tao, R}]$$

where R is any random system, we can also say ~[Green Tao, R] is an attribute of the Cross and thus conclude the Cross is indeed *perfectly uncertain*.

```
                         Light
                           |
          --------------------------------
          |                              |
          |                              |
      Godhead          ‹~~›         Symmetries     ‹~~›      Cross
          |                              |                      |
          |                              |                      |
          --------------------------------              perfectly uncertain
                           |
                  perfectly uncertain
```

Among other things this means the Cross is *unknowable*, just like the corresponding symmetries, and so any search for a fully consistent and complete description of this *action*, outside the bounds of at-one-ment in the *current* mosaic, will likely be futile.

Just as Heisenberg's *Uncertainty Principle* prevents us from ever knowing completely the behavior of sub-atomic particles ...... so too does *perfect uncertainty* place limits on our understanding of the Fundamental Theorem of Creation. In particular, this means the Cross will always remain a mystery, no matter where you are beyond the membrane, and thus it becomes both a notion and a symbol one accepts by *faith*, and not by *knowledge*.

```
knowledge   |‹--- ∂ ---›  perfect uncertainty  ‹--- ∂ ---›|   faith
                                 |
                                 |
                               Cross
```

When Christ said

> '... unless you change and become like little children, you will never enter the kingdom of heaven ...' [Matthew, 18th chapter]

he was really making a comment about the Cross, and more specifically, the attribute of perfect uncertainty within the context of the Fundamental Theorem.

In the final analysis, then, it behooves each of us to set aside our own prejudices and our arrogance, by accepting what we cannot understand and believing what we cannot comprehend. In other words, becoming that little child who doesn't know any better, were it possible. Theory, it seems, will only take you so far ...

Coexistence and The Fundamental Theorem of Creation

Up until now, we have, for the most part, used *equivalency* to explain different ideas associated with the Fundamental Theorem of Creation, whether we were looking at *foundational* or *layered* components ... all of which are woven together into a unified whole through connectedness ...

```
Godhead ‹~~› membrane ‹~~› cosmos
                 |
                 |
       duality ‹~~› action ‹~~› laws
                 |
                 |
               Cross
```

For example, in the diagram above, there is *no* difference between the cosmos and the laws, mathematically speaking, where the former is defined to be the union of souls, realms and the physical ring. Thus, any boundary dividing the symmetries from the laws can *also* be thought of as a boundary dividing the Godhead from the cosmos, thereby making the notion of *at-one-ment*, in particular, much more palatable.

In reality, it is much more likely the foundational components, defined to be the union of symmetries, laws and the action, *coexist* with the layered components [cosmos], where each has its own *matter* space, according to the diagram below …

```
   ---------------------------------------------------------------------------------
   |                  |                  |            ◊
 souls   <~~>   physical ring   <~~>   realms         ◊     matter
   |                  |                  |            ◊
   ----------- α --------------------- α -------------------------------------------
   |                  | β                |            ◊
   |                  |                  |
Symmetries  <~~>    Action     <~~>    Laws           ◊    antimatter
   |                  |                  |
   |                β |                  |            ◊
   ---------------------------------------------------------------------------------
```

Under this scenario, all foundational components in the *antimatter* world are equivalent to one other, as we know, and a very *real* membrane β exists which divides the symmetries from the laws, in accordance with Noether. Similarly, all layered components in the *matter* world are equivalent to one another, and a very *real* membrane α exists, which divides this space from its *antimatter* counterpart.

But because everything is woven together into a unified whole through connectedness and equivalency, the α and β membranes are *also* equivalent to each other, and thus crossing over the α

boundary and into the Godhead is really no different than passing through the β boundary. At-one-ment, it seems, can actually be achieved *even under* coexistence.

As such, the Fundamental Theorem of Creation remains the best compact description of things, whether or not we choose coexistence as the preferred frame of reference. A fully consistent and complete prescription for the grand design itself, you might say, regardless of the vantage point ...

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
         |                   |                  |
         |                   |                  |
         |                   |                  |
         --------------------- Cross ---------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

Godel's Theorem and The Green Tao Theorem Revisited, Part II

In Part I we gave a rather heuristic argument for the equivalency of Godel and Green Tao. Here we are going to tighten things up a bit by examining processes within the Godhead [G] initially, and then moving out from there. So let's start by assuming G actually uses the prime numbers P to make positive decisions, which we'll label *yes*, and that all other decisions are made in the complement space, excluding *evens*, which we'll label ~P.

```
{2,  3,  5,  7,  11,  13,  ....}
 |   |   |   |   |    |
 y   y   y   y   y    y    ....
```

The entire set of *yeses* is indeed random, but not perfectly random according to Green Tao, and so contains regions of predictable regularity of *arbitrary* size, throughout. In other words, within P there are regular, contiguous intervals or *epochs* [if you will], of arbitrary size, at which the Godhead *always* responds by saying *yes*, even though it might not have wanted to.

```
  p -------------- p + k ------------- p + 2*k -------------- p + 3*k  ...
  |                |                   |                      |
  y                y                   y                      y
  |                |                   |                      |
prime            prime               prime                  prime     ...
```

Such a constraint infringes unfairly on the Godhead's right to choose freely now, thereby creating inconsistencies within this domain that, by definition, cannot exist. The Godhead, by default, is both fully consistent and complete according to the Fundamental Theorem of Creation, and so cannot admit *any* random system R containing predictable regions of arbitrary size, anywhere, it seems. Thus, we may conclude

```
   ~[Godel]     ≡    ~[Green Tao, R]
       |                   |
       |                   |
fully consistent     no regularity in
   & complete          randomness
```

within G, implying, in turn, the equivalency of these two theorems.

Within the current mosaic, then, we could describe the Godhead a little more accurately by saying it is probably made up of *perfectly uncertain* symmetries of Light, absent the familiar dimensions of time and space. In other words, a light-filled, perfectly random and perfectly symmetric geometry of infinite expanse, you might say, were you able to see this structure.

```
                    Light
                      |
         ---------------------------------
         |                               |
         |                               |
      Godhead         ‹~~›          Symmetries
         |                               |
         |                               |
         ---------------------------------
                      |
              perfectly uncertain
```

If we are correct here, the Godhead is *unknowable*, at least in the differentiable mosaic, even though earlier writings hint at the simultaneous experience of at-one-ment and atonement ... within this particular domain. It's a paradox which can't be reconciled easily, to be sure, but for the vast majority of souls living and operating beyond the membrane in the current mosaic, *knowing* God completely would probably seem like a strange idea; so strange, in fact, that it is doubtful they would ever take it seriously ...

On The Equivalency Between The Cross and The Membrane

The Fundamental Theorem of Creation tells us there is an equivalency between the membrane and the Cross, but leaves that more intimate relationship between the two open to interpretation. Here we offer an approach by selecting a *frame of reference,* and analyzing things accordingly --

```
                              membrane
                                 |
                                 |
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
         |                       |                      |
         |                       |                      |
         |                       |                      |
         -------------------- Cross --------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

Noether's Theorem tells us symmetries and laws can be unpacked from an *action* ... ... where the symmetries imply the laws and vice-versa. In the case of the Fundamental Theorem, it should be clear that if there were no action there would be no Cross, since the Cross itself is equivalent to the law of h&s. Thus the Cross necessarily implies the existence of an action. Conversely, the existence of an action necessarily implies the Cross [law of h&s], and so the Cross and the action are equivalent to one another.

```
Cross  ==>  Action  ==>  Cross
  |           |            |
  |           |            |
  ------- law of h&s --------
```

From a spiritual perspective the *action* really is the membrane that divides the Godhead and its symmetries from the laws of creation, but from a *physical* perspective this same *action* becomes the Cross. The two are equivalent to one another, according to the Fundamental Theorem, but now appear to be different structures depending on your frame of reference.

```
  membrane  ‹~~›  action  ‹~~›  Cross
      |                           |
spiritual frame            physical frame
```

Seen in this light, Christ's sufferings on the Cross become the *catalyst* that caused *all* of the encoded energy and information to come into being, and in the process, build and sustain all of Creation itself, according to God's simultaneous *now*.

```
energy ‹~~›  membrane ‹~~› action ‹~~› Cross ‹~~› sufferings
                 |                           |
          spiritual frame             physical frame
```

Such a conclusion seems reasonable, especially when you think of the *generalized* law of h&s, which tells us that every negative energy caused by an act of harm must be cancelled by a positive energy caused by an act of suffering [Cross]. From here, it is not much of a leap to conclude that this same suffering extends to *all* laws, symmetries, realms, souls, the physical ring and, of course, the membrane itself.

We can't be completely sure of our result, but it should be remembered that the Fundamental Theorem of Creation is not only both fully consistent and complete, but also optimal because of least action. As such, and by equivalency, the Cross has no replacement and so, is the *only* physical expression of suffering allowed by the *action* leading to laws.

Thus, the membrane uniquely implies the Cross [fully and completely] and vice-versa, and together these two pieces become the *same* action seen differently in two different frames of reference. What we saw 2000 years ago is simply what the dimensions of time and space let us see, but in reality, we were very likely witnesses to the unfolding and authenticity of creation itself, which concluded with the words of Christ 'It is finished'...

```
                 Spiritual Laws
                       |
                       |
creation ‹~~› Cross ‹~~› suffering
                       |
                       |
              Symmetries Godhead
```

Whether we look at symmetries in the Godhead, the action or the spiritual laws of creation, there seems to be a common thread running through each component; namely, one of *duality*. Symmetries correspond to the Father & Son, for example, while every physical law seems to have a spiritual counterpart. It shouldn't come as a surprise, then, that the *action* also has a dual nature, thereby bolstering our view that the *frame of reference* approach may be the correct way to see things after all ...

The Continuum Hypothesis, Godel and Green Tao

It struck me recently that *undecidability*, in all its many forms, could never be an attribute of the Godhead, and if so, this means we have an equivalency chain as shown in the diagram below --

~[Godel] < ≡ > ~[Green Tao, R] < ≡ > ~[Continuum Hypothesis] (†)

All three are, in fact, attributes of the Godhead, and thus we may conclude, more specifically --

[Green Tao, R] ≡ [Continuum Hypothesis] (*)

The Continuum Hypothesis tells us we can never decide if there is or isn't a class of numbers S sitting between the integers and the reals, given our current understanding of axiomatic set theory. Typically, mathematicians assign a cardinal number of **N** to the integers and **c** to the reals, and tell us we cannot know if there is a cardinal number **r** corresponding to S such that

$$\mathbf{N} < \mathbf{r} < \mathbf{c}$$

the integers	|	the set S	|	the reals
N		**r**		**c**
	|	*undecidable*	|	

According to (*), *undecidability* exists *because* of [Green Tao, R] and conversely. Thus, it is the *very* principle of regularity within randomness, along the number line, that leads to the notion of undecidable theorems, and similarly, undecidable theorems over the reals imply embedded regularity within randomness, literally everywhere. They are equivalent ideas, if you will ...

undecidable theorem T ‹~~~› regularity within random set R

.

.

.

This kind of correspondence has to make one wonder just how large the set U of undecidable theorems really is, if it can even be ascertained. For example, if card(U) ≈ **c** there would be no way to order the elements of this set, and we would be forced to conclude, necessarily, that an *uncountable* number of theorems over the reals are, in fact, *inconclusive*.

But regardless, the main thrust of this particular note is to see the connections in (†), and realize the Godhead contains many attributes, all of which are equivalent to one another by virtue of symmetry. By taking their complements and operating beyond the membrane, as it were, we can, from time to time, learn more about the world we live in and perhaps see things in this space from a fresh perspective. The relationship depicted in (*) is but one example ...

On The Equivalency of Attributes in The Godhead

This could well be our swan song, at least in so far as contributions to the *Essay* folder are concerned. It's been a long road -- these last five years -- with many wonderful concepts and ideas flowing from a simple thought (regarding Noether) that struck my mind back in the summer of 2009. Now, 45 articles later, I'd like to summarize the various attributes in the Godhead I've become aware of, and see if there isn't some way to tie it all together. So let's start with the following equivalency diagram, which depicts these properties as a simple chain --

```
~[Godel]      < ≡ >     ~[Green Tao, R]  < ≡ >  ~[Continuum Hypothesis]
    |                         |                          |
    |                         |                          |
    |                         |                          |
~[Time]       < ≡ >        ~[Space]      < ≡ >       ~[Light]
```

Following a *two-state* model (coexistence) where the Godhead [G] is actually part of *antimatter* space, we can see here that G is a pretty strange locale, wholly comprised of symmetries with the following characteristics --

- fully consistent and complete
- perfectly uncertain
- no undecidability
- no familiar dimensions of time
- no familiar dimensions of space
- anti-photonic light *de auro purissimo*

The list above is certainly not exhaustive, but does give us some feel for what it might be like to step into this rather exotic region, were we able to pass through the *α-β* boundaries discussed in earlier writings, and shown below --

```
  ---------------------------------------------------------------------------------
  |                    |                  |           ◊
souls   <~~>   physical ring   <~~>   realms          ◊     matter
  |                    |                  |           ◊
  ----------- α --------------------- α -------------------------------------------
  |                    | β                |           ◊
  |                    |                  |
Symmetries  <~~>    Action     <~~>    Laws           ◊   antimatter
  |                    |                  |
  |                  β |                  |           ◊
  ---------------------------------------------------------------------------------
```

Curiously, the photon and anti-photon, in particular, are *one* and the *same* thing, according to physics, and thus we can say, with some assurance, that light *de auro purissimo* [of the purest form] is not only an attribute of the Godhead; but also, the *action* and the *laws*, because of equivalency.

In other words, the *foundational* components of Creation are, in all likelihood, made from this very *same* substance -- *liquid gold light*, if you will -- a notion which ties in nicely with current research linked to the Mereon Matrix, for example.

Beyond the Godhead, and outside the bounds of at-one-ment [in the *matter* world], we are largely encumbered by the inverse of any attributes associated with G, and so can never know with any precision how the laws function, for example, or how the Godhead thinks and communicates with its constituent parts [symmetries]. Indeed, our view of reality is constrained *in part* by things like incompleteness, *imperfect* uncertainty, undecidability, time, space and even light itself ... to the degree we understand these *equivalent* impressions.

When I see an equation like

$$E = mc^2$$

am I really any closer to understanding the spiritual laws of creation, or are these opaque transcriptions merely dim, shadowy reflections of the pristine structures that exist behind the α boundary in this strange, if not bizarre world of the Godhead ?

It's difficult to say, but one thing is certain and will always remain -- until and unless we find at-one-ment with the Creator, we will never unravel the many mysteries which have been laid before us ... and so will have to go on being content with discontent itself ... Creation, it seems, would have it no other way ...

A New Heaven and a New Earth, Part II

In Part I, and according to the Law of Purification of the Void, we noted that every soul will eventually return to the Light of God, and attain a parallel state of complete at-one-ment and atonement ... ... relative to the simultaneous now. As well, we saw that Creation would eventually morph into a new kind of structure where all rings beyond the Godhead will have merged with the Light, thereby creating a more homogeneous framework which we'll call the *unified* mosaic.

Primordial Fabric ~~› Differentiable Mosaic ~~› Unified Mosaic

In the *simultaneous now* this transition has already occurred, but in the *ordered now* [our reality] things are unfolding just as they should, leaving us with the question as to what exactly the unified mosaic will really look like when it's all said and done.

```
                  souls    <~~>    physical ring   <~~>   realms
                    |                    |                    |
                    |                    |                    |
      Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
                    |                    |                    |
                   life                 way                 truth
                    |                    |                    |
                     -------------------- Cross --------------------
                                         |
                           consistent, complete, optimal
```

> 'In truth I tell you, till heaven and earth disappear, not one dot,
> not one little stroke, is to disappear from the Law until all its
> purpose is achieved ...' [Matthew, 5th chapter]

When reading from the Book of Matthew above, the implication seems to be that the differentiable mosaic and its laws are going to undergo a change of sorts [or disappear altogether], leaving us with a unified structure that has actually merged with the Light. Indeed, in this new state the law of h&s won't even be needed, for example, and the axiom of free choice, by default, will be fully consistent and complete because of at-one-ment.

In fact, by losing the law of h&s, either Creation collapses, which is no longer possible, or the *action* extends out to infinity where the laws as a whole become marginalized, so to speak, and [mathematically] tend to zero in the limit. As such, the most likely scenario is one in which the Godhead literally *expands* into the current mosaic, everywhere, thereby creating a symmetrical union with all that exists here today -- a new heaven [realms] and a new earth [physical ring], as it were, where the spiritual laws of creation have been completely subsumed by the Light [life] itself.

```
souls    <~~>    physical ring   <~~>   realms
  |                    |                   |
  |                    |                   |
 ---------- Symmetries Godhead -----------
                       |
                       |
                     Light
```

`For look, I am going to create new heavens and a new earth, and the past will not be remembered and will come no more to mind` [Isaiah 65:17]

If the passage above from Isaiah is correct, we can look forward to a day when there no longer is a boundary separating the Godhead from its creation. Gone will be the days of suffering and hardship, the void and any other traces of darkness altogether. In those days, however far off, the Light [life] will become the Law, and the Cross -- the very root of Creation itself, according to the Fundamental Theorem -- will vanish.

In this sense the Cross [law of h&s] should be seen as a temporary structure of sorts, used initially to bring Creation into being, and then discarded once the process of purifying the existing mosaic has been completed. No doubt the wounds of Christ will also vanish once the law has served its purpose, and so they should, for by any norm or standard the long-suffering nature of God has already exceeded any reasonable bound or limit, no matter the measure.

In fact, if the symmetries [in the Godhead] disappear as the *action* tends to infinity -- something we expect -- there will be no Father and Son in the unified mosaic -- only the pure Light of God everywhere holding everything together. As such, this familiar duality could also be seen as a temporary arrangement, used initially to uphold the spiritual laws of creation within the current mosaic, and then discarded once these laws are no longer needed. An illusive means to an end, you might say, all made possible through the magic of Noether ...

```
Primordial Fabric  ~~>  Differentiable Mosaic  ~~>  Unified Mosaic
              |              |                 |
              |              |                 |
            death          ~~~>           resurrection
              |              |                 |
              |              |                 |
            Cross          ~~~>              Life
                             |
                             |
                         law of h&s
```

And Jesus said, 'I am the resurrection and the life ....' [John, 11th chapter]
'All shall bend the knee to me, by me every tongue shall swear ...' [Isaiah, 45th chapter]

A Brief Summary of The Various Fabrics in Creation

In the beginning, before there was anything else, there was the *primordial* fabric, containing the undifferentiated energy of God, everywhere. Previous studies tell us this particular manifold M was probably equivalent to the n-sphere, and endowed with the attribute of *perfect certainty* ...

```
M ~ S(n)   < ≡ >   Perfect Certainty
              |
              |
           ζ-space
```

To understand this attribute more fully, think of travelling through number line, only to find that no matter what point you land on, it is always {0}. Such would also be the case for the primordial sphere; a uniformly distributed energy field which looked the same no matter where you were on this manifold.

At some point, though, a shift took place, with the Cross acting as a *catalyst* of sorts, causing symmetries and laws to come into being, according to Noether and the Fundamental Theorem of Creation. We call this the *differentiable* or *current* mosaic, and adopting a two-state model of *coexistence*, can see some of the detail in the following diagram --

```
      ---------------------------------------------------------------------------------
      |                    |                  |            ◊
    souls    <~~>    physical ring   <~~>   realms         ◊      matter
      |                    |                  |            ◊
      ----------- α --------------------- α -------------------------------------------
      |                    | β                |            ◊
      |                    |                  |
 Symmetries   <~~>       Action     <~~>    Laws           ◊    antimatter
      |                    |                  |
      |                  β |                  |            ◊
      ---------------------------------------------------------------------------------
```

In the β-world all things are *perfectly uncertain*, for example, whilst in the α-world they are *imperfectly uncertain*. The two attributes are, in fact, complementary and, among other things, determine the behavior of the laws in these respective domains.

Someday, according to the Fundamental Theorem of Creation, a *unified mosaic* will emerge, where the α-β boundaries have disappeared altogether, along with the action and the laws, and the symmetries will recombine into a single essence [Light of God] which permeates a *purified* α-space, literally everywhere.

```
                      γ-space
                         |
souls    <~~>    physical ring   <~~>   realms
  |                      |                  |
  |                      |                  |
  ---------- Symmetries Godhead -----------
                         |
                         |
                       Light
```

Just what attributes this new mosaic will be endowed with is unclear to me, but to be sure, all living things will have a clear sense of *at-one-ment* with both the Light *and* each other in those days, that might best be described as a *commingling* of sorts ...

******

In a nutshell, this is the general sequence of events which depicts the unfolding of Creation, according to our studies. Nothing is cast in stone, and indeed, in α-space we have at best an incomplete picture of things. Still, if the Fundamental Theorem holds true, it is probably the best description we are ever going to get, whether we look to the past or the future for a better understanding of how it all came to be ...

******

Undecidability and The Fundamental Theorem of Creation

Up until now, we have seen the Fundamental Theorem of Creation largely as a prescription for the grand design itself -- a prescription which is both fully consistent and complete according to our studies ...

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
          |                    |                    |
          |                    |                    |
          |                    |                    |
          --------------------- Cross ---------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

However the Cross actually resides in α-space, if we adopt the two-state model of coexistence, and so we are forced to conclude, necessarily, that *imperfect uncertainty* $[\sim\mu]$ must be one of its attributes. Thus, both $\sim\mu$ and its complement μ are consistent with the Cross simultaneously, implying *a fortiori* that up to attribute, anyway, the Cross must be *indeterminate.*

By equivalency, then, the same is true of the Fundamental Theorem [F] -- the set of attributes A in β-space, and their complements ~A in α-space, are consistent with F simultaneously, and thus F itself is also *inconclusive.*

$$\sim A\,,\ A \quad <\equiv> \quad F \qquad (*)$$

In particular, and as a result, the spiritual laws of creation, embedded in the *matter* world, would appear to be incomplete or inconsistent, if you will, showing through more as shadows or weak impressions, which come to us from the pristine structures that exist behind the α-boundary in the rather bizarre *antimatter* world of the Godhead.

Heisenberg's *Uncertainty Principle* is a shining example of this effect, for it tells us we can never know exactly the position and momentum of a sub-atomic particle at the same time. The more we know about one the less we know about the other. Similarly, solutions to Einstein's field equations of *general relativity* will always have a singularity, according to the work of Hawking and Penrose, and the axiom of free choice, both in mathematics and in our daily lives, leads to many inconsistencies and conflicts.

In the same vein, we know from previous research *at-one-ment* and *atonement* are equivalent notions, and that both are equivalent to the Cross, and therefore F. Hence the relationship in (*) must be true for both, and so *at-one-ment* in particular, embedded in the *matter* world, would also appear to be incomplete or inconsistent, just like the spiritual laws of creation.

Forming a complete union with Godhead, it seems, is only possible in the *antimatter* world, assuming it's possible at all ...

As a final example, consider the law of h&s. It too, embedded in the *matter* world, would appear to be incomplete or inconsistent, but in reality, is simply mirroring our own actions determined by *choice*. A feedback loop, if you will, which was mentioned in the original essay --

thought ~~> action ~~> deed ~~> consequence

If (*) really is true can we ever talk sensibly about unified theories in physics, at least in α-space ? Probably not, for in our domain there is just enough *regularity* running through the randomness to create the kind of uncertainty that is needed to prevent this from ever happening. Unlike the β-world of the Godhead, where all things are *perfectly uncertain*, such is not the case for us, and so any attempt to unify laws or offer a fully consistent and complete description of Creation ... will ultimately fail ... necessarily so ...

regularity in randomness < ≡ > imperfect uncertainty

.

.

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On The Nature of Coexistence and Invertible Transforms

For a while now, I have been thinking a little more deeply about the two-state model of coexistence, and how the α-space, in particular, came to be ...

```
 ---------------------------------------------------------------------------------
 |                       |                    |            ◊
souls   <~~>   physical ring  <~~>  realms          ◊      matter
 |                       |                    |            ◊
 ----------- α --------------------- α --------------------------------------------
 |                       | β                  |            ◊
 |                       |                    |
Symmetries  <~~>   Action   <~~>   Laws             ◊    antimatter
 |                       |                    |
 |                     β |                    |            ◊
 ---------------------------------------------------------------------------------
```

In order for the matter world to exist, and inherit the properties that it has, some *mapping* or *transform* [Λ] would have been used, in all likelihood, to create the α-instance from β and endow it with an attribute such as *imperfect* uncertainty...

```
                          Λ
        β-world       ------------›       α-world
           |                                 |
           |                                 |
                      ‹------------
                          Λ′
           |                                 |
   perfectly  uncertain            imperfectly uncertain
```

Thus, we can say with some assurance *coexistence* necessarily implies Λ. On the other hand, if we assume *invertibility*, which is not unreasonable, then a transform Λ′ must also exist that does the opposite, taking us from the α-world back to β. In other words, the *very* fact that Λ′ *exists* implies *a priori* the existence of α, and so we have the following equivalency --

'For some invertible transform Λ, coexistence implies Λ,
and conversely its inverse Λ′ implies coexistence'

We can't know much, if anything, about Λ directly, since here we are operating over a perfectly uncertain space β, putting us in no man's land. On the other hand, α and its attributes are something we are familiar with, giving us some hope the search for an invertible transform Λ′ is not totally in vain ... assuming coexistence ....

If Λ' can ever be found, we may be able to recast problems in a different light, more the way they might be viewed in β-space, for example, were it at all possible. Intractability may yield to solvability, in ways we've never seen before, and mysteries along the number line might vanish altogether. Indeed, finding Λ' could take us one step closer to finding out who or what God really is ...

$$\text{coexistence} \quad <\equiv> \quad [\ \Lambda\ ,\ \Lambda'\ ]$$

On The Nature of Coexistence, Spin and Vibration

In the essay, which I went back and read recently, we spoke about the notion of *resonance*, and suggested that realms within the current mosaic possess this attribute in varying degrees. Here we want to offer a little more detail on how the entire process might work ...

If I stand by the edge of a pond on a still day and throw a stone into the water, waves will be created that move with a certain *frequency*. Throwing more stones into the water at the same time generates much more vibration, and tossing an *infinite* number of stones into the pond, simultaneously, will create so much *randomness* that there will be no *regularity* left to speak of.

infinite number of stones ---› ---› perfectly random

This is probably what β-space is like -- perfectly random structures made from liquid gold light that seem *frothy* or *foamy* once all the [infinitely many] superpositions have been added together.

As you move out from the β-world and into the α-world, via the Λ transform, say, these vibrations will begin to diminish, causing *more* regularity within the randomness as you go down through the realms, like rungs on a ladder. We are throwing fewer and fewer stones into the pond at the same time, so to speak, until finally things become *still*, and there are *no* vibrations at all. This endpoint is the *ground* state, also referred to as the *primordial* sphere, or ζ-space.

α-world

| ↑ spin-states ‹---› ↓ regularity within randomness

β-world ---› Λ ---›

| ↓ spin-states ‹---› ↑ regularity within randomness

Strangely, ζ-space has *no* spin and may have no attributes at all, other than *perfect certainty*, and so is very likely a uniformly distributed *dark* energy field which is constant throughout. A curious structure indeed, from whence the differential mosaic may have been built by mapping an *infinite* number of spin-states to β-space initially, and in the same breath, a *finite* number of spin-states to each of the α-subspaces that reside in the *matter* world.

$$\Psi : \zeta \dashrightarrow \beta \;\; \cap \;\; \Omega : \zeta \dashrightarrow \alpha$$

or

$$\Psi : \zeta \dashrightarrow \beta \;\; \cap \;\; \Lambda : \beta \dashrightarrow \alpha$$

Notice in the diagram above the first pair of transforms taking us from ζ to β and α, respectively, are *not* the same thing. In fact, the latter [Ω] is a *convolution*, namely $\Lambda \circ \Psi$, and thus we may conclude that *antimatter* space [β] really was built using mathematical principles separate and apart from the *matter* space [α], even though both have a common origin in ζ.

As an alternative, one could use the second pair of transforms above, and produce the same result. Either path is acceptable, mathematically speaking, and so without loss of generality we will focus exclusively on the Λ transform, going forward. It is, after all, the more interesting object to study.

To create a subspace within the α-domain, Λ must operate over an infinite number of spin-states, or superpositions in β-space, and generate from this an α-realm, with a particular level of *vibration* and *regularity*. In the extreme case, where $\Lambda : \beta \dashrightarrow \zeta$, this is easily done by mapping to {0,1}, and on the other end, where $\Lambda : \beta \dashrightarrow \beta$, by mapping to {∞,0}.

Somewhere in between these extremes is the α-subspace we are looking for; one in which the vibration [ν] and measure of regularity [ρ] within the randomness reinforce each other --

$$\underset{\Sigma\,\nu}{\Lambda : \beta} \;\dashrightarrow\; \alpha\,(\nu, \rho)\,, \quad \Sigma\,\nu < \infty\,, \;\; 0 < \rho < 1$$

Can such a thing be simulated in the real world we live in ? Can we find a transform Λ which takes us from an infinite number of spin-states over β to a finite number of spin-states over α, thereby creating an α-subspace with a certain amount of regularity ρ running through the randomness ?

It's hard to tell, but if it can be done, we would be nicely on our way to understanding more fully how *coexistence* came into being, and might even be able to say something about the inverse transform Λ′ as well. Indeed, knowing more about Λ′ could open up a whole new world of discovery for us, by shifting our frame of reference to a simulated β-space, for example, and solving problems there -- problems which, up until now, would seem intractable by any recent standard or measure ...

On The Undecidability of Attributes in The Unified Mosaic

For a while now, I've been wondering more and more what the *unified* mosaic [*γ-space*] is really going to look like. It's odd enough to me, anyway, that such a mosaic is inevitable, but supposing it to be true, can we say anything about the corresponding attributes ?

```
                         γ-space
                            |
souls    <~~>    physical ring   <~~>   realms
  |                         |                      |
  |                         |                      |
  ---------- Symmetries Godhead -----------
                            |
                            |
                          Light
```

The unified mosaic is a peculiar *marriage* between the α and β spaces, where the α-β boundaries have been broken, so-to-speak, and the two domains have *merged* according to the diagram above. As such, one could say the Fundamental Theorem of Creation [F] no longer applies here, and thus ~F is all that remains. If so, then one might also argue

$$\sim A \, , \, A \quad <\equiv> \quad \sim F \quad (*)$$

where A is the set of attributes in β-space and ~A are the complements in α-space.

In other words, since F is already *indeterminate* [up to attribute], so is its complement ~F, ergo (*) must hold, logically speaking. A very peculiar result, to say the least, but if it's true we can't say *anything* about the attributes of *γ-space* -- the matter is *undecidable* in mathematical parlance.

On the one hand, *γ-space* is the complete union between God and Creation; a blending together of *all* that is, if you will. On the other hand, all living things still need to retain their own individuality, make choices, learn, develop, discover, and so on, no matter what space they finally dwell in. The endpoint cannot be a utopia *devoid* of the qualities and characteristics that make life worth living, for if such were the case, I suppose a reversion to ζ-space [the primordial fabric] might be our only alternative.

```
M ~ S(n)     <≡>    Primordial Fabric
                |
                |
             ζ-space
```

But who wants to live on a spinless, stateless sphere coated in a uniformly distributed *dark* energy field throughout, which is probably freezing cold !! Even God decided at some point he'd had enough, and alas ... a meaningful creation was born according to the words ... 'Let there be light ...'

How Large Is The Physical Ring

We know from inflation theory that any universe initially expanded very rapidly in the first $1 / 10^{36}$ seconds, roughly, and that Christ spent about 6 hours on the cross.

According to the essay, during this time on the cross, the physical ring was lit up with an almost infinite number of big bangs, and if these occurred in rapid-fire succession, we can estimate just how many universes would have been created:

$$N \sim [\, 2 \text{ x } 10^{4} * 10^{36} \,] / 2 \sim 10^{40} \quad \text{universes}$$

In this calculation, inflationary periods could be longer than $1 / 10^{36}$ seconds, and so we divide by 2 to get some sort of average.

Recent studies seem to indicate our own universe is somewhere around 10^{11} light-years in diameter, and so comprises a volume of roughly:

$$U \sim 1/2 * (D^{3}) \sim [10^{33}] / 2 \qquad \text{cubic light-years}$$

Allowing for both future expansion and for a certain amount of distance or sparseness in the physical plane, we see to the nearest *exponential* order of magnitude, the volume of the physical ring can be estimated at:

$$V \sim [\, 10^{40} \ * \ 10^{33} \,] / 2 \approx 10^{100} \ \text{cubic light-years}$$

If now, we take *all* souls ever made and move them to the physical plane, we see that a reasonable measure for the number of souls, to the nearest *exponential* order of magnitude, is *also*

$$\text{est number of souls} \approx 10^{100},$$

for the lower bound of 10^{10} would lead to too much sparseness, and the higher estimate of 10^{1000} to too much overcrowding.

****

These calculations are *very* rough, but if they are in the right ballpark, there are about 10^{100} brothers and sisters throughout creation -- in higher realms, lower realms and the physical ring. Of this number, some 10^{10} are here on earth -- a virtually insignificant percentage. Indeed, the number of souls on the earth is one full *exponential* order of magnitude *below* the total number of souls ever made.

Recent studies also show there are roughly 6 x 10^{22} stars in our universe, out of which one in six contains an earth-sized orb that may be suitable for life. If so, about 10^{22} stars in our universe meet this criterion, which means some 10^{62} stars might be found in the physical ring that have earth-like planets. And, if we assume around 10^{10} inhabitants on each planet, this would imply about 10^{72} beings in the physical plane, all told, but still well below the very rough estimate of $\sim 10^{100}$ souls ever created.

****

One reasonable conclusion to draw from all of this is that most souls have *never* incarnated, even once. Indeed, the number of souls that have descended into the physical plane since its inception is virtually insignificant, with the ratio being somewhere around

$$\text{est number of incarnations} \;\approx\; 1 / 10^{30}$$

And we can also infer most darker entities (about one-third in all) have never incarnated --- they simply moved into darker realms from the outset.

Such a massive transition, involving both souls and angelic beings, was probably brought on by a large-scale rebellion in the higher realms initially, for it is doubtful these beings would have gone willingly. And so, the transition was more likely an *expulsion* than anything else, giving rise to the notion of 'fall from grace', for which there is some biblical evidence.

The fate of darker angelic beings caught up in this rather unfortunate tragedy is unclear to me --- beings in this category are not considered in the essay .. and are not seen as a fundamental component of creation. Souls, on the other hand, are a fundamental component, and could be protected by the Law of Purification of the Void under any condition. But regardless, darker entities as a whole seem determined to damage those parts of creation that are still available to them, including the physical ring, where many innocent, well-intentioned souls live and operate according to the various cycles of life.

$$\text{est number of beings in physical plane} \;\approx\; 10^{72}$$

So how long will it take God to purify the void ? According to the essay, God has no choice but to purify the void, and other research suggests about one in every three souls has been lost to the darker regions already. If we make the reasonable assumption that purification starts with lessons learned in the physical ring, some

$$10^{100} / 3$$

souls need to descend across ~ 10^{62} planets in parallel.

Yet even if we allow for one descent every second, or thereabouts, it's going to take approximately 10^{38} / 3 seconds to complete the task, assuming one incarnation does the trick. However, for a darker soul one descent is usually not enough, and so a fairer estimate might be three passes, in which case 10^{38} seconds is the approximate [minimum] time needed to purify the void. In earth years, this number is approximately

$$\text{time to purify the void est} \;\approx\; 3 \times 10^{30} \;\; \text{yrs,}$$

an almost incomprehensible span no matter the measure ...

I say minimum time needed to purify the void because shells can ultimately be occupied by higher or lower-ordered beings, and so there will be competition for these bodies which is ultimately decided by God. On the one hand, higher-ordered beings have every right to make a descent and learn the lessons they've mapped out for themselves, but on the other, the void also needs to be purified. Creation, it would seem, has some very serious challenges at this juncture.

******

If nothing else, these calculations tell us just how vast creation really is, how magnificent the Creator really is, and how critically important Christ's role in creation really is. Indeed, without Christ, and according to the essay, you don't have a creation -- only the primordial fabric that existed before it was all brought into being ...

Is Heaven At Risk Of Becoming A Ghost Town

In our last addendum, we addressed the issue, to the degree that we can, of just how large the physical ring might be. One of the interesting consequences of this note was that a massive expulsion in the higher realms likely occurred early on in creation, brought on, in turn, by a rebellion or failed coup attempt that resulted in the fall of about one-third of *all* angelic beings and souls.

Truly an incomprehensible event when, to the nearest *exponential* order of magnitude, you realize some 10^{100} such beings may have been created originally.

We also know, from the essay, that incarnations can be either progressive or regressive, and may not necessarily return a soul back to the safety of the higher realms upon death. Indeed, a full third of all such descents may be regressive, in which case souls run the risk of winding up in some region of the void -- at least among those who choose to incarnate.

The total number of incarnations since the beginning of creation is very small indeed; in fact, virtually insignificant with an estimate of only one soul in ~ 10^{30} that has ever made a descent into the physical ring. Even so, as the process continues more and more souls are being lost to the void, and at some point, unless things are reversed, a *critical mass* will be reached where more than half of all souls ever created can no longer call the higher realms their home.

Assuming at each iterative cycle one-third don't actually make it back [where a cycle is defined as the length of time it takes for *all* higher-ordered souls to have incarnated at least once before starting anew], we can see that after n such cycles you have, fractionally,

(2/3) ^ n

souls left in the upper realms, and

(1/3) * [1 + (2/3) + (2/3)^2 + ... + (2/3) ^ (n - 1)]

souls caught up in the void.

Initially, with n = 0, and before the expulsion, all beings would have been in the higher realms, but this changes to (2/3, 1/3) at n = 1, and to (4/9, 5/9) at n = 2, and so on.

God has a two-fold problem here; namely (a) how to extricate souls from the void regardless of what put them there in the first place and (b) how to slow down or even reverse the flow of souls into the void because of failed incarnations.

It's a massive undertaking for which a mere mortal such as myself has no useful solution. Indeed, about all I can do is appreciate the complexities here and, in turn, do my best to walk with God on a daily basis. Perhaps when I get back to the higher realms [assuming I do], there will be more of an opportunity to make a contribution ...

Is Heaven At Risk Of Becoming A Ghost Town, Part II

In previous research connected to this topic .. .. we learned that it was going to take approximately

$$3 \times 10^{30} \text{ yrs}$$

to purify the void, if we assumed 10^{100} souls throughout creation, and didn't factor in *any* incarnations originating in the higher realms, which almost surely will lead to additional failure.

Here we are going to construct a simple flow model which allows for incarnations originating in *both* the higher and lower realms simultaneously, where available shells are *evenly* apportioned, and the axiom of free *choice* functions in a normally distributed fashion.

axiom of free choice ~ [2/3, 1/3]

Under these assumptions, a higher-order being is expected to succeed about two-thirds of the time, by descending into the physical ring, and a lower-ordered being about one-third of the time. Since two-thirds of *all* souls still dwell in realms *above* the physical plane, according to our recent studies, the expected *flow rate* Θ can be calculated as follows, with $\lambda = 1/2$ ...

$$\Theta \approx \lambda * [\, 2/3 - 1/3 \,] + (1 - \lambda) * [\, 1/3 - 2/3 \,] = 0 \qquad (*)$$

In other words, the expected flow rate is *zero*, which is bad, because it means that over the eons and eons of what I call *cosmic* time, on balance the void will *not* be purified ... regardless of where in α-space the incarnation originated. Dividing the shells evenly is a wash, so to speak and *positive* values of Θ imply an *expanding* void, strangely enough.

*****

Also, Θ will change if available shells aren't divided equally -- by weighting them 2:1 in favor of higher-ordered descents, for example, the expression above becomes 1/9, and this can only make one wonder what the deeper meaning behind reincarnation truly is. Could it be that the Creator, all along, intended to use the physical plane as the primary mechanism for purifying the void, according to some bias or preference, by asking many souls in higher rings to eventually play a part, even if there was some risk ?

It's hard to tell, but (*) is a rather sensitive calculation, which relies heavily on *some* participation in the higher realms to begin with, otherwise the flow rate could easily vanish or turn positive. To see this more clearly note that (*) can be rewritten as

$$\Theta \approx [\, 2\lambda - 1 \,] / 3$$

where λ is the proportion of shells assigned to higher-ordered incarnations. In order to have a *negative* flow rate, we must always have $\lambda < 1/2$, and thus it becomes a balancing act of sorts. If, for example, *two-thirds* of all potential incarnations originated in higher realms, and the other *third* in lower realms, and shells were *evenly* apportioned, λ would be 1/2, and Θ would be zero --

the void would never be purified in this case, irrespective of how many higher-ordered beings wanted to descend into the physical plane at any point in time ...

```
o  o  o  o  ᴥ  ᴥ     o  o  o  o  ᴥ  ᴥ     o  o  o  o  ᴥ  ᴥ       ...
☺           ☺           ☺        ☺              ☺        ☺       ...
```

It seems learning *harm* may be fundamental to our progress after all .... no matter where we are in the virtual space of creation, for without this knowledge there are no viable gains to speak of, and of course, those in the void would continue to languish ...

*****

It is now an easy matter to calculate the time needed to purify creation, using our previous knowledge of things, and assuming *many* beings eventually become part of the reincarnation cycle. Applying a 2:1 weighting in favor of *lower*-ordered descents, which seems to be the more 'natural' choice, | Θ | becomes 1/9 and so --

$$\text{time to purify the void est} \approx [\,(10^{100})\,/3\,]\,/\,[\,\Theta * 10^{62}\,]$$

$$\approx 9 \times 10^{30} \text{ yrs}$$

```
higher  ‹-------------- | -------›  lower
              souls     |     souls
               2/3             1/3
```

```
o  o  o  o  ᴥ  ᴥ     o  o  o  o  ᴥ  ᴥ     o  o  o  o  ᴥ  ᴥ      ...
   ☺        ☺                    ☺  ☺           ☺        ☺      ...
```

This number is in good agreement with the original calculation, where we omitted higher-ordered incarnations altogether .... Either way, assuming a second-order exponential estimate for the total number of souls ever made, we're looking at an incomprehensible span of time, no matter the measure. But perhaps this too is good, for if God really wants to make the unified mosaic a risk-free reality, it's probably better to be safe than sorry, by waiting for as long as it takes to make things right again ...

```
                      γ-space
                         |
souls    <~~>    physical ring    <~~>   realms
  |                      |                  |
  |                      |                  |
  ---------- Symmetries Godhead -----------
                         |
                         |
                       Light
```

On The Nature of Coexistence, Spin and Vibration, Part II

In Part I, we discussed spin-states inside the α-β context, and saw that higher-vibrational realms contained *less* regularity within randomness, while the *opposite* was true for lower-vibrational worlds. Let us begin by revisiting the diagram which depicts this relationship in some detail --

α-world

| ↑ spin-states ν ‹---› ↓ regularity ρ within randomness

β-world ---› Λ ---›

| ↓ spin-states ν ‹---› ↑ regularity ρ within randomness

In the *ground* state, or ζ-space, we are on the primordial sphere, so to speak, and here we said this was very likely a uniformly distributed *dark* energy field, which is constant throughout. In other words, a *perfectly* certain structure, if you will, as show in the diagram below ...

M ~ S(n)	< ≡ >	Primordial Fabric	< ≡ >	perfectly certain
|		|		|
|		|		|
ζ-space	< ≡ >	Dark Energy	< ≡ >	no spin-states

Looking at these illustrations, it only makes sense to associate ρ with dark energy, in varying amounts, which *diminishes* as you move *up* into the higher realms, for example, and finally in β-space, *vanishes* altogether.

dark energy ‹~~~› regularity within randomness
ξ ρ ν

One conclusion to be drawn from this is that anti-photonic light, in the β-world, is of the *purest* form possible because there is *no* dark energy here; apparently, it only exists as an 'attribute' in α-space where light, at least to some degree, appears to be degraded by its presence.

As such, and in passing, we shouldn't think of dark energy as an attribute at all, because it really doesn't have a legitimate complement in β-space. In other words, in order to make sense of the expression

[~A , A] < ≡ > [~F , F]

regardless of the fabric, any attribute *and* its complement *must* have *meaning* in some sense of the word, otherwise it's rather like taking the dual of {0}, which yields nothing.

While it is true that the *familiar* dimensions of *time* and *space* don't exist in β, which might have some bearing on the quality and purity of light here, it is probably more correct to say their *complements*, along with the *absence* of dark energy, contribute synergistically to light in β-space, just as any corresponding properties or attributes would in α-space ...

```
~[Time]    < ≡ >    ~[Space]    < ≡ >    ~[Light]  .....  β-space

                          or

 [Time]    < ≡ >     [Space]    < ≡ >     [Light]  .....  α-space
             |                    |
             |                    |
             ----  dark energy  ----
```

In our domain, for example, scientists tell us about two-thirds of the universe is indeed dark energy. If this is so, a reasonable estimate for ρ within the physical ring might also be ~ 2/3, which would make the physical plane, in turn, predominantly *regular* and thus of relatively *low* vibration. Most α-realms, it seems, are of a much *higher* resonance, while roughly one-third -- the so-called darker regions -- are of a much *lower* resonance, by comparison.

This shouldn't come as a surprise to us now, because smoothed out over long periods, even the axiom of *free choice* is going to lead to unfortunate consequences about one-third of the time, assuming things are normally distributed.

axiom of free choice ~ [2/3, 1/3]

In a peculiar way, then, the distribution of light and dark realms within α-space could be seen as a reflection of our own thoughts and actions; that is to say, our *choices*. Somewhere along the line, the Creator knew this -- most likely in the primordial fabric, before *anything* was built, but *everything* considered -- and fashioned a design accordingly. It appears there are no coincidences after all, for even when looking at the large-scale structure of realms and rings throughout creation, a distribution emerges which seems to be wholly compatible with our evolution as souls, no matter where we are, beyond the membrane ...

On The Nature of Coexistence, Spin and Vibration, Part III

In this particular note we are going to study, in a little more detail, the attributes of *time* and *space*, to the degree that we can, as we move through the various fabrics in creation. So let's start with the following diagram, familiar to us now from previous studies --

$$
\begin{array}{lll}
 & & \qquad\qquad \alpha\text{-world} \\[1em]
 & & |\ \ \uparrow\ \text{spin-states } \nu \ \ \langle\text{---}\rangle\ \ \downarrow\ \text{regularity } \rho \text{ within randomness} \\
\beta\text{-world}\ \ \text{---}\rangle\ \ \Lambda\ \ \text{---}\rangle & & \\
 & & |\ \ \downarrow\ \text{spin-states } \nu \ \ \langle\text{---}\rangle\ \ \uparrow\ \text{regularity } \rho \text{ within randomness}
\end{array}
$$

As you descend the various α-realms, dark energy increases, light diminishes and time & space 'intensify', in so much as any experiencer will *feel* these two attributes more acutely with each downward step, like rungs on a ladder. Eventually, in the *ground* state, or ζ-space, we are on the primordial sphere, and here the 'intensity' becomes infinite, in that the 'distance' between *any* two points on this manifold becomes {0}, whether we are measuring clock ticks, proper length, or even dark energy itself. There is no separation of variables in ζ-space, so to speak, irrespective of your position, because this is a perfectly *certain* structure throughout.

$$\lim\ \text{dist}(p, q)\ \text{---}\rangle\ 0\ \text{ as }\ \nu\ \text{---}\rangle\ 0 \qquad \forall\ \alpha\ \ (\text{for all } \alpha)$$

In β-space, just the opposite is true. Dark energy vanishes altogether, because this is a perfectly *random* structure; light is of the purest form possible and the 'distance' between any two points is now infinite, whether we are looking at clock ticks or proper length, say. A second may as well be a minute or an hour or a billion years, and similarly, an inch may as well be a foot or a mile or trillions of miles. In an instant, you are able to grasp any fragment of creation or the whole of creation, simultaneously, and no doubt, this is how God perceives things, at least within the β-domain.

$$\lim\ \text{dist}(p, q)\ \text{---}\rangle\ \infty\ \text{ as }\ \nu\ \text{---}\rangle\ \infty \quad \forall\ \alpha\ \ (\text{for all } \alpha)$$

In α-space, we do not have this luxury, for in any subspace you are going to have the presence of dark energy which will diminish the light coming to us from the β-world. Time and space will intensify as the vibrations lessen, for example, and any experiencer in these lower realms could feel 'pressed in' or even 'bored' as a result. A mile might seem like half-a-mile, say, and one hour like two hours, giving any observer here the feeling of living in 'cramped quarters' or that time is 'dragging itself along'. Needless to say, just the opposite would be true in higher-vibrational realms ...

To see this more clearly consider sitting in a prison cell that *feels* like it's 6' x 6'. In reality the cell is double this size, but in your mind there has been a length *contraction* which produces this illusion. Similarly, if you are bored at work, one hour might *feel* like two, because in your mind the 'distance' between perceived clock ticks has *shortened* -- for every two units of time you believe went by, only one elapsed, in actuality, giving the impression of a 'slow day on the job'. In all

likelihood this is what experiencers feel as you descend the various α-realms, and just the reverse as you go up - the 'distance' between meaningful points on the manifold will decrease or increase, accordingly ...

```
|  unit of time   |   unit of time  |         ...   perceived
|                                   |
 ‹---------  unit of time  -----------›         ...      real
```

Are time and space really attributes within the α-β domains ? I think so, provided we interpret things correctly, for they seem to be more *modes* by which we experience reality than *actual* dimensions. To say that time and space have *no* complements in β is the equivalent of saying they both take on a value of {0} here.

Such a constraint seems unreasonable to me, especially when you consider that even β-space itself was fashioned from the primordial sphere by transforming a *layer* of the dark energy, and stamping it as antimatter. Why, in turn, should this mean forfeiting time and space ? Doesn't God have the right to enjoy his own reality as he sees fit, even if it means making these attributes more elastic ? Me thinks he does ...

$$\Psi : \zeta \longrightarrow \beta \quad \cap \quad \Omega : \zeta \longrightarrow \alpha$$

or

$$\Psi : \zeta \longrightarrow \beta \quad \cap \quad \Lambda : \beta \longrightarrow \alpha$$

The Heart, Mind and Soul of God - A Contextual Interpretation

For a long time now, I have wondered about the heart, mind and soul of God, if they really exist, and if they do, what we might be able to say about each within the current mosaic. Let us begin by offering the following diagram, as a prelude to this short note --

```
              heart                  soul                 mind
                |                      |                    |
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
                |                      |                    |
                |                      |                    |
                |                      |                    |
                ------------------- Cross -------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

In James T's NDE, for example, the Holy Ghost is seen as the *mind* of God and made up of antimatter that coexists with matter everywhere. Thus it only makes sense to associate this attribute with the spiritual *laws* of creation, in accordance with our previous research.

> '... for He is not a man that He should change His mind ...' [1 Samuel, 15th Chapter]

The lifeblood of the Fundamental Theorem, however, really is the Godhead, in so much as symmetries here give rise to the laws by way of some mysterious action. It becomes the beating *heart* of the entire system, you might say, and so it only makes sense to associate this attribute with these *symmetries*, as shown in the diagram above.

> '... The Lord has sought out for Himself a man after His own heart ....'   [1 Samuel, 13th Chapter]

That leaves us with the *soul* of God, at least within the current context, and here we can associate this peculiar attribute with the *action* -- a boundary or membrane of sorts, containing encoded energy and information, which is equivalent to the Cross.

> '... He hath poured out His soul, even unto death ...'
> [Isaiah, 53rd Chapter]

Naturally our interpretation of things is purely metaphorical, but it does serve to differentiate the notions of heart, mind and soul of God, at least within the *current* mosaic. Not much can be said about corresponding properties when we consider the primordial fabric, for example, and according to our recent studies, attributes in the unified mosaic are undecidable, mathematically speaking.

```
                    γ-space
                       |
souls    <~~>    physical ring   <~~>   realms
  |                    |                  |
  |                    |                  |
   ---------- Symmetries Godhead -----------
                       |
                       |
                     Light
```

Thus, when trying to understand God, or label God, as it were, we have to remember that no language, set of symbols, or detailed constructions are ever going to give us an accurate picture of things ... Indeed, even the Fundamental Theorem of Creation [F] itself is *undecidable*, according to our more recent research, and that, on its own, ought to be enough to give us some pause for thought, always ...

$$[\ \sim A\ ,\ A\ ] \quad < \equiv > \quad [\ \sim F\ ,\ F\ ]$$

$$\vdots$$

The Continuum Hypothesis, Godel and Green Tao, Part II

In Part I, we established a relationship between various attributes in the Godhead [G], and showed that

~[Godel] < ≡ > ~[Green Tao, R] < ≡ > ~[Continuum Hypothesis] (*)

Among other things, this led to the interesting conclusion that *undecidable* theorems over a random set R exist because of [Green Tao, R] and conversely. Here we are going to study the primes [P] a little more closely and 'show' the following --

' If U is the set of undecidable theorems over P then
the cardinality of U is at least **c** '

Let us begin by defining any *irrational* number θ on the interval between 0 and 1 as

θ = **.**xyzw....,

where the individual digits run from 0 to 9 and form an infinite sequence as shown above. Now define the set of primes P' ⊂ P according to the following algorithm --

skip x primes in P and take the following two, say
skip y primes in P and take the following two, say

.... and so on .....

As an example, if θ = 0.314159 ... [π/10], we would skip the first three primes, but retain the next two; skip the next prime but retain the following two; then skip the next four and retain two, etc. In so doing, our set P', which is *unique* to θ, would look like --

P' = {7, 11, 17, 19, 41, 43, ...}

We will begin by assuming there are *one* or *more* undecidable theorems U' over P'. There has to be at least one, otherwise P' itself would be perfectly random according to (*), which is *not* allowed beyond G. Now write P' = S' ∪ T', where U' is *defined* to be undecidable over T' if at least one member of U' is undecidable over any element in P' [if the test passes, the element is moved to T', otherwise it is moved to S', and so on, for each element in P']. Under this construction, S' cannot be infinite, or else it too would be perfectly uncertain according to (*), which is *not* possible in ~G.

Either T' corresponds to some θ' in [0,1] or it doesn't. If *no*, set θ' = θ and continue on, as we now have a unique association in place. If *yes*, then either θ' = θ because S' is null or T' hashes to some value of θ' ≠ θ.

In the latter case, since S' is finite, there must be a point ω, beyond which *all* elements of T' agree with P', and if θ' is the number in [0,1] corresponding to T', there must also be a relative point β, beyond which *all* digits in θ' agree with θ [the ω-point is the largest number in S'].

$$T' = P' \text{ beyond } \omega \implies \text{digits}(\theta') = \text{digits}(\theta) \text{ beyond } \beta$$

This means, in turn, the digits in θ and θ' differ in only *finitely* many places, and hence θ' itself must *also* be irrational, in the case where T' maps to some value of $\theta' \neq \theta$ initially [if θ' can be derived from $\Theta \neq \theta$, then the digits in θ and Θ must also differ in only *finitely* many places … an important notion when contemplating uncountability].

To see this more clearly, suppose S' = {7, 11} and T' = {17, 19, 41, 43, ...}, in our example above, where

P' = {7, 11, 17, 19, 41, 43, ...}

Then beyond ω = 11, T' and P' are identical, and comparing θ and θ' now, we have …

θ' = 0.64159 ...   θ = 0.314159 ...

Thus, θ' is indeed irrational and *uniquely* defines T', in the particular case where T' maps to some value of $\theta' \neq \theta$ initially. Notice too that the digits in θ and θ' differ in only *finitely* many places.

More generally, since there are *uncountably* many irrationals in the unit interval [0,1], even after excluding derivatives like θ' or mirrors like Θ, this approach tells us the cardinality of U has to be $\geq$ **c**, for with each set T' there must always be at least one corresponding undecidable theorem, otherwise T' itself would be perfectly random according to (*), which is not allowed in ~G.

It seems undecidability is everywhere, and indeed, there could easily be *more* theorems of this kind over the reals than statements which can actually be proved. From a more spiritual perspective, indeterminacy implies regularity within randomness, and here we know from previous studies the latter leads to conflict *and* inconsistency, especially when considering the axiom of free choice, for example. All the more reason to guard our thoughts carefully, as they say, and do our best on a daily basis to make the best choices that we can, given the many obstacles that we face ...

undecidable theorem T ‹~~~› regularity within random set R

.

.

.

OTHER CONSIDERATIONS

It could happen that we need to skip more than nine primes in our construction of a particular set. For example, if θ = 0.$\underline{31}$ $\underline{41}$ 59 ... [π/10], we would skip the first 31 primes, then take 2; skip the next 41 primes and take 2 again; then skip 5 primes and take 2, and so on.

By underlining where applicable, we are effectively creating *another* instance of the same irrational number [π/10 in this case], and so long as we are willing to count it as such, forming sets and establishing uncountability, as we did above, should give us the result we are looking for … …

Relativity Theory and The Spiritual Laws of Creation, Part II

In Part I, we said there was a fundamental curvature-stress duality [Ɗ] originating in the spiritual domain [β], and with a physical counterpart, which goes something like this:

> *For any curvature tensor C there exists a stress tensor T such that T induces C, and conversely. Properties of C are inherited by T, and vice versa, because of duality.*

From this law we were able to show, rather easily, that the field equations of general relativity [G] in α-space take the form

$$C(u,v) = k * T(u,v) + K \qquad (*)$$

where C and T are curvature and stress tensors, respectively, k is a numerical scaling constant, and K is also a tensor associated with *dark* energy, for which cov(K) = 0. Thus,

Ɗ ==> G

as our frame of reference *shifts* from the pristine structures in β to the weak impressions in α, caused in part by the presence of dark energy [ρ] in the latter. This much is true from previous studies ...

Also, from previous research, we know

$$\text{coexistence} \quad <\equiv> \quad [\, \Lambda \,,\, \Lambda' \,]$$

where the Λ transform and its inverse Λ' are defined according to the following relationship --

$$\underset{\Sigma\, \nu}{\Lambda :} \beta \;\; \text{--->} \;\; \alpha\,(\nu, \rho)\,, \quad \Sigma\, \nu < \infty\,, \;\; 0 < \rho < 1$$

Since there is *no* dark energy in β-space, one might reasonably conclude that $\Lambda' \circ K = 0$, and so, operating on (*) via Λ′, we have

$$\Lambda' \circ C(u,v) \;\equiv\; \Lambda' \circ T(u,v) \qquad (\dagger)$$

In other words, within the β-domain, (†) gives rise to a unique curvature-stress duality [Ɗ] implied by both (*) and *equivalency*, and thus we may say with some assurance the following relationship holds ...

G ==> Ɗ

Taken together, Ɗ and G now become *equivalent* notions, where the fundamental curvature-stress duality in β-space implies general relativity in α-space, *and* conversely. A first, you might say, in terms of understanding at a deeper level, how symmetries and laws within the pristine world of the

Creator give rise to these weak impressions beyond the membrane ... ... in realms where *imperfect* uncertainty reigns supreme ... and a first, too, in terms of understanding at a deeper level, how these opaque transcriptions [equations] in our domain can be traced back to *perfectly* uncertain structures [laws] in the rather strange, if not completely bizarre world of the Godhead ...

```
  Ɗ   <=>   G
  |         |
β-space   α-space
```

Relativity Theory and The Spiritual Laws of Creation, Part III

In Part II we saw that curvature and stress form a duality [Ɗ] in β-space, according to the following relationship ...

$$\Lambda' \circ C(u,v) \equiv \Lambda' \circ T(u,v) \qquad (*)$$

Here we'd like to dig a little deeper into the Fundamental Theorem of Creation [F] and discover what we can about its various components, given our current understanding of General Relativity [G] ...

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
         |                  |                |
         |                  |                |
         |                  |                |
         --------------------- Cross ---------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

In order for F to be a *minimal* energy theorem over a manifold M ~ S(n), it is both necessary and sufficient that *dual* energy strands exist within the Godhead, which can take on various symmetrical configurations leading to laws. No more than a duality is required, and indeed, *law equivalency* guarantees it when you consider these things more deeply ...

Since (*) holds in β we may now formulate a 'covariant derivative' here, assuming the [Λ , Λ′] transforms commute in the usual way. To wit,

$$\nabla \circ \Lambda' \circ C(u,v) = \Lambda' \circ \nabla \circ C(u,v) = 0 \qquad (\dagger)$$

which implies

$$\nabla \circ \Lambda' \circ T(u,v) = 0 \qquad (\dagger)$$

Thus, the 'covariant derivative' of both curvature *and* stress is zero in β, and this can only mean *either* strand of energy, in some symmetrical configuration, may take on the role of C, while the other adopts the role of T concurrently. The particular strand, it seems, is of no relevance ...

As a result, the curvature-stress duality [Ɗ], seen as a law in β, is conserved, in the sense that these coupled strands of energy, in some symmetrical configuration, are *interchangeable* representations of C and T, respectively. In other words, under this interpretation

$$Ɗ \equiv C \oplus T \equiv T \oplus C$$

Needless to say, when mapping back to α-space with ξ equal to C or T, (†) implies the following ...

$$\Lambda \circ \nabla \circ \Lambda' \circ \xi(u,v) \;=\; \nabla \circ \Lambda \circ \Lambda' \circ \xi(u,v) \;=\; \nabla \circ \xi(u,v) \;=\; 0$$

and so, the curvature-stress duality, seen as a law in α, is *also* conserved, but now becomes *tainted* by the presence of *dark* energy, for example, along with other attributes such as incompleteness.

More generally, then, when looking at laws in β-space, we should view them as pristine *dualities* -- generated, in turn, by the double strands of energy ... in some symmetrical arrangement ... where corresponding derivatives are zero, and conservation principles apply in *both* the α and β domains, respectively ...

The law of h&s would work this way, for instance ... where now *harm* [H] becomes one of the symmetrical strands in β, and *suffering* [S] the other, with conservation determined by

$$\nabla \circ \Lambda' \circ H \;=\; \nabla \circ \Lambda' \circ S \;=\; 0$$

implying, in this case,

$$\mathcal{D} \;\equiv\; H \oplus S \;\equiv\; S \oplus H$$

In α-space we would see that

$$\nabla \circ H \;=\; \nabla \circ S \;=\; 0$$

and so conclude

$$H \;=\; k * S \;+\; K$$

where K can be associated with dark energy [ρ] and cov(K) = 0. In other words, the law of h&s, embedded in the *matter* world, would appear to be incomplete or inconsistent because of ρ, but in reality, is simply mirroring our own actions determined by *choice.* A feedback loop, if you will, which was mentioned in the original essay --

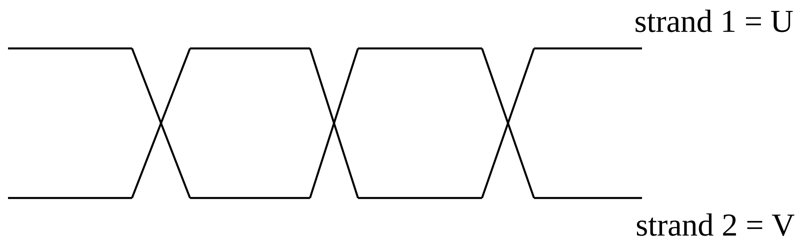
thought ~~> action ~~> deed ~~> consequence

Our best interpretation of the Fundamental Theorem, then, seems to lie in understanding that laws emerge and are conserved in β through symmetries which *always* represent dualities. Whether it's curvature and stress, particles and waves, harm and suffering or choice and consequence -- to name a few -- this *dual* nature in Creation seems to be a foundational principle ... upon which literally everything rests.

As such, when considering problems in α-space and mapping them over to β-space via Λ′, our best hope for a solution depends heavily on finding a β-law [duality] which reflects the very essence of the problem itself. This is the kind of thinking that is required, even in our domain, if we are ever to make any real progress ...

Relativity Theory and The Spiritual Laws of Creation, Part IV

In Part III we learned that laws in β-space are formed through symmetries in the Godhead, from double strands of energy [light], leading to *dualities*, which are conserved ...



$$Ɗ \equiv U \oplus V \equiv V \oplus U$$

$$\nabla \circ U = \nabla \circ V = 0$$

Here we wish to show that *all* laws in β are equivalent to one another, according to the following principle ...

> *If* $\mathcal{D}$ and $\mathcal{D}'$ *are any two laws in β-space .. then an invertible transform pair (ω, ω') exists with cov(ω) = cov(ω') = 0, and*
>
> $$\mathcal{D}' = \omega' \circ \mathcal{D} \;,\; \mathcal{D} = \omega \circ \mathcal{D}'$$
>
> *A similar relationship also holds for corresponding laws in α-space induced by Λ*

To prove this interesting result, begin by letting S be the symmetry associated with Ɗ and similarly let S' be the symmetry associated with Ɗ', where

$$\begin{tikzcd} S \arrow[r, squiggly, "\omega'"] \arrow[d, dash, "Ɗ"'] & S' \arrow[d, dash, "Ɗ'"] \\ S & S' \arrow[l, squiggly, "\omega"] \end{tikzcd}$$

Note there is *always* an invertible transform pair (ω, ω') taking us between the symmetries,

$$S = \omega \circ S' \;,\; S' = \omega' \circ S$$

and so by association, using Leibniz as our guide, we may write, for either strand,

$$\nabla \circ U = \nabla \circ (\omega \circ U') = \omega \circ (\nabla \circ U') \oplus U' \circ (\nabla \circ \omega) = 0 \qquad (*)$$

Since cov(U') = 0, (*) reduces to

$$U' \circ (\nabla \circ \omega) \;=\; 0$$

implying cov(ω) = 0, which proves the first part of the theorem. Transforming to α-space via Λ, we have

$$\Lambda \circ \nabla \circ \omega \;=\; \nabla \circ \Lambda \circ \omega \;=\; 0$$

and thus the covariant derivatives of the transform pair [Λ ∘ ω, Λ ∘ ω'] in α are also 0.

Since ω takes us from S' to S and ω' does the reverse, it stands to reason that in α-space you have a *similar* behavior for the corresponding symmetries induced by Λ, *and* in turn, the laws induced by these symmetries ...

$$\begin{tikzcd}
\Lambda \circ S \arrow[r, squiggly, "\Lambda \circ \omega'"] \arrow[d, dash, "\Lambda \circ \mathcal{D}"'] & \Lambda \circ S' \arrow[d, dash, "\Lambda \circ \mathcal{D}'"] \\
\Lambda \circ S & \Lambda \circ S' \arrow[l, squiggly, "\Lambda \circ \omega"]
\end{tikzcd}$$

Because $\omega \circ \omega' \;=\; 1$ we may now conclude

$$\Lambda \circ (\omega \circ \omega') \;=\; 1 \;=\; (\Lambda \circ \omega) \circ (\Lambda \circ \omega')$$

implying the transform pair [Λ , Λ′] is indeed *multiplicative* with respect to (ω, ω'). As well,

$$\Lambda \circ \mathcal{D}' \;=\; \Lambda \circ (\omega' \circ \mathcal{D}) \;=\; (\Lambda \circ \omega') \circ (\Lambda \circ \mathcal{D})$$

and this completes the second part of the theorem.

******

The result is quite deep, in that it unifies all of the spiritual laws of creation by way of transforms linked back to symmetries, in both α and β-space, at least theoretically. Since the odds of actually finding [Λ , Λ′] are slim to none, about the best we can do here is say this remarkable equivalency exists, and in the end, state what may have been obvious all along ... all laws really are reflections of one another, no matter the vantage point ...

******

On The Nature of Coexistence and Invertible Transforms, Part II

In previous research … … we studied the [Λ , Λ′] transform pair [π] and realized the following equivalency …

'For some invertible transform Λ, coexistence implies Λ,
and conversely its inverse Λ′ implies coexistence'

Here we wish to show that up to attribute, anyway, π is *indeterminate* with cov(π) = 0. So let's begin by stating the obvious --- when talking about the Fundamental Theorem of Creation [F], it doesn't matter whether we do *or* don't adopt a two-state model of coexistence [Σ] because the elemental building blocks are *equivalent* to one another, regardless.

Thus, without loss of generality, *sans* F there is no *action,* and so ~F ==> ~Σ, or equivalently, Σ ==> F, assuming Noetherian principles apply. On the other hand, *all* things flow from the action, which means F ==> Σ, and hence F <=> Σ. Consequently,

$$\Sigma \quad <\equiv> \quad [\,\Lambda\,,\Lambda'\,] \quad <\equiv> \quad F \qquad\qquad (*)$$

and since

$$\sim\!A\,,A \quad <\equiv> \quad F$$

where A is the set of attributes in β-space and ~A are the complements in α-space, π must, perforce, be *undecidable* ...

$$[\,\sim\!A\,,A\,] \quad <\equiv> \quad [\,\Lambda\,,\Lambda'\,]$$

From previous research we also know *all* the spiritual laws of creation [$\mathcal{D}$] in β are equivalent to one another and that cov($\mathcal{D}$) = 0. Thus by equivalency, cov(F) = 0, and so by (*), necessarily cov(π) = 0. Ergo, the following theorem applies, provided ∇ and π do, indeed, commute … …

If π is the invertible transform pair Λ , Λ′ which maps from
β to α and conversely, then π is undecidable up to attribute,
and cov(π) = 0

******

As a result, there isn't much we can *ever* know about π, outside the bounds of at-one-ment, and so, even though the laws [$\mathcal{D}$] really are unified into a whole through symmetries via (ω, ω'), it becomes virtually impossible to codify [Λ , Λ′] in any meaningful way, and hence the spiritual laws of creation, themselves.

******

About the best we can do is write down a generic, first-order relationship between the two strands which make up the duality, just as was done in the case of *curvature* and *stress*, say, or the law of

harm and *suffering*, and let it go at that. Coding the actual mathematics, it seems, is beyond both our reach and our grasp, dashing once again any hope that someday, just maybe, a complete description of Creation might finally appear on the horizon ....

******

We mentioned ∇ and π needed to commute in order to prove the theorem above. In fact, it turns out commutativity is both a *necessary* and *sufficient* condition for the case where $cov(\Sigma) = 0$. To see this more clearly, let C_α be the curvature tensor associated with general relativity in α-space, and let C_β be its counterpart in β. Assume now the operators commute, so that $[\nabla , \pi] = [\pi , \nabla]$. Then

$$\nabla \circ C_\beta = \nabla \circ \Lambda' \circ C_\alpha = \Lambda' \circ \nabla \circ C_\alpha = 0$$

implying covariance in Σ. On the other hand, if $cov(\Sigma) = 0$, then by equivalency (*)

$$cov(\mathcal{D}) = cov(\pi) = 0$$

and so,

$$\nabla \circ C_\beta = \nabla \circ (\Lambda' \circ C_\alpha) = \Lambda' \circ (\nabla \circ C_\alpha) \oplus C_\alpha \circ (\nabla \circ \Lambda') = \Lambda' \circ (\nabla \circ C_\alpha)$$

Thus

$$\nabla \circ (\Lambda' \circ C_\alpha) = \Lambda' \circ (\nabla \circ C_\alpha) = 0$$

implying commutativity, and hence we have the following conservation theorem ... …

The operators ∇ and π commute if and only if $cov(\Sigma) = 0$, where Σ is coexistence and π the Λ , Λ' transform pair

Notice, too, that if the energy in Creation is conserved [$cov(\Sigma) = 0$], then the condition $\nabla \circ C_\beta = 0$ automatically implies $\nabla \circ C_\alpha = 0$, because of the theorem above. This is precisely what Einstein was looking for – a fully covariant curvature tensor which could be paired with the stress tensor T_α, that would ultimately lead to the celebrated field equations of general relativity.

The fact that C_α exists, and does so uniquely, as a contracted form of the Riemann curvature tensor, is nothing short of a miracle, in my view, and indeed, you could almost say it is divine … …

$$C_\alpha \equiv T_\alpha$$

Covariance and The Fundamental Theorem of Creation

In earlier studies we learned that

$$\Sigma \quad <\equiv> \quad [\, \Lambda \,, \Lambda' \,] \quad <\equiv> \quad F \qquad (*)$$

assuming Noetherian principles, where Σ denotes coexistence and the other symbols their standard interpretations. From this, it was shown that cov(F) = 0, and so by (*)

$$F \quad ==> \quad \mathrm{cov}(\Sigma) = 0$$

Thus, if creation is seen as a system, based on Noether, the *energy* of this system will be conserved. On the other hand, if God decides to build creation over a simply-connected manifold M ... from first principles ... and insists on the *conservation* of energy, we know from previous writings M ~ S(n), implying a Noether solution everywhere. Hence,

$$\mathrm{cov}(\Sigma) = 0 \quad ==> \quad F$$

and so the following theorem (†) emerges rather naturally ...

> *The energy in Creation is conserved if Noether applies*
> *and conversely, with cov(Σ) = 0 <=> F, where Σ is*
> *coexistence and F some minimal energy theorem*
> *over the manifold M ~ S(n)*

There are some interesting ideas that come out of this, the first of which concerns salvation-based religions such as Christianity. In more fanatical variants the *bad* soul is doomed to hell and ultimately destroyed by God at the appropriate time. However, if this were really true, and any trace of the soul's energy was *completely* extinguished, (†) tells us *either* ~F holds *or* the fanatical interpretation is incorrect.

If the former, Creation becomes a system *without* accountability on both sides. Symmetries leading to laws [dualities] such as *choice* and *consequence*, or *harm* and *suffering* needn't exist anymore, and God can destroy whatever he wants. In effect, you now have a system running on *chaos*, where neither God nor the soul needs to respect anything ... and in my view, anyway, such a system is doomed to fail, even in the short run.

As such, the fanatical view *has* to be tossed in favor of something which makes more sense, and so far the best approach seems to be a combination of ideas involving the ancient writings, near-death research, and of course, the mathematics and physics sitting in between the two. In other words, the Fundamental Theorem of Creation, no matter the vantage point ...

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
        |                   |                  |
        |                   |                  |
        |                   |                  |
        -------------------- Cross --------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

PUTTING IT ALL TOGETHER

The following equivalency diagram shows the relationship between commutativity, covariance and Noetherian principles. All three are identical notions, mathematically speaking …

```
          ---------  cov(Σ) = 0  ---------
          |                              |
          |                              |
[∇, π]  =  [π , ∇] ------------ F ⊃ M ~ S(n)
```

The Continuum Hypothesis and Unified Theories in Physics, Part II

In Part I, we reasoned that unified theories in physics, involving Quantum Mechanics and General Relativity, were *not* possible because of the Continuum Hypothesis [Ƕ]. Let us refresh our memory of things by recalling the diagram which illustrates this idea in simple detail ...

Quantum Mechanics | bridging theory | General Relativity

N **r** **c**

| *undecidable* |

Recent studies indicate there actually *is* a unified theory in creation, which ties together *all* laws [Ɗ] in β-space via the invertible (ω, ω') transform pairs that act on the various symmetries here. Thus we can define the attribute *decidability* [Đ] in β as

$$Đ \ = \ U \{ \omega \mid \omega \ \varepsilon \ \beta \}$$

since the ω construction is always definite. Moving over to α-space, and knowing Λ**:** β ---› α itself is undecidable up to attribute, the complement can be written as

$$\sim Đ \ = \ \Lambda \circ U \{ \omega \} \ = \ U \{ \Lambda \circ \omega \} \qquad (*)$$

and this, consequently, becomes the definition of *undecidability* in α. Now since all attributes in α-space are equivalent to one another, by virtue of symmetry in β, necessarily

$$\sim Đ \ < \equiv > \ Ƕ$$

and so we may conclude from (*) that *any* or *all* ω-constructions induced by Λ in the α-domain are *indeterminate* because of Ƕ.

In other words, bridging theories in α-space, which take us from one law to another, by way of symmetries, are undecidable *because* of the Continuum Hypothesis .... and so a constructible, unified theory of physics spanning Quantum Mechanics to General Relativity, in particular, simply cannot exist in our domain. This was the premise established in Part I, but there we didn't have enough machinery at the time to demonstrate the result more rigorously. Now, it seems, we do ...

Quantum Mechanics | bridging theory | General Relativity

N **r** **c**

| *undecidable* |

$$\Lambda \circ Ɗ' \ = \ \Lambda \circ (\omega' \circ Ɗ) \ = \ (\Lambda \circ \omega') \circ (\Lambda \circ Ɗ)$$

Godel's Theorem, Green Tao Theorem and The Godhead, Part II

In Part I, long ago, we made a loose association between Godel and Green Tao, surmising the two were indeed equivalent by looking at their counterparts in the Godhead [G] ...

$$\text{Godel's Theorem} \;\equiv\; \text{Green Tao Theorem}$$

This led us to the idea of *perfectly uncertain* symmetries [S] in β-space -- an idea which has taken center stage in all of our writings ever since. Here we'd like to expand a bit on the notion of perfect uncertainty [μ], and offer up a more rigorous definition of what it might actually be ... physically speaking...

From earlier writings, we know *all* attributes in β are equivalent to each other, by virtue of symmetry, and because the ω constructions are equivalent to S, one may write

$$Đ \;=\; U\{\,\omega \mid \omega\;\varepsilon\;\beta\,\} \;=\; U\{\,S \mid S\;\varepsilon\;\beta\,\} \;=\; \mu \qquad (*)$$

Thus, under this arrangement, decidability [Đ] corresponds to *perfect* uncertainty [μ] via (*), and the latter *becomes* a union of the symmetries, formed from dual strands of light in G, relative to the simultaneous now.

Moving over to α-space, and knowing $\Lambda\!: \beta \dashrightarrow \alpha$ itself … … is undecidable up to attribute, the complement can be written as

$$\sim\!\mu \;=\; \Lambda \circ U\{\,S\,\} \;=\; U\{\,\Lambda \circ S\,\}$$

and this, consequently, becomes the definition of *imperfect* uncertainty from our perspective.

In all likelihood, then, β-space was made by transforming double strands of light into a myriad of vibrational patterns [symmetries], which *in toto*, displaced the dark energy here, altogether, as the frequency of these patterns approached infinity ...

$$\beta : \Sigma\,\nu \;\dashrightarrow\; \infty \underset{\mu}{\Longrightarrow} \rho \;\dashrightarrow\; 0 \qquad (\dagger)$$

In α-space, however, the presence of dark energy, in varying amounts, forces these symmetries to vibrate at a *lower* resonance, and at best, we only have a weak impression of what they might be, because of the undecidable nature of [Λ, Λ'] ...

$$\alpha : \Sigma\,\nu \;<\; \infty \underset{\sim\mu}{\Longrightarrow} \rho \;>\; 0 \qquad (\ddagger)$$

So when thinking back to the 'pond and stones' analogy, tossing an infinite number of stones into the pond is analogous to (†), in so much as infinitely many stones [symmetries], all vibrating at specific frequencies, is no different than finitely many stones, all vibrating at an infinite frequency.

Either way, you get the same thing, so to speak, giving rise to the simultaneous now and hence, the notion of perfect uncertainty.

In the ordered now, such isn't possible, since here you are only tossing a finite number of stones into the pond, and symmetries induced by Λ are vibrating at a finite rate because of dark energy (‡). It's an odd dichotomy which isn't easy to understand, but still, if we are to accept the idea that G is a fully consistent and complete domain within β-space, then we must also come to grips with the idea of *simultaneity* ... spanning laws, symmetries, transforms ... indeed, the whole of Creation itself you might say ...

'infinite number of stones ---› ◌ ---› perfectly random'
$\nu < \infty$

or

'finite number of stones ---› ◌ ---› perfectly random'
$\nu = \infty$

Geometry, Groups and The Fundamental Theorem of Creation

Our recent studies on *Relativity and The Spiritual Laws* led to the discovery of a *unified* theory of Creation, according to the following principle (†) ...

> *If $\mathcal{D}$* and *$\mathcal{D}'$ are any two laws in β-space .. then an invertible transform pair (ω, ω') exists with cov(ω) = cov(ω') = 0, and*
>
> $$\mathcal{D}' = \omega' \circ \mathcal{D} \ , \ \mathcal{D} = \omega \circ \mathcal{D}'$$
>
> *A similar relationship also holds for corresponding laws in α-space induced by Λ*

However the proof of this statement never really went beyond the Godhead [G] ... when considering symmetries [S] and laws [$\mathcal{D}$], so here we'd like to expand on things a bit by seeing these laws as geometric *reflections* of the symmetries, via the action [β-membrane] that divides the two ...

$$\begin{tikzcd} S \arrow[r, squiggly, "\omega'"] \arrow[d, dash, "\mathcal{D}"'] & S' \arrow[d, dash, "\mathcal{D}'"] \\ S & S' \arrow[l, squiggly, "\omega"] \end{tikzcd} \quad \overset{\beta}{<\sim\sim\sim>} \quad \begin{tikzcd} \mathcal{D} \arrow[r, squiggly, "\tau'"] \arrow[d, dash, "S"'] & \mathcal{D}' \arrow[d, dash, "S'"] \\ \mathcal{D} & \mathcal{D}' \arrow[l, squiggly, "\tau"] \end{tikzcd}$$

$$\begin{array}{c} | \\ \text{action} \\ \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim> r \\ r^* <\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \end{array}$$

Let us start by defining the reflection [r] as that transform which maps any symmetry in G across β, and define r* as its inverse. Then

$$\mathcal{D} = r \circ S \ , \ S = r^* \circ \mathcal{D}$$

and

$$\tau = r \circ \omega \circ r^* \ , \ \tau' = r \circ \omega' \circ r^*$$

Taking the covariant derivative of any law, we have

$$\nabla \circ \mathcal{D}' = \nabla \circ (\tau' \circ \mathcal{D}) = \tau' \circ (\nabla \circ \mathcal{D}) \oplus \mathcal{D} \circ (\nabla \circ \tau') = 0 \qquad (*)$$

.
.
.

Since $\nabla \circ \mathcal{D} = 0$, (*) reduces to

$$\mathcal{D} \circ (\nabla \circ \tau') = 0$$

implying $cov(\tau') = 0$. Thus (†) holds, even under reflection, where the (ω, ω') transform pair now becomes (τ, τ') and laws exist as *separate* geometries, beyond the β-membrane [action] ... induced by (r, r^*).

******

No longer are these laws just descriptions of the various symmetries in the Godhead; but rather, living, breathing entities ... perfectly *uncertain* vibrational patterns, if you will, which act as agents on behalf of G, performing their tasks without bias, fairly and ethically, at all times. And so, establishing this fact lends even more *credence* to the Fundamental Theorem of Creation, in so much as each of its components now becomes a reality, grounded in reasonable mathematics, that seems to justify the whole of things, according to Noetherian principles ... in other words, a unified theory of all that is ...

```
Symmetries Godhead  <~~>  Action  <~~>  Law of Truth
        |                    |                   |
        |                    |                   |
        |                    |                   |
        -------------------- Cross --------------------
```

Symmetries in the Godhead uphold the Law of Conservation of Truth by way of an action equivalent to the Cross, and vice-versa. Truth, necessarily, is defined as the union of the spiritual laws of creation, which are equivalent.

$$\Lambda \circ \mathcal{D}' = \Lambda \circ (\tau' \circ \mathcal{D}) = (\Lambda \circ \tau') \circ (\Lambda \circ \mathcal{D})$$

Geometry, Groups and The Fundamental Theorem of Creation, Part II

In Part I, we saw that laws, seen as geometries in β-space, beyond the Godhead [G], could be written as

$$Ɗ \ = \ r \circ S \ , \ S \ = \ r^{*} \circ Ɗ \qquad (*)$$

where (r, r*) is the transform pair connecting symmetries [S] to laws [Ɗ] via the β-boundary. Here we'd like to write down the fuller form of any law in α-space, by incorporating (r, r*) as something other than the *identity* operator.

If U and V are any two strands of a symmetrical pattern in G, then (*) implies

$$Ɗ \ \equiv \ r \circ (U \oplus V) \ \equiv \ r \circ (V \oplus U)$$

which means in α we must have

$$(\Lambda \circ r) \circ (\Lambda \circ U) \ = \ (\Lambda \circ r) \circ (\Lambda \circ V) \ + \ K \qquad (\dagger)$$

where cov(K) = 0. To see this more clearly, simply operate on (†) with Λ' and observe that Λ' ∘ K ‘vanishes’ in β. With ξ = U or V, the left and right-hand sides now become

$$\begin{aligned} \Lambda' \circ ((\Lambda \circ r) \circ (\Lambda \circ \xi)) \ &= \ (\Lambda' \circ (\Lambda \circ r)) \circ (\Lambda' \circ (\Lambda \circ \xi)) \\ &= \ ((\Lambda' \circ \Lambda) \circ r) \circ ((\Lambda' \circ \Lambda) \circ \xi) \\ &= \ r \circ \xi \end{aligned}$$

and this takes us back to the laws in β, which is the expected result.

Thus, the first-order relationship for *all* laws in α induced by both Λ and (r, r*) is indeed (†), and the covariant derivative of either side may be written as

$$\begin{aligned} \nabla \circ (\Lambda \circ r) \circ (\Lambda \circ \xi) \ &= \ (\Lambda \circ r) \circ (\nabla \circ (\Lambda \circ \xi)) \ + \ (\Lambda \circ \xi) \circ (\nabla \circ (\Lambda \circ r)) \\ &= \ (\Lambda \circ r) \circ (\Lambda \circ (\nabla \circ \xi)) \ + \ (\Lambda \circ \xi) \circ (\Lambda \circ (\nabla \circ r)) \end{aligned}$$

Since $\nabla \circ \xi \ = \ 0$ the above expression reduces to

$$(\Lambda \circ \xi) \circ (\Lambda \circ (\nabla \circ r)) \qquad (\ddagger)$$

Invoking (*), we have

$$\nabla \circ Ɗ \ = \ \nabla \circ (r \circ S) \ = \ r \circ (\nabla \circ S) \ \oplus \ S \circ (\nabla \circ r) \ = \ 0$$

Since $\nabla \circ S \ = \ 0$ it follows that $\nabla \circ r \ = \ 0$ and thus the expression (‡) is also 0. Finally, then,

$$\nabla \circ (\Lambda \circ r) \circ (\Lambda \circ \xi\,) \;=\; 0$$

for ξ = U or V and so (†) preserves covariance, just as one would expect.

As an application of this theory, consider the field equations of general relativity. Originally, with the transform pair (r, r^*) equal to the *identity*, these equations in α took the form

$$C(u,v) \;=\; k * T(u,v) \;+\; K$$

but now become

$$(\Lambda \circ r) \circ C(u,v) \;=\; k * (\Lambda \circ r) \circ T(u,v) \;+\; K$$

where k is a numerical scaling constant and cov(K) = 0. Because Λ is undecidable up to attribute, it is highly unlikely we will ever know the exact expression for these statements, but at least to first-order, anyway, one can now see what they might look like, mathematically speaking, after translating the law to a geometry in β, beyond the membrane [action], and mapping it back to α.

In fact, because of the undecidable nature of Λ, in general, (†) implies *all* laws in α are incomplete … an idea which is surely not new to us -- and so the search for unified theories will go on and on, just as it should ...

Stones, Ponds and The Green Tao Theorem

In this note, we are going to align more closely the Green Tao Theorem with the vibrational patterns [symmetries and laws] that exist in α-β space. Let us begin, then, with the following diagram from previous research and assume, for now, we are only dealing with a single law or symmetry [ξ] ...

α-world

| ↑ spin-states ν ‹---› ↓ regularity ρ within randomness

β-world ---› Λ ---›

| ↓ spin-states ν ‹---› ↑ regularity ρ within randomness

Tossing an infinite number of stones into the pond represents *all* of the vibrational levels for ξ and thus in β we can think of this as a 'sum over frequencies', leading to an *infinite* number of superpositions, and hence *perfect* uncertainty itself, in the absence of dark energy ...

$$\xi \;=\; \beta : \underset{\mu}{\Sigma\, \nu \;\text{---›}\; \infty} \;==>\; \rho \;\text{---›}\; 0$$

$$\mu \;=\; U\,\{\, \xi \,|\, \xi \;\varepsilon\; \beta \,\}$$

Aligning with Green Tao, the infinite number of states 'can be seen' as the prime numbers [P] and dark energy [ρ] as the arithmetic sequences [S] of arbitrary length, which vanish as ν ---› ∞. In fact, the *only* arithmetic sequence allowed in β must have a period [k] which is infinite ..... implying that the distance between any two β-points must also be infinite. Conversely, the *only* arithmetic sequence allowed in ζ [primordial fabric] must have a period [k] of length 0, since here regularity running *through* randomness does not exist at all -- only dark energy.

To see these things more clearly, imagine all of the primes being 'relocated' to ∞, or β-space. It would be akin to 'stretching out' the number line until the distance between any two elements in P was also infinite, and similarly, if all of the primes were compressed into a single point, say {0}, the distance between any two would now be zero as well, putting us in ζ-space.

$$\beta : \;\; f_\nu \;\text{---›}\; \infty \quad \lambda_\nu \;\text{---›}\; 0 \quad f_\rho \;\text{---›}\; 0 \quad \lambda_\rho \;\text{---›}\; \infty$$

$$\zeta : \;\; f_\rho \;\text{---›}\; \infty \quad \lambda_\rho \;\text{---›}\; 0 \quad f_\nu \;\text{---›}\; 0 \quad \lambda_\nu \;\text{---›}\; \infty$$

Moving over to α, by way of Λ, any subspace will inherit ξ ... but now the number of vibrational states associated with ξ is *finite* in this subdomain and so

$$\xi \;=\; \alpha : \underset{\sim\mu}{\Sigma\, \nu \;<\; \infty} \;==>\; \rho \;>\; 0$$

$$\sim\mu \;=\; U\,\{\, \Lambda \circ \xi \,\}$$

In order to align with Green Tao, we must have 'an *infinite* number' of α-subspaces, where each one, seen as a partition Π, resonates on a finite number of levels, which can be associated, in turn, with a finite set of unique primes. Thus,

$$\alpha = U \{ \Pi \mid \nu \in P, \Sigma \nu < \infty \} \qquad (*)$$

and viewed as a continuum, the α-domain *in toto* now becomes P, and dark energy [ρ] the arithmetic sequences [S] of arbitrary length running through it. Without partitioning in this way, we'd essentially have β-space, and so one of the distinct features separating the transforms

$$\Psi : \zeta \dashrightarrow \beta \;\cap\; \Omega : \zeta \dashrightarrow \alpha$$

or

$$\Psi : \zeta \dashrightarrow \beta \;\cap\; \Lambda : \beta \dashrightarrow \alpha$$

from each other is the notion of *decomposition*.

An observer in β sees Creation in one shot, where *all* the partitions Π in α are meshed together as an infinite set of superpositions in the simultaneous now, while in α any partition is a finite construction, according to (*). Either way, there is a distinct correspondence with Green Tao, when each space is seen as a continuum in the limit, and so one has to wonder ... are *all* vibrational states a reflection of the prime numbers ? Stay tuned ...

$$\underset{\Sigma \nu}{\Lambda : \beta} \dashrightarrow \alpha(\nu, \rho), \quad \Sigma \nu < \infty, \quad 0 < \rho < 1$$

Stones, Ponds and The Green Tao Theorem, Part II

In Part I, we tried to align more closely the Green Tao Theorem with the vibrational patterns that exist in α-β space [Σ], to better understand the notion of *perfect uncertainty* [μ], in particular. In this note, we'd like to continue the discussion … by seeing the attribute [μ] … as more of a *geometry* [Γ], endowed with a peculiar distance metric [Δ], where

$$\Delta(p, q) \;=\; \infty \quad \text{for all } p, q \;\varepsilon\; \beta \qquad (*)$$

Let us begin, then, with a single law or symmetry [ξ] and assume for now there is only *one* subspace in α. Clearly ξ vibrates at a *finite* rate in α because of the presence of dark energy [ρ], but when moving back to β, the frequency of these vibrations, magically, becomes *infinite*. How is this possible, or is it, in fact, an illusion ?

To answer the question, it is important to understand that (*) really *is equivalent* to μ. In other words, if β is perfectly uncertain, then λ_ρ ---› ∞, which implies Δ(p, q) = ∞ for any two points here. On the other hand, if (*) is true it is no longer possible to 'measure the distance' between any two points in β, and so the notion of regularity within randomness disappears altogether. Thus we have the following theorem ...

If β is any space of points endowed with a distance metric Δ
according to some geometry Γ, then β is perfectly uncertain
if and only if Δ(p, q) = ∞ for all p, q ε β

The result is quite deep in that it allows us, now, to see β-space as more of a geometry [Γ] endowed with a strange metric [Δ] where (*) is always true, and laws or symmetries here 'seem' to vibrate at a frequency f_v ---› ∞. A masterful trick if there ever was one, you might say, which creates the illusion of a very *real* infinity, and thus *perfect uncertainty* itself, in the absence of ρ. Strictly speaking ... the *simultaneous now* ... referred to in many earlier writings ...

In α-space such is not the case because Δ(p, q) < ∞ for any two points in this domain, giving rise to *imperfect uncertainty*, and hence the *ordered now*. The notion of infinity becomes a 'far off ' concept in α, with no satisfactory explanation of how you 'really get there', and indeed, you never actually reach this point [∞] since neither Γ nor Δ, in our world, allow for it. Infinity, then, becomes more an artifact of the human imagination and should be treated as such, accordingly.

To this end, Γ could be seen as an *attribute* of Σ, inducing a metric Δ which is always infinite in β, and with a complement ~Γ in α inducing a metric Δ which is always finite. Such an approach would neatly explain everything we've discussed so far in previous research, and perhaps pave the way for any discoveries yet to come ... we'll just have to wait and see, as they say ...

$$\Sigma \;\;<\equiv>\;\; [\,\sim\Gamma\,,\,\Gamma\,] \;\;<\equiv>\;\; [\,\sim\mu\,,\,\mu\,] \;\;<\equiv>\;\; F \qquad (\dagger)$$

In the meantime, by taking our cue from (†), we do have the following rather interesting corollary ...

> *If Γ is any geometry in α-space endowed with a metric Δ for which $\Delta(p, q) < \infty$ for all $p, q \ \varepsilon \ \alpha$, then Γ is both incomplete and inconsistent*

Hilbert's problem is a good example of this corollary in action, for it tells us there is no complete, regular surface S, of constant negative Gaussian curvature K, in Euclidean 3-space. And, as a second example, if Γ is *any* space-time geometry associated with the field equations of general relativity, it too will be both incomplete and inconsistent, since here the distance metric [Δ] is always a solution to these equations. If Δ is well-behaved the corollary applies, and if it isn't, Γ will contain singularities. Either way, it seems, Einstein's theories are, at best, provisional ...

And there would be many more illustrations of this kind; in fact, too many to number ... from all that we know ... to say nothing of undecidability ...

Stones, Ponds and The Green Tao Theorem, Part III

In Part II, we developed our understanding of α-β space just a little deeper, by seeing each domain as more of a *geometry* [Γ] ... endowed with a *distance* metric [Δ], where

$$[\ \sim\Gamma\ ,\ \Gamma\ ] \quad <\equiv> \quad \Sigma$$

Here we'd like to expand on things a bit by looking at the fundamental *axioms* of set theory [Z] in α, and discovering what we can about their counterparts in β, assuming such exist ...

So begin by letting S be the statement 'Ƕ is true', where Ƕ is the Continuum Hypothesis, and let ~S be the corresponding statement 'Ƕ is false'. From Cohen et al,

$$[\ \sim S,\ S\ ] \quad <\equiv> \quad Z$$

which means Z admits *both* ~S and S simultaneously, implying, as a consequence, the set Z must be *indeterminate*. On the other hand, Ƕ follows from Z and thus Z implies [~S, S], which leads us to the chain below ...

$$Z \quad ==> \quad [\ \sim S,\ S\ ] \quad ==> \quad Z\ \text{ indeterminate}$$

In other words, the fundamental axioms of set theory are *undecidable* up to S, and so by equivalency the pair [~Z, Z] becomes an attribute of Σ, suggesting in turn, that *countability* does not exist in β at all. Ergo, the following, rather compact, theorem applies ...

If T is any set of points in β-space then the cardinality of T is unmeasurable

If, now, we define the set T to be {1, ∞, 2, ∞, 3, ∞, ...}, then T = V ∪ W, where V = {1, 2, 3, ...} and W = {∞, ∞, ∞, ...}. Viewed on the boundary [∂αβ] between α and β, T appears to be enumerable [V], with card(T) ≈ **N**, but when viewed on the boundary between β and α [∂βα], T appears to be unmeasurable [W].

In fact, the two perspectives are *equally* valid since ∂αβ = ∂βα, with the only difference being 'which side' of the boundary you are on. In the case of ∂αβ an observer would say the α-viewpoint is correct, and in the case of ∂βα an observer would opt for the β-viewpoint ...

******

If, indeed, our analysis of things is correct here, T becomes an example of a set in Σ whose cardinality is undecidable, up to attribute. It is both measurable and unmeasurable at the same time, so to speak, and thus exists as a boundary condition in principle, if it exists at all. Strangely enough, the Cross is *also* a boundary condition, according to the Fundamental Theorem of Creation, and so, at least in a religio-philosophical sense of the word, its import would seem immeasurable in β-space, whilst to someone in α-space, it may have little or no meaning at all ...

Stones, Ponds and The Green Tao Theorem, Part IV

In Part III we looked at the fundamental axioms of set theory [Z] and saw that they formed an attribute in Σ, which led to the notion of *unmeasurability* in β-space. Here we'd like to expand on the self-titled Consistency Hypothesis [Ꝁ], which simply posits that Z is consistent.

[~Z, Z]   < ≡ > Σ (*)

So let's begin by defining S to be the statement 'Ꝁ is true' and define ~S to be the statement 'Ꝁ is false'. From Godel, we know that if Z is consistent it cannot be proved from within, implying that up to S, anyway, Z must be indeterminate. In other words, the axioms themselves cannot actually determine whether they are or are not consistent, because they are not self-aware, and so

[~S, S]   < ≡ >   Z

Thus, relative to the axioms, Ꝁ must also be undecidable, and so by *equivalency* the pair [~Ꝁ, Ꝁ] becomes an attribute of Σ, just like Ƕ, suggesting in turn that Z is indeed inconsistent in α-space. Ergo, the following theorem applies ...

The fundamental axioms of set theory are neither complete
nor consistent in α-space, with Ꝁ an attribute of Σ, where
Ꝁ is the Consistency Hypothesis

The Hausdorff and Banach-Tarski paradoxes are good examples of this theorem in action, for they tell us you can take a 3-dimensional ball of radius 1, dissect it into 5 parts, and then move and rotate these parts to get two balls of radius 1. And similar logic allows one to assign a measure of both **0** and **c** to the unit circle simultaneously, assuming the axiom of free choice, by creating orbits under a group of rational rotations and invoking the principle of 'countable additivity'.

Both of these ill-conceived constructions, however, depend heavily on the *axiom of choice*, mentioned above, which, in turn, is only made possible *because* of the regularity running through the randomness of the number line. Without this embedded regularity, such choices leading to bizarre outcomes would not be possible in mathematics, and in fact, without embedded regularity you would be in β-space ... where all decisions are made *simultaneously*, according to (*) ...

regularity within R   ‹~~~›   axiom of choice

Similarly, it is the presence of *dark* energy running through the randomness of light in α-space that gives rise to poor choices in our daily lives, and indeed, the laws themselves are affected not only by the whereness of ρ, but also, our thoughts and actions ...

dark energy	‹~~~›	regularity within randomness	‹~~~›	choice
ξ		ρ ν		c

******

These illustrations, and many more like them, can only bolster our view that the axioms of set theory are indeed both incomplete *and* inconsistent in our domain ... and that more generally the [Godel] attribute is functioning just as it should under coexistence. The fabric of reality, it seems, is filled with holes and oddities everywhere, which should come as no surprise to us now, for Creation, it appears, would have it no other way ...

Stones, Ponds and The Green Tao Theorem, Part V

In Part IV we looked at the self-titled Consistency Hypothesis [Ꝁ] and saw that [~Ꝁ, Ꝁ] formed an attribute of Σ, implying the axioms of set theory [Z] are neither complete nor consistent in α-space.

[~Ꝁ, Ꝁ]   < ≡ >   Σ

Now we'd like to extend things a bit by examining the Quantum Hypothesis [Q], which simply posits that light is a wave, as opposed to a particle.

So begin by letting S be the statement 'light is a wave' and let ~S be the statement 'light is a particle'. According to previous research, the Hausdorff-Banach-Tarski paradoxes [Ϻ] tell us the measure of a unit circle [C] is both **0** and **c** simultaneously ... depending on how the *axiom of choice* is used to generate the various constructions here. Both outcomes are consistent with Z, but lead to different results because of this axiom, which, in turn, is only made possible because of the regularity running through the randomness of the number line. We know as much already ...

regularity within R   ‹~~~›   axiom of choice

Turning to Quantum Mechanics, let W be a wave of light on C with measure **c** and now 'perform an experiment' which collapses this wave into a particle P, yielding a measure of **0** on the same circle. Evidently, such can be done according to quantum theory, but can it, really ?

If we draw on the paradoxes above as our analogy, the answer is actually *indeterminate*, since the Measure Hypothesis [Ϻ] itself is undecidable relative to Z, and thus an attribute of Σ ...

[~Ϻ, Ϻ]   < ≡ >   Σ

In other words, the Quantum Hypothesis [Q] is essentially *no different* than Ϻ and so the statements [~S, S] are consistent with Z simultaneously, which means now Q is also an attribute of Σ, by virtue of equivalency ...

[~Q, Q]   < ≡ >   Σ   (*)

Depending on 'how you perform the experiment' in α-space, via the axiom of choice, a circle can be *made* to look like either a wave of light [W] or a particle [P], without conflict, relative to Z. Both are acceptable outcomes, but because of (*) we have the following theorem ...

If Q is the Quantum Hypothesis then Q is both
incomplete and inconsistent in α-space, and
by equivalency, undecidable up to Z

Light, it seems, is *neither* a wave *nor* a particle on C until some 'experiment' or construction is carried out, according to Z, that allows us to 'make a decision' one way or the other. Without invoking the axioms of set theory, and more specifically the axiom of *choice*, it appears we can't

know *anything* about this peculiar attribute of the Godhead, and even then, with these tools, our understanding of its true nature is woefully incomplete, just as it should be ...

The β-Space Duality

' For any wave W there exists a particle P such that P induces W, and conversely.
Properties of W are inherited by P, and vice versa, because of duality '

The α-Space Equations

$$(\Lambda \circ r) \circ W \;=\; k * (\Lambda \circ r) \circ P \;+\; K$$

$$\nabla \circ W \;=\; 0 \;=\; \nabla \circ P$$

```
      |                    |
 symmetries          cons laws

Ꜧ -------------------------- Ƙ
|            \     /            |
|             \ /               |
                Z
|             / \               |
|            /     \            |
Ϻ -------------------------- Q
```

... but the axioms you believe is the reality you perceive ...

*****

By the way, the same argument above can be applied to the 'dimensions' of space and time in our world. If Ꞇ is the Space-Time Hypothesis which posits that these dimensions are continuous, as opposed to discrete, then the following theorem holds ...

If Ꞇ is the Space-Time Hypothesis then Ꞇ is both incomplete and inconsistent in α-space, and by equivalency, undecidable up to Z

Covariance and The Fundamental Theorem of Creation, Part II

In Part I, we discussed the notions of *energy* and *covariance* under coexistence, and saw that cov(Σ) vanished if and only if F was true over the manifold M ~ S(n). To wit,

> *The energy in Creation is conserved if Noether applies and conversely, with cov(Σ) = 0 <=> F, where Σ is coexistence and F some minimal energy theorem over the manifold M ~ S(n)*

Now we'd like to better understand the meaning of energy in each of the three fabrics we've studied so far, and then make an educated guess as to how the transition to γ-space is really going to occur. So let's begin with the ground state, namely ζ-space, and realize that here in a *perfectly* certain equilibrium the total energy $E_\zeta = 0$. This much we can say, without loss of generality ...

Moving over to Σ, and recalling the generalized law of h&s, one can rewrite the energy of each space according to the following decomposition,

$$\underbrace{E_\beta = E'_\beta + \Delta_\beta}_{positive} \quad , \quad \underbrace{E_\alpha = E'_\alpha + \xi}_{negative} \qquad (*)$$

where Δ_β is the incremental *positive* energy stored in β, which is used by the law of h&s to cancel *negative* energy associated with acts of harm in α, and ξ is, of course, dark energy.

As these acts of harm are erased over a long period of time in the *ordered now*, ξ ---› 0 since it is our thoughts and actions, by *way* of dark energy, that lead to harmful outcomes initially. Finally then ... with the last act of harm erased, both Δ_β and ξ vanish altogether, and so purifying α-space becomes a *zero-sum* game played out in Σ over the eons of cosmic time, with $\Delta_\beta = |\xi|$ originally. Hence the following theorem (‡) applies ...

> *Acts of harm exist in α-space if and only if ξ < 0, where ξ is dark energy*

As a natural consequence of this theorem, we offer the following definitions of dark energy on the primordial sphere, for the reader to ponder ...

> *If ζ is the primordial sphere and η the negative rest energy in ζ, then η is the sum total of all negative thoughts and actions associated with the collective consciousness in Σ, where η = ξ, up to transform*

* * * * * *

If ζ is the primordial sphere and π the positive rest energy in ζ ,
then π is the sum total of all positive thoughts and actions
associated with the collective consciousness in Σ, where
$\pi = \Delta_\beta$, up to transform

Note that by combining these definitions, $E_\zeta = 0$ *and* the positive incremental energy [Δ_β] used by the law of h&s to cancel negative energy associated with acts of harm in α .... is actually a *copy* of our positive thoughts and actions [π], regenerated by the Cross in Σ. It appears, therefore, that we are really healing ourselves, under this scenario ...

$$\underset{\pi}{\overset{\beta}{\dashrightarrow}} \quad \text{law of h\&s} \quad \underset{\eta}{\overset{\alpha}{\dashrightarrow}}$$

Since cov(Σ) = 0, the *total* energy of Σ doesn't change, from start to finish, as it goes through the purification process, and so the energy carried into the *unified* mosaic must be, necessarily,

$$E_\beta + E_\alpha = E'_\beta + E'_\alpha \qquad (\dagger)$$

In other words, when moving from Σ to γ, the energy is actually *conserved*, which means that if ~F holds in γ, the symmetries, laws *and* the action in β must recombine into a single essence, which permeates a purified α-space literally everywhere ... an essence we call the Light of God ...

```
                    γ-space
                       |
souls    <~~>    physical ring   <~~>   realms
  |                    |                   |
  |                    |                   |
 --------  Symmetries Action Laws  --------
                       |
                       |
                     Light
```

$$[\,\sim A\,,\,A\,] \quad <\equiv> \quad [\,\sim F\,,\,F\,]$$

Also, (†) implies that not a single thing will be lost in the transition ... every living being will ultimately return to the Source, the Light, pure Love ... something Juliet Nightingale was told in her NDE. And from (‡) we may deduce that purifying α-space is really up to *us*, since ξ is intimately tied to our thoughts and actions, according to the following chain ...

dark energy	‹~~~›	regularity within randomness		‹~~~›	choice
ξ		ρ	ν		c

If ξ vanishes in α it is *because* of *our* choices and conversely; either way the Law of Purification of The Void will have reached its fulfillment, which is something God would have desired in the beginning ... before anything was ...

A Message From The Light

You save, redeem and heal yourself. You always have ... you always will. You were born with the power to do so from before the beginning of the world ... so says the Light ...

Covariance and The Fundamental Theorem of Creation, Part III

In Part II we discussed in some detail the transition from Σ to γ, largely from the perspective of an observer in α-space, viz the *ordered now*. Here we'd like to repeat the exercise, but this time from the viewpoint of an observer in β-space, meaning the *simultaneous now*, and then compare notes on the two approaches.

So let's start by realizing that the Cross [Ꞡ] is actually an *energy generator* or I/O device, where the input [sufferings] exactly matches the output [energy] relative to the diagram below, with cov(Ꞡ) = cov(Ⱥ) = 0.

$$\begin{tikzcd} \text{sufferings} \arrow[r] & \text{Ꞡ} \arrow[r] \arrow[d, dash] & \text{energy} \\ & \text{Ⱥ} & \end{tikzcd}$$

In β-space the 'distance' between any two points is always infinite and so Ꞡ persists indefinitely, which means, by equivalency, the action [Ⱥ] is also persistent, much like water flowing through a pipe -- it is simply there until you shut off the supply. Thus, the incremental *positive* energy [Δ_β] used by the law of h&s to cancel acts of harm in α is really an *integral* part of Ⱥ ... and remains so until the purification process is complete. It never 'shuts off ' ... so to speak ... until it's time to flip the switch ...

As such, to an observer in β, the energy of the action isn't changing, which means $\Delta_\beta = 0$. Indeed, to this onlooker, it would seem as though $\mathrm{cov}(\beta) = 0$, implying the energy of β-space itself is *constant.* But since

$$E_\beta = E'_\beta + \Delta_\beta \ , \quad E_\alpha = E'_\alpha + \xi$$

and $\mathrm{cov}(\Sigma) = 0$ ... we may also infer $\xi = 0$, and so from the β-perspective, anyway, an observer would say α-space is *already* purified, with the energy carried into the *unified* mosaic being equal to

$$E_\beta + E_\alpha = E'_\beta + E'_\alpha$$

In other words, whether we adopt the α or the β view, the energy in transitioning from Σ to γ doesn't change; what *does* change is the way in which the transition is actually perceived. In the *ordered now* it really is a zero-sum game played out in Σ over the eons of cosmic time, with $\Delta_\beta = |\xi|$ originally, but in the *simultaneous now* the game has already been played, with $\Delta_\beta = \xi = 0$. Both views are equally valid and each has a story to tell, to be sure, but in the final analysis one has to remember ...

```
Ƕ --------------------------- Ƙ
|            \     /            |
|             \ /              |
               Z
|             / \              |
|            /     \            |
M --------------------------- Q
```

... the axioms you believe is the reality you perceive ...

ϮϮϮ ϮϮϮ

Covariance and The Fundamental Theorem of Creation, Part IV

Our recent studies on covariance in Σ relied heavily on being able to decompose the energy in α and β according to the following equations ...

$$E_\beta = E'_\beta + \Delta_\beta \quad , \quad E_\alpha = E'_\alpha + \xi \qquad (*)$$

At the same time, we know β is *perfectly random* and so, at this point, anyway, it behooves us to define more rigorously just what is meant by (*) when talking about energy.

Let us start, then, by noting that in canonical form

$$\beta = \Lambda' \circ \alpha$$

and since $\nabla \circ \Lambda' = 0$, the covariant derivative in product form may be written as

$$\nabla \circ \beta = \nabla \circ (\Lambda' \circ \alpha) = \Lambda' \circ (\nabla \circ \alpha) \oplus \alpha \circ (\nabla \circ \Lambda') = \Lambda' \circ (\nabla \circ \alpha)$$

To compute the energy E_β now, simply integrate the expression above and allow the operators to commute where applicable; viz ...

$$E_\beta = \int \nabla \circ \beta = \int \Lambda' \circ (\nabla \circ \alpha) = \Lambda' \circ \int \nabla \circ \alpha = \Lambda' \circ E_\alpha$$

or, in the alternative, merely integrate by parts and note again that $\nabla \circ \Lambda' = 0$ ...

$$E_\beta = \int \nabla \circ \beta = \int \Lambda' \circ (\nabla \circ \alpha) = \Lambda' \circ E_\alpha - \int E_\alpha \circ (\nabla \circ \Lambda')$$

Ergo, the following definition for energy holds in Σ, which we'll use going forward ...

If E_β and E_α are the energies associated with β and α in Σ, then $E_\beta = \Lambda' \circ E_\alpha$ and $E_\alpha = \Lambda \circ E_\beta$, where Λ and Λ' are the invertible transform pair under coexistence

As an application of this definition, note that

$$\nabla \circ E_\beta = \nabla \circ (\Lambda' \circ E_\alpha) = \Lambda' \circ (\nabla \circ E_\alpha)$$

and so cov(β) = 0 if and only if cov(α) = 0. In other words, to an observer in β-space, who lives in a world of *persistence*, the energy in *both* domains will appear to be *constant*, which means

$$\Delta_\beta = \xi = 0 \qquad (**)$$

from this vantage point, with $E_\beta = E'_\beta$ and $E_\alpha = E'_\alpha$.

* * * * * *

Thus, when thinking back to *souls* and the Law of Leaving The Godhead, for example, which we discussed in the original essay, it is more than likely this law is just a technicality -- souls probably didn't migrate from β to α initially, for if they had, it would actually violate conservation theorems discussed here, at least from the β perspective. This assumes, of course, souls have non-zero, measurable energy in β.

And since (**) holds, *all* events R in α have *already* taken place, relative to β, implying $\Lambda' \circ R$ exists, and indeed, becomes a reasonable definition for the *simultaneous now,* when considering the Σ to γ transition.

* * * * * *

As a second application of the definition, note that for an observer in α-space ... with E'_β and E'_α remaining constant,

$$\nabla \circ E_\beta = \nabla \circ (\Lambda' \circ E_\alpha) = \Lambda' \circ (\nabla \circ E_\alpha) = \Lambda' \circ (\nabla \circ \xi) = \nabla \circ \Delta_\beta$$

and so

$$\Delta_\beta = \int \nabla \circ \Delta_\beta = \int \Lambda' \circ (\nabla \circ \xi) = \Lambda' \circ \int \nabla \circ \xi = \Lambda' \circ \xi$$

Thus, dark energy [ξ] and the incremental *positive* energy [Δ_β] used by the law of h&s to cancel acts of harm in α, are tied together through the [Λ , Λ′] transform pair, just as one would expect. An onlooker in α-space really believes both energies exist *and* are changing because of this connection ... even though β-space itself is perfectly uncertain ...

Notice too,

$$\Lambda' \circ \xi = -\xi \quad , \quad \Lambda \circ \Delta_\beta = -\Delta_\beta$$

making both Δ_β and ξ *invariant* with respect to [Λ , Λ′]. Any time this sort of thing happens it should be considered significant, and perhaps a sign or tip that we may be on the right track, at least in terms of our understanding of these matters ...

* * * * * *

Along these lines, and recalling the general form of the laws in α, viz ...

$$(\Lambda \circ r) \circ (\Lambda \circ U) = (\Lambda \circ r) \circ (\Lambda \circ V) + K$$

it may be more correct to say

$$\Lambda' \circ K = -K \qquad (\dagger)$$

since K is a representation of ξ with $\mathrm{cov}(K) \approx 0$. Originally, the right-hand side of (†) mapped to zero in β, which seems reasonable, but because of the invariant nature of both Δ_β and ξ ... the expression above might be the better choice. Either way, the general form of the laws in our domain doesn't change ...

* * * * * *

Finally, what are we to do with God ? I have always wanted to know who God is, but sadly, it's looking more and more like we'll never know. Indeed, the following equivalency diagram is pretty much the last nail in the coffin, at least as far as I'm concerned ...

$$\begin{tikzcd} \Sigma \ \ldots \arrow[r, phantom, "<\equiv>"] \arrow[d, dash] & \ldots \ F \arrow[d, dash] \\ {[\ \Lambda\ ,\ \Lambda'\ ]} \arrow[r, phantom, "<\equiv>"] & {[\ \sim A\ ,\ A\ ]} \end{tikzcd}$$

From this diagram we can see that God, at a minimum, must be more than *just* the Noetherian components found in β, and indeed, the minute we step away from Σ altogether, and consider ζ or γ, all bets are off. Undecidability reigns supreme in these fabrics and our knowledge of God diminishes significantly.

Even in Σ it seems we are forced to choose between the *ordered* and *simultaneous* frames, which only adds to the confusion, and now, using conservation theorems, are learning that an observer in α could very well see things quite differently than an observer in β. Maybe in the end it's better to go fishing ... and find someone to love ...

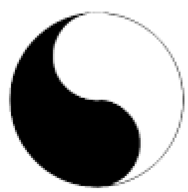
Covariance and The Fundamental Theorem of Creation, Part V

In Part II of this series we came up with a rather interesting set of definitions for the *primordial* sphere. Here we'd like to recall that interpretation and then compare notes with the *yin-yang* symbol found in Eastern Traditions. This is a rather brief note, left for the reader to ponder ... but at the same time, one cannot help but see the striking parallels between the two ...

If ζ is the primordial sphere and η the negative rest energy in ζ ,
then η is the sum total of all negative thoughts and actions
associated with the collective consciousness in Σ, where
$\eta = \xi$, up to transform

* * * * * *

If ζ is the primordial sphere and π the positive rest energy in ζ ,
then π is the sum total of all positive thoughts and actions
associated with the collective consciousness in Σ, where
$\pi = \Delta\beta$, up to transform



Yin and yang can be thought of as complementary, rather than opposing forces, that interact to form a dynamic system in which the whole is greater than the assembled parts.

Everything has both yin and yang aspects; for example, shadow cannot exist without light. Either of the two major aspects may manifest more strongly in a particular object, depending on the criterion of the observation.

The yin-yang symbol shows a balance between two opposites, with a portion of the opposite element in each section ...

$$\pi + \eta = 0$$

Covariance and The Fundamental Theorem of Creation, Part VI

In previous studies within this series we learned that ultimately Δ_β and ξ will vanish once α-space is purified. For the observer in β-space this has already happened, as we know, but in the *ordered now* $\xi \dashrightarrow 0$ over the eons of what I call cosmic time. Either way, both observers agree on the endpoint, namely,

$$\Delta_\beta = \xi = 0 \qquad (*)$$

So what happens next ? Do we really transition into a *unified* mosaic [γ] where the α and β domains have merged in some sense of the word, or do these two spheres remain separate and apart from each other, literally forever ?

To answer the question, note that if α is now *perfectly* uncertain, because of (*), then from previous theorems,

$$\Delta(p, q) = \infty \quad \text{for all } p, q \;\varepsilon\; \alpha$$

implying the distance between any two points in the *physical* ring must be infinite. An absurdity, to say the least. On the other hand, (*) tells us α-space can no longer be *imperfectly* uncertain, and so we find ourselves in a contradictory position where neither attribute describes α adequately. The α-domain, as things stand, is now in an *untenable* state.

Such a conflict can *only* be resolved by shifting gears yet again, and transitioning into a fabric [γ] which is *undecidable* up to attribute, according to the following chain ...

$$\sim\!A\,,\,A \quad <\equiv> \quad \sim\!F \quad <\equiv> \quad \gamma = \alpha + \beta \qquad (\dagger)$$

By doing so all conflicts are removed in the sense that they are overshadowed by *undecidability*, thereby making it impossible to say anything at all about γ. About all we can say, because of (†), is that necessarily ~F *must* hold because F and Σ are equivalent to one another. Without the ~F condition, the indeterminate nature of γ would *never* emerge, leaving us with the kind of contradiction we alluded to earlier.

$$\begin{array}{ccc}
\Sigma \;\ldots & <\equiv> & \ldots\; F \\
| & & | \\
| & & | \\
| & & | \\
[\,\Lambda\,,\,\Lambda'\,] & <\equiv> & [\,\sim\!A\,,\,A\,]
\end{array}$$

~F is equivalent to breaking the boundary
that divides α from β under coexistence

Thus, the Σ to γ progression is not an arbitrary result. It must happen if we are to resolve any inconsistencies here, which means, in turn, our original interpretation of the *unified* mosaic ... is probably correct. When Christ made the statement shown below, from the Book of Matthew, he wasn't kidding ...

```
                      γ-space
                         |
souls    <~~>    physical ring   <~~>   realms
  |                      |                  |
  |                      |                  |
  --------  Symmetries Action Laws  --------
                         |
                         |
                       Light
```

'In truth I tell you, till heaven and earth disappear, not one dot, not one little stroke, is to disappear from the Law until all its purpose is achieved ...' [Matthew, 5th chapter]

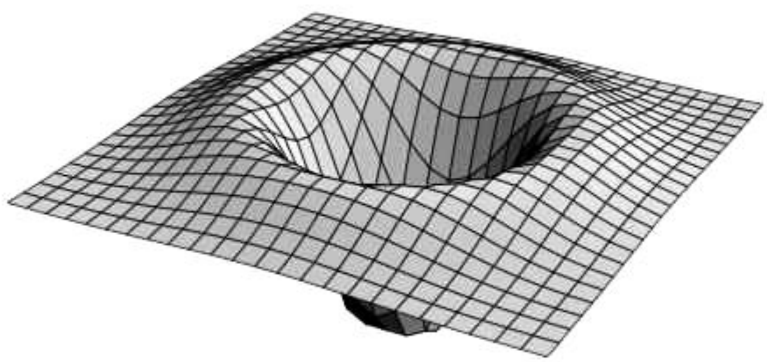
Searching For That Elusive Dark Energy

In previous research connected to general relativity, we saw that the broader form for the equations could be written as

$$(\Lambda \circ r) \circ C^{u,v} \approx k * (\Lambda \circ r) \circ T^{u,v} + K \qquad (*)$$

where (r, r*) is the transform pair taking us between symmetries and laws in β-space, and K is some tensor associated with dark energy, for which cov(K) ≈ 0. Here we'd like to learn a little more about this dark energy tensor, by developing estimators for both $\Lambda \circ r$ and K in α-space, and then drawing some rather interesting conclusions.

So let's begin by considering the large family of *integral transforms* that include Laplace, Fourier, Mellin, Z, and even the wavelets. Such transforms lead to ripples in the space-time fabric, as shown below, and could well serve as reasonable approximations for $\Lambda \circ r$, among other things.



Moreover, they are, mathematically speaking, roughly equivalent to one another, and so, without loss of generality, we will restrict our analysis to £, the Laplace transform. As well, we'll define K to be the 'tensor' $\lambda(s)g^{u,v}$, in keeping with Einstein's original approach, where s is any point in the α-domain. Writing $G^{u,v} = C^{u,v} - kT^{u,v}$, and defining £' as the Laplace inverse, (*) now becomes, for any point t in α,

$$G^{u,v} \approx £' \circ \lambda(s)g^{u,v} = \kappa\int_{\gamma} e^{st}\,\lambda(s)g^{u,v}\,ds \qquad (\dagger)$$

where κ is some constant, and γ denotes some form of contour integration, according to Bromwich protocol, that envelops singularities associated with K(s).

Either $G^{u,v} \equiv 0$ or it isn't. If so, there is *no* dark energy in this case, and indeed, this is something that is confirmed by setting λ(s) = 0 in (†). On the other hand, if $G^{u,v} \neq 0$, then complex variable theory tells us λ(s) must have *poles* or *singularities* where it becomes infinitely large, otherwise the integration in (†) would always be zero.

Today, there is no agreement whatsoever on just what the density of dark energy might actually be. Quantum field theories, for example, predict a value which is 10^{120} times *greater* than current measurements, and this discrepancy has been called “the worst theoretical prediction in the history of physics!” In fact, without using a process called 'renormalization', these theories go even further, and calculate a value for dark energy which is literally infinite.

But in some sense, this disparity gives us a bit of comfort, in so much as the analysis above tells us $\lambda(s)$ is *not* well-behaved, assuming we can use integrable transforms to model $\Lambda \circ r$. Indeed, if you think about it a little more deeply, dark energy, in our universe alone, has to come from somewhere, and so perhaps the singular nature of $\lambda(s)$ points to a space-time fabric which is full of holes and oddities, literally everywhere … just as it should be …

OTHER CONSIDERATIONS

In their original form, the field equations of general relativity include the term $\Lambda g^{u,v}$, where Λ is the cosmological constant. Including $g^{u,v}$ in this term seems to be for housekeeping purposes only, in so much as it allows us to separate one field equation from another, and at the same time, preserve the tensorial nature of these expressions across the board. In other words, $g^{u,v}$ functions as more of an appendage here.

If we do the same thing with respect to (*) above, by referencing K with Λ, then (†) becomes, for any point t in α,

$$G^{u,v} \approx \kappa g^{u,v} \int_{\gamma} e^{st} \lambda(s) ds \qquad (‡)$$

where γ denotes some form of contour integration that envelops singularities associated with $\lambda(s)$, the new estimate for K.

Such an approach leads to consistency, especially when considering other dualities that might incorporate dark energy, and in this particular case … … makes solving the equations for $g^{u,v}$ in (‡) above, much easier.

Indeed, you could ask just how one goes about solving (†) as it stands, for in that case, it seems as though foreknowledge of $g^{u,v}$ is required at different points in α-space … unless, of course, you treat $g^{u,v}$ as an appendage to the dark energy component …

Moving Between The Laws In α-Space

In a previous research note … titled *Geometry, Groups and The Fundamental Theorem of Creation* [see pp 144-145], we saw that laws were connected to one another in α-space according to the following equation:

$$\Lambda \circ Ɗ' = \Lambda \circ (\tau' \circ Ɗ) = (\Lambda \circ \tau') \circ (\Lambda \circ Ɗ)$$

Recall the generator τ' can be written as $r \circ \omega' \circ r^*$, and since Λ is multiplicative, our impression of τ' in α-space now becomes

$$\Lambda \circ \tau' = (\Lambda \circ r) \circ (\Lambda \circ \omega') \circ (\Lambda \circ r^*) \ ,$$

where (r, r^*) takes us between symmetries and laws in β-space, and (ω, ω') moves us from one symmetry to another here.

Thus, when moving *between* laws in our domain, the expression above tells us how to do this, *without* any consideration for dark energy [K]. We are simply shifting from one law to another, and once we have the new law in hand, can append K to the corresponding equation(s) and interpret things, accordingly.

In the last addendum we did just this for the field equations of general relativity, by providing estimators for $\Lambda \circ r$ and K …. and saw that the Bromwich integral could well serve as an approximation for the *inverse* of $\Lambda \circ r$, when considering dark energy and its singularities [note that the inverse of $\Lambda \circ r$ really is $\Lambda \circ r^*$].

When rotating between the laws, however, we *still* need to be consistent, in so much as estimators for $\Lambda \circ r$ cannot be one thing for a rotation, without respect to K, and another when incorporating K into any particular equation. In this sense, the Bromwich integral has the interesting property of becoming an *inverse* Fourier transform in the *absence* of singularities, and so, almost magically, we don't have to do anything here. Our original estimate for $\Lambda \circ r$, namely the Laplace transform, actually plays a *dual* role when considering its inverse; to wit, a contour integration where singularities are present, otherwise Fourier.

Armed with this encouraging knowledge, we can now provide a guesstimate for $\Lambda \circ \tau'$, realizing too that it is always going to be difficult to approximate $\Lambda \circ \omega'$ [Ɯ], no matter the approach. Broadly speaking, then, and denoting F the Fourier transform under discussion (with F' its inverse), one has

$$\Lambda \circ \tau' \approx F \circ \text{Ɯ} \circ F'$$

Interestingly enough, later on in life, Einstein felt that the physical laws of creation might be unified by looking at the various components of a Fourier transform. In light of this, perhaps we should be satisfied and feel somewhat confident that we may be on the right track after all ...

Moving Between The Laws In α-Space, Part II

In Part I, we discussed the theoretical form of the generator $\Lambda \circ \tau'$, which takes us between laws in α-space; namely

$$\Lambda \circ \tau' \;=\; (\Lambda \circ r) \circ (\Lambda \circ \omega') \circ (\Lambda \circ r^{*}) \,,$$

and saw that it could be approximated by

$$\Lambda \circ \tau' \;\approx\; F \circ \text{ɰ} \circ F' \qquad (\dagger)$$

where F is a Fourier transform, and F' its inverse, derived from the Bromwich integral. Recall too that ɰ is an estimator for $\Lambda \circ \omega'$, where the latter is our impression of ω', which moves us from one symmetry [S] to another [S'] in β-space. Given this setup, how then should we interpret the laws, from a geometric perspective, in our reality ?

To answer the question, first note, from the last addendum, that laws in β-space [Ɗ] are connected to one another in α-space via the following relationship …

$$\Lambda \circ \text{Ɗ}' \;=\; \Lambda \circ (\tau' \circ \text{Ɗ}) \;=\; (\Lambda \circ \tau') \circ (\Lambda \circ \text{Ɗ}) \,,$$

and so from (†) we may infer that $\Lambda \circ$ Ɗ can be thought of as a ‘wave-like’ pattern in our domain, which can be traced back to a ‘pulse-like’ symmetry [$\Lambda \circ S$], after operating on it with F'. Once we have the pulse train for $\Lambda \circ S$ in hand, ɰ maps it to $\Lambda \circ S'$ [another ‘pulse-like’ symmetry in α-space], and finally, F maps $\Lambda \circ S'$ to the ‘wave-like’ pattern for $\Lambda \circ$ Ɗ'.

$$\begin{array}{ccccccccc}
 & & & \Lambda \circ S & & \Lambda \circ S' & & & \\
 & & & | & & | & & & \\
\Lambda \circ \text{Ɗ} = \text{waves} & \sim\sim> & & \text{pulses} & \sim\sim> & \text{pulses}' & \sim\sim> & & \text{waves}' = \Lambda \circ \text{Ɗ}' \\
 & & & & & & & & \\
 & F' & & & \text{ɰ} & & F & &
\end{array}$$

For an illustration of what a ‘pulse-like’ symmetry might look like in our domain, refer back to page 135. Also note that these pulses and waves will be distorted somewhat, by the presence of dark energy in α-space, because of the commingling effect.

Can we search for echoes in the antimatter world of the Godhead that might tell us more about the nature of these symmetries in our reality, and thus, the laws themselves ? If so, it would be a significant step forward in terms of understanding how the Godhead thinks and communicates with its constituent parts, and more generally, could lead to a universal classification scheme in our domain, as we pursue our relentless quest for a grand unification theory of all that is …

Time will tell, I suppose, if such a thing can be done ... …

A Brief Note On The Primordial Transforms

Not much has been said about the transforms mapping from ζ to Σ so in this short note we're going to offer just a little more information and then leave it with the reader. Recall these transforms behave according to the diagram shown below, but one could ask ... what do they really do ?

$$\Psi : \zeta \dashrightarrow \beta \quad \cap \quad \Omega : \zeta \dashrightarrow \alpha$$

or

$$\Psi : \zeta \dashrightarrow \beta \quad \cap \quad \Lambda : \beta \dashrightarrow \alpha$$

To answer the question let us first note that both Ψ and Ω are intimately tied to Λ through *convolution* and so are undecidable up to attribute. In fact, these convolutions are $\Lambda' \circ \Omega$ and $\Lambda \circ \Psi$ respectively, and thus we have the following theorem ...

If Ψ and Ω are the primordial transforms, then
both are undecidable up to attribute with
$\Psi = \Lambda' \circ \Omega$ *and* $\Omega = \Lambda \circ \Psi$

We can see too, from the theorem, that the covariant derivatives of Ψ and Ω are also connected via the [Λ , Λ′] transform pair, and indeed, cov(Ψ) = 0 if and only if cov(Ω) = 0.

Thinking back to our series on *Stones and Ponds*, we learned that Σ could be characterized as separate geometries [Γ], along with a distance metric, according to the following chain ...

$$\Sigma \quad <\equiv> \quad [\, \sim\Gamma , \Gamma \,] \quad <\equiv> \quad [\, \sim\mu , \mu \,] \quad <\equiv> \quad F$$

Thus, if we view the action [A̸] as one giant, highly efficient piece of machine code, and [Ψ , Ω] as sub-routines callable by A̸, then it is possible these routines were responsible for *hollowing out* a space for both α and β by creating these geometries, and endowing each with a measure [Δ] of some kind.

Once this was done, both α and β could be populated with the elemental building blocks we have become familiar with, via the algorithms in A̸, and alas, a creation is born according to Noetherian principles.

There is no way to know if this actually happened, but common sense tells us you can't put furniture into a room before the room itself is built. Since the primordial sphere is largely a structure comprised of *thought energy,* which contains all of the plans for Creation in blueprint form, that energy, and the corresponding plans, need to be translated into something *concrete* in order to make the cosmos an eventual reality. In this sense, perhaps Ψ and Ω played a pivotal role ...

$$\zeta \quad \dashrightarrow \quad \Sigma$$

thought *reality*

A Brief Note On The Unified Transforms

In this short note we wish to extend the idea of transforms to the *unified* mosaic, so a complete picture of all three fabrics emerges, simultaneously. To wit, ...

$$\begin{tikzcd}
\zeta \arrow[r, "\Omega"] \arrow[d, dash] & \alpha \arrow[r, "\omega"] \arrow[d, shift left, "\Lambda'"] & \gamma \arrow[d, dash] \\
\zeta \arrow[r, "\Psi"'] & \beta \arrow[r, "\pi"'] \arrow[u, shift left, "\Lambda"] & \gamma
\end{tikzcd}$$

In the diagram above, the unified transforms [ω, π] take us from α and β to γ, respectively, and so

$$\pi = \omega \circ \Lambda \;,\; \omega = \pi \circ \Lambda'$$

Thus, the energy carried into γ from each domain in Σ is

$$\pi \circ E'_{\beta} = (\omega \circ \Lambda) \circ E'_{\beta} = \omega \circ (\Lambda \circ E'_{\beta}) = \omega \circ E'_{\alpha}$$

and similarly,

$$\Omega' \circ E'_{\alpha} = \Psi' \circ E'_{\beta} \qquad (\dagger)$$

In other words, for *any* unified transform pair [ω, π], the energy mapped to γ from *either* α *or* β is always the *same*, which means, necessarily, we must have

$$E'_{\beta} = E'_{\alpha} = 0 \qquad (*)$$

and ditto in reverse for all primordial transform pairs [Ω, Ψ].

To see this more clearly note that any *canonical* transform ought to be *independent* of the object it's acting on and so, for example, if $E'_{\beta} = E'_{\alpha}$ for any value E'_{β} we care to choose, then $\pi = \omega$ holds. On the other hand, if $E'_{\beta} = -E'_{\alpha}$ for arbitrary E'_{β}, then $\pi = -\omega$, and thus we have a dependency conflict, in so much as the transforms, now, are bound by the energy states in α and β. Such a conflict can only be resolved if we assume (*) ...

* * * * * *

To prove the second result observe that Λ and Λ' are both the *negative identity* operator and so,

$$E'_{\alpha} = \Lambda \circ E'_{\beta} = -1 \circ E'_{\beta} = -E'_{\beta}$$

Continuing,

$$\omega' \ = \ \Lambda \circ \pi' \ = \ -1 \circ \pi' \ = \ -\pi'$$

Ergo $\pi = -\omega$, up to reflection, but this is not an acceptable solution because among other things, Λ and Λ' are no longer *undecidable* up to attribute ... both in this case and in the case where $\pi = \omega$.

* * * * * *

Thus $E_\gamma = 0$ and by extension, $E_\Sigma = E_\zeta = 0$, and hence one has the following theorem ...

For any fabric in Creation the total energy in that fabric is always 0. That is to say,

$$E_\zeta \ = \ E_\Sigma \ = \ E_\gamma \ = \ 0$$

Such a result makes eminent sense since all along we felt Creation, no matter the fabric, was built on a manifold $M \sim S(n)$. Ergo, there can be no leaks -- energy can be rearranged on M ... but it cannot be destroyed or manufactured out of nothing ...

$$\zeta \quad \langle\text{---}\rangle \quad \Sigma \quad \langle\text{---}\rangle \quad \gamma$$

thought *reality* *exotic*

And, this theorem only bolsters our view that *no* living being will ever be destroyed as a result of the purification process. In good time, each and every being will return to the Source, the Light, pure Love ... just as Juliet Nightingale was told in her NDE ...

* * * * * *

Finally, if Ω and Ψ do play a role in mapping the *negative* and *positive* rest energies from ζ to Σ, at least from the α-perspective, with

$$\Omega \circ \vartheta- \ = \ \xi \quad , \quad \Psi \circ \vartheta+ \ = \ \Delta_\beta$$

then

$$\vartheta- \ = \ \Omega' \circ \xi \ = \ (\Psi' \circ \Lambda') \circ \xi \ = \ \Psi' \circ (\Lambda' \circ \xi) \ = \ \Psi' \circ \Delta_\beta \ = \ \vartheta+$$

This seems to imply these energies are indeed *mirror images* of one another, but with opposite sign, and indeed, since ζ is a *perfectly* certain structure or singularity, such a conclusion would make sense ... …

Notice too from (†) the *undifferentiated* energy of God in ζ is 0, implying that this energy somehow *commingles* with $\vartheta-$ and $\vartheta+$, whilst at the same time remaining separate and apart from the two …

The Case For Negative Mass In α-Space

Our recent studies on the *unified* transforms led us to the conclusion that both E'_β and E'_α are 0, no matter the frame of reference. In particular, $E'_\alpha = 0$, which means the energy of α-space itself must be zero, excluding the dark energy component [ξ]. In this short note, we wish to investigate the case for negative mass in α-space, and see if it is even feasible.

Suppose now we let $\mathfrak{m}_+$ be the positive mass associated with α-space, and $\mathfrak{m}_-$ the negative mass, if such exists. According to Einstein,

$$E'_\alpha = (\mathfrak{m}_- + \mathfrak{m}_+)c^2 = 0 \qquad (*)$$

implying that negative mass *must* reside within the α-domain, which exactly *balances* its positive counterpart. Indeed, (*) can only be true under this condition.

Oddly enough, recent research on this subject has shown solutions to Einstein's general theory of relativity also exist, that allow for negative mass, without breaking any essential assumptions. See, for example, the work of Mbarek and Paranjape, where they treat $\mathfrak{m}_-$ as a perfect fluid, which leads to a reasonable Schwarzschild metric over the de Sitter manifold.

And then there are those wormholes. Today, theoretical physicists can actually show how negative mass could be used to create these strange objects, providing the so-called *energy condition* is relaxed. This condition simply asserts that only reasonable states of matter and non-gravitational fields are admissible when using general relativity.

If (*) is true then we should expect to see negative mass everywhere throughout both our universe and α-space, at large. In turn, this will lead to many new ideas and even bizarre notions that, up to now anyway, would seem more like science fiction.

But these conclusions rely heavily on our interpretation of the unified transforms, discussed in previous research; namely, that

$$\pi \circ E'_\beta = (\omega \circ \Lambda) \circ E'_\beta = \omega \circ (\Lambda \circ E'_\beta) = \omega \circ E'_\alpha$$

for all possible [π, ω]. Our best understanding of these identities leads us to believe E'_β and E'_α must always be zero, no matter where you are standing in Σ …

$$E'_\beta = E'_\alpha = 0$$

The Case For Negative Mass In α-Space, Part II

In Part I, we put forward the idea of negative mass in α-space, excluding the dark energy component [ξ], and saw that

$$E'_{\alpha} = (\mathfrak{m}_{-} + \mathfrak{m}_{+})c^2 = 0$$

Here we'd like to explore the *energy-mass* connection a little more, according to the following duality, which [we'll assume] finds its origins in β-space.

For any energy E there exists a mass ɱ such that ɱ induces E, and conversely.
Properties of E are inherited by ɱ, and vice versa, because of duality.

Previous theorems now tell us the corresponding equations for *energy-mass* in α-space take the form

$$(\Lambda \circ r) \circ E = k * (\Lambda \circ r) \circ \mathfrak{m} + K \qquad (\dagger)$$

where $\Lambda \circ r$ is our impression of the transform [r], which moves us between symmetries and laws in β-space, k is a numerical scaling constant, and K is a density function of sorts, associated with dark energy [ξ]. Furthermore, these theorems imply the covariant derivative of both E and ɱ is zero in α-space,

$$\nabla \circ E = 0 = \nabla \circ \mathfrak{m}$$

suggesting, in turn, conservation laws with respect to this coupling.

And along these lines, because we are dealing with Noetherian principles throughout, one can say that symmetries associated with E imply conservation laws in ɱ, and similarly, symmetries associated with ɱ imply conservation laws in E.

In the case where K is 0, there is no dark energy here, and (†) reduces to the familiar Einstein equation, with $k = c^2$, where c is the speed of light.

$$E = \mathfrak{m}c^2$$

In the case where $K \neq 0$, we know from previous studies this density function very likely contains singularities, and so, using the Bromwich integral as our estimator for the inverse of $\Lambda \circ r$ … (†) now becomes, for any point t in α,

$$E \approx \mathfrak{m}c^2 + \kappa \int_{\gamma} e^{st} K(s)ds \qquad (*)$$

where κ is some constant, and γ denotes some form of contour integration, in multiple dimensions, that envelops any singularities associated with K(s).

Now let us 'convert' the rightmost term in (*) to a 'mass equivalent', at least mathematically speaking, by doing the following,

$$[\mathfrak{m}_\xi]c^2 \;=\; \kappa\int_\gamma e^{st}K(s)ds$$

where $\mathfrak{m}_\xi$ can be thought of as a 'special mass' associated with dark energy. Then for any point t in α, we have …

$$E \;\approx\; (\mathfrak{m} \;+\; \mathfrak{m}_\xi)c^2 \quad (\ddagger)$$

which is the covariant form of the *energy-mass* equation in our domain, that incorporates dark energy, and is a reflection of the corresponding energy-mass duality found in the antimatter world of the Godhead; that is to say, β-space.

Is Einstein's famous equation just an 'accidental discovery' or can it be traced back to fundamental symmetries and laws in the β-world, where all things began ? I guess it depends on your philosophical view here, but in the end, physics could well be the study and pursuit of all things spiritual …

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'All physical things are first formed in the spiritual world … including laws …'

---from the NDE research, Emanuel Swedenborg, et al ...

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OTHER CONSIDERATIONS

One can apply de Broglie's law to (‡) above by imagining a *single* particle, such as a photon, situated at a point t in α (or a train of photons passing through t). Then at this point,

$$E \;=\; h\nu \;\approx\; (\mathfrak{m} \;+\; \mathfrak{m}_\xi)c^2$$

which is the extended form of the law that incorporates dark energy, where h is Planck's constant, $\mathfrak{m}$ is the 'mass' of the photon, and ν its frequency. Note that if there is *no* dark energy you recover the classical expression for de Broglie, and if there *is*, it seems to have an influence on ν, all else being equal …

The Wave-Particle Duality, General Relativity and Covariance

In our *Stones and Ponds* section [pp 155-6], we discussed the Quantum Hypothesis and formulated an expression for the wave-particle duality, according to the following principle:

The β-Space Duality

' For any wave W there exists a particle P such that P induces W, and conversely. Properties of W are inherited by P, and vice versa, because of duality '

The α-Space Equations

$$(\Lambda \circ r) \circ W = k * (\Lambda \circ r) \circ P + K$$

$$\underbrace{\nabla \circ W}_{\text{symmetries}} = 0 = \underbrace{\nabla \circ P}_{\text{cons laws}}$$

Here, as in all other dualities, $\Lambda \circ r$ is our impression of the transform [r], which takes us between symmetries and laws in β-space, k is a 'bridging connector', and K is a density function of sorts, associated with dark energy. As well, previous theorems tell us the covariant derivative [∇] of both W and P is always 0, however we wish to invoke the notion of covariance for various representations of 'wave' and 'particle'.

But is there any evidence to support the α-space equations above, thus giving us some confidence that they really do find their origins in the antimatter world of the Godhead [β-space] ? The answer is actually yes.

Some years ago, the physicist Christopher Hill gave a talk on symmetry in physics and noted that the law of conservation of charge, with respect to an electron, could be traced back to symmetries in the corresponding wave (typically a phase shift or translation). For a long time, he says, the actual symmetry wasn't known or understood, but today that is no longer the case.

And indeed, this is what the α-space equations above are actually saying, in the broader sense, when considering their behavior over a manifold M ~ S(n) in creation:

> 'for *any* wave-particle duality … symmetries associated with conserved currents in W imply conservation laws in P .. and conversely .. symmetries associated with conserved currents in P imply conservation laws in W'

Now let us turn our attention to general relativity. Here the broader form of the equations is …

$$(\Lambda \circ r) \circ C^{u,v} \approx k * (\Lambda \circ r) \circ T^{u,v} + K$$

$$\nabla_u C^{u,v} = 0 = \nabla_u T^{u,v} \qquad (\dagger)$$

where $C^{u,v}$ is the Einstein curvature tensor, $T^{u,v}$ the relativistic stress tensor, and ∇_u the covariant derivative in u or v, indicating a sum over indices. Given this setup, how then are we to interpret (†) above ?

To answer the question, imagine an arbitrarily small, four-dimensional space-time cube or block situated at a *fixed* point p in α-space, whose volume is *not* changing. As material flows through this cube, say *into* the block along a *spatial* direction, and *out* of the block along the *temporal* direction, the block's shape is going to change, as its interfaces become deformed because of the flow. An *inward* flow will naturally cause an inward deformation of the corresponding interface(s), and just the reverse, in the case of an *outward* flow.

However, because the volume of the cube is *not* changing at p, flow in *must* match flow out, and this is precisely what the right-hand side of (†) is saying. The stress tensor $T^{u,v}$ exhibits covariance for both momentum and energy, across spatial and temporal directions, irrespective of the cube's shape, *because* the volume of the block at p is constant.

$$\nabla_u T^{u,v} = 0$$

Now let's deal with the left-hand side of (†). As material flows *into* the cube along a spatial direction, say, the corresponding interface suffers an *inward* deformation, which means the cube itself sees its volume *decrease* here – an effect which is exactly balanced by material *leaving* the block along the temporal direction … … where the corresponding interface suffers an *outward* deformation, leading to a volume *increase* here. In other words … 'sectional volume' lost along one interface, due to inward flow, is exactly *matched* by 'sectional volume' gained along another, due to outward flow.

Although the left-hand side of (†) references a curvature tensor, it can be thought of as a metric which measures the change in *sectional* volumes of a cube, as material moves in and out of the block along different directions.

To see this more clearly, let us move over into *differential* notation, and multiply the left-hand side of (†) by the volume of the block; namely, ΔxΔyΔzΔt. With u = 1 (the x-axis) and v travelling between 1 and 4, one has

$$\underbrace{\Delta C^{1,1}\Delta y\Delta z\Delta t}_{\substack{\text{sectional vol}\\ \text{across I/F 1}}} + \underbrace{\Delta C^{1,2}\Delta x\Delta z\Delta t}_{\substack{\text{sectional vol}\\ \text{across I/F 2}}} + \underbrace{\Delta C^{1,3}\Delta x\Delta y\Delta t}_{\substack{\text{sectional vol}\\ \text{across I/F 3}}} + \underbrace{\Delta C^{1,4}\Delta x\Delta y\Delta z}_{\substack{\text{sectional vol}\\ \text{across I/F 4}}} = 0$$

since the sectional volumes exist and are unique for
any given flow, the same must be true of $C^{u,v}$

where $\Delta C^{u,v}$ is to be interpreted as the covariant differential along a particular direction. It is these pieces in the summation above that represent the various sectional volumes, when considering material flow through the cube, in the primary direction of u, where $1 \le u \le 4$.

Thus we have a mathematical object which exhibits covariance, and indeed, it is the *only* contracted form of the Riemann curvature tensor that does this *uniquely*, after factoring in a shadow term [its algebraic description is most easily discovered by studying Riemannian properties of the 2-sphere].

Writing $C^{u,v} = R^{u,v} - \frac{1}{2} R\, g^{u,v}$, where $R^{u,v}$ is the Ricci tensor and R the Ricci scalar … one has the following, remarkable identity …

$$\nabla_u C^{u,v} = 0$$

Thus, general relativity is really an accounting law, where the books are exactly balanced by comparing the law of conservation of 'sectional volumes' with the law of conservation of momentum and energy, when examining material flow through an arbitrarily small space-time cube. In my opinion, this is the simplest way to understand the curvature-stress duality, in its broadest sense.

And finally, because we are dealing with Noetherian principles throughout, one can say that symmetries associated with conserved currents in $C^{u,v}$ … imply conservation laws in $T^{u,v}$, and conversely, symmetries associated with conserved currents in $T^{u,v}$ … imply conservation laws in $C^{u,v}$ … a general notion that is true for all foundational dualities …

The Wave-Particle Duality and The Riemann Hypothesis

The Riemann Hypothesis is a long-standing problem in mathematics, which seeks to understand just where the *non-trivial* zeroes of the Riemann zeta function [$\zeta(s)$] lie in the critical strip of the complex plane $\mathbb{C}$. The critical strip S is the entire space for which $0 < Re(z) < 1$, where z is any complex number, and the critical line ϒ is defined to be $Re(z) = 1/2$.

Here we are going to offer our thoughts on this conjecture, by drawing parallels to the wave-particle duality, and then put forth some interesting conclusions. So let's begin by recalling Riemann's functional equation in $\mathbb{C}$; to wit,

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin(\pi s/2) \qquad (*)$$

where $\Gamma(s)$ is the gamma function.

We can see from (*) that there are two distinct components which form a 'wave-particle' duality of sorts, with $W = \sin(\pi s/2)$ playing the role of the wave, and $P = \zeta(s)$ the particle. Previous studies now tell us that

> 'for *any* wave-particle duality … symmetries associated with conserved currents in W imply conservation laws in P .. and conversely .. symmetries associated with conserved currents in P imply conservation laws in W'

and so, for example, the *continuous* set T of *simple* translations s ---› s + δ, of the wave function W, lead to conservation laws in P. These conservation laws, or conserved currents in P, can be thought of as the non-trivial zeroes of $\zeta(s)$, forming vertical strands in S, where the strands are symmetric under reflection about ϒ, and also the x-axis. Thus, the zeroes here are *conserved* under reflection – a remarkable thing, in and of itself, and indeed, it is these symmetries (reflections) in P that imply conservation laws in W, like energy, say.

Since T is an *uncountable* set, we cannot associate, with *each* translation in T, a *unique* current in P, for if we could, $\zeta(s)$ would have uncountably many, non-trivial zeroes – a contradiction. The simplest alternative, using Noether and *least* action principles, is to map T to a *single* current γ in P, and, conversely, we choose the *simplest* symmetry in P, tied to γ, that leads to conservation laws in W. This symmetry would be reflection across the x-axis, which follows naturally from properties of $\zeta(s)$ … implying, in turn, that γ really is a *single-stranded* current.

In other words, reflection across the critical line ϒ, in the critical strip S, *isn't* needed, in order to satisfy the requirements of the wave-particle duality associated with (*), when considering Noether and *least* action principles. It can be discarded altogether, without losing any information, and to me, at least, this seems to be the best approach when considering the Riemann Hypothesis.

Indeed, it may be the way forward, and if so, tells us that *all* non-trivial zeroes of $\zeta(s)$ must lie *exactly* on ϒ …

Consciousness, Mach's Principle, and The Dark Energy Contour Integral

What is dark energy [ξ] ? In previous research, we offered a definition [pp 157-8], which is shown below …

> *If ζ is the primordial sphere and η the negative rest energy in ζ , then η is the sum total of all negative thoughts and actions associated with the collective consciousness in Σ, where η = ξ , up to transform*

Thus, dark energy is really our *own* negative thoughts and actions, made material in the current mosaic [Σ], and is slowly dissipating over very long spans of time, as we learn to heal from acts of harm we have caused one another. In other words, dark energy is part of our *consciousness,* and so, its presence 'over there' throughout our universe, say, affects us 'over here'.

This is Mach's Principle, in a nutshell, and a postulate Einstein vigorously embraced in his early years, when working with general relativity.

But what does 'over there' mean in this context ? Well, previous studies [pp 168-9] tell us the *density* of dark energy is probably *quantumlike*, containing many singularities [S] throughout our universe alone, and so 'over there' could reasonably be identified with these singularities. On the other hand, 'over here' is any point t in our space-time fabric affected by S, and thus, if we are looking for a mathematical object that *embraces* Mach's Principle, within the context of dark energy … the Bromwich integral quickly becomes a very interesting candidate.

Recalling the highlights from these studies, we have, in the case where $g^{u,v}$ is coupled *directly* to the dark energy density [λ(s)] itself …

$$\underbrace{G^{u,v}}_{\text{over here}} \;\approx\; £' \circ \lambda(s)g^{u,v} \;=\; \kappa\underbrace{\int_{\gamma} e^{st}\,\lambda(s)g^{u,v}\,ds}_{\text{over there}}$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, and £' denotes the Laplace inverse.

Thus, we can collapse 'over there' into a *single* point by way of a contour integration, according to Bromwich protocol, and so preserve both the *conscious* and *quantumlike* natures of dark energy *simultaneously,* when incorporating it into the field equations of general relativity. To me, at least, such an approach seems very promising indeed, and merits further investigation …

Covariance and The Dark Energy Contour Integral

Einstein would probably be rolling over in his grave if we didn't say something about the covariant nature of the dark energy contour integral, so in this note, which is largely a follow-on to the last few addenda, we are going to outline an approach that seeks to demonstrate this property.

Let us begin, then, with the simplest option, where $g^{u,v}$ is *not* coupled directly to the dark energy density function itself [$\lambda(s)$], in which case [see pp 168-9] …

$$G^{u,v} \approx \kappa g^{u,v} \int_{\gamma} e^{st}\, \lambda(s) ds \qquad (*)$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, κ is some constant, and γ denotes some form of contour integration, according to Bromwich protocol, that envelops any singularities associated with $\lambda(s)$. Further to this, we will assume that $\lambda(s)$ has *no* branching points, but that it does have both *countably* many zeroes and *simple* poles.

With $\{\zeta_1, \zeta_2, \zeta_3, \ldots\}$ representing the zeroes, and $\{\omega_1, \omega_2, \omega_3, \ldots\}$ the poles, we may write, over all $\boldsymbol{i}$ [and for illustrative purposes only] …

$$\lambda(s) = \Pi(s - \zeta_i) / \Pi(s - \omega_i)$$

Now define

$$\lambda(s \mid \omega_j) = (s - \omega_j)\lambda(s)$$

so that $\lambda(s \mid \omega_j)$ is just $\lambda(s)$, less the singularity at ω_j. Then for any point t in α-space, the right-hand side of (*), at least symbolically and up to any constants, can be written over all $\boldsymbol{i}$ as …

$$\Psi(t) \sim g^{u,v} \Sigma\, \lambda(\omega_i \mid \omega_i) e^{t\omega_i} \qquad (\dagger)$$

As an example, suppose j = 1. Then

$$\lambda(s \mid \omega_1) = (s - \zeta_1)(s - \zeta_2)(s - \zeta_3) \ldots / (s - \omega_2)(s - \omega_3) \ldots$$

and thus,

$$\lambda(\omega_1 \mid \omega_1) = (\omega_1 - \zeta_1)(\omega_1 - \zeta_2)(\omega_1 - \zeta_3) \ldots / (\omega_1 - \omega_2)(\omega_1 - \omega_3) \ldots$$

At the origin, or even near the origin, assuming we can recover something here, one can discard the exponential term in (†), since it is exactly (or approximately) 1. Doing so gives us, over all $\boldsymbol{i}$ …

$$\Psi(0) \sim g^{u,v} \Sigma\, \lambda(\omega_i \mid \omega_i)$$

But notice, now, in the expression for $\Psi(0)$ above, that (a) $g^{u,v}$ is surely *covariant*, no matter the frame of reference (as is $G^{u,v}$), and (b) $\lambda(\omega_i \mid \omega_i)$ is always *invariant* under a coordinate translation, for any ***i***. Thus, if I am at the point t in the current frame S, it can always be made to 'look like' the origin in another frame S', through translation, and so $\Psi(0)$, in *any* frame of reference, exhibits covariance.

To see this more clearly, suppose in the frame S

$$\lambda(s) = (s - 6)(s - 7) / (s - 1)(s - 2)$$

Then the residue at $\omega = 1$ is $-30e^t$ and the residue at $\omega = 2$ is $20e^{2t}$. If we now shift the origin to 10, say, then in this new frame S',

$$\lambda(s) = (s + 4)(s + 3) / (s + 9)(s + 8)$$

and the residue at $\omega = -9$ is $-30e^{-9t}$, whilst the residue at $\omega = -8$ is $20e^{-8t}$. Notice that at $t = 0$, in either frame, the corresponding residues are equal, and of course, sum to the same value in both S and S', respectively !!

Thus, $\Psi(0)$, in *any* frame of reference, generated by translation, really *does* exhibit covariance, according to (a) and (b) above, which is what we wanted to establish initially.

A similar argument can be made where $g^{u,v}$ is coupled *directly* to $\lambda(s)$ [see pp 168-9], in which case

$$G^{u,v} \approx £' \circ \lambda(s)g^{u,v} = \kappa\int_\gamma e^{st}\,\lambda(s)g^{u,v}\,ds$$

Here again, $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, and £' denotes the Laplace inverse.

In conclusion, the point of this addendum was to try and show, rather heuristically, that the case for a dark energy contour integral, which exhibits covariance, can in fact, be made. However, the approach is not complete, nor is it guaranteed to be perfectly accurate; rather, more of a roadmap or sketch, that might be used for demonstrating the property of covariance, in general, when dealing with very real, *quantumlike* densities. As such, we should feel optimistic about things, going forward.

And finally, it should be pointed out that the residues are really a measure of our conscious or subconscious perception of dark energy. As such, solving the field equations of general relativity in different frames of reference could lead to different solutions. But this is to be expected, because we are no longer dealing with a uniform density here. Dark energy singularities that are nearer to the origin in one frame of reference [say S], may be further away from the origin in another frame [S'], and this, most certainly, will have an impact on these solutions. Needless to say, more research needs to be done on this very interesting topic …

Covariance and The Dark Energy Contour Integral, Part II

In the last addendum, we outlined a method that might be used to illustrate the covariant nature of the dark energy contour integral. However, the approach was somewhat heuristic, so here we'd like to tighten things up a bit by looking more closely at the following expression, over all $\boldsymbol{i}$ …

$$\Psi(t) \sim g^{u,v} \sum \lambda(\omega_i \mid \omega_i) e^{t\omega_i} \quad (\dagger)$$

Our goal is to show that $\Psi(0)$, in any frame of reference, exhibits covariance, but in order to do this, we really need to formulate a proper derivative for (†), which we'll label $\nabla_u\Psi$, and demonstrate that under certain circumstances, $\nabla_u\Psi(0) = 0$.

Using the product form for covariant differentiation, we may write, at least symbolically, and over all $\boldsymbol{i}$ …

$$\nabla_u\Psi(t) \sim (\nabla_u g^{u,v}) \sum \lambda(\omega_i \mid \omega_i) e^{t\omega_i} + g^{u,v} \sum \omega_i \lambda(\omega_i \mid \omega_i) e^{t\omega_i} \quad (*)$$

Since the foundational tensor $g^{u,v}$ is fully covariant, the first term in (*) vanishes at $t = 0$, and the second term will *always* vanish at $t = 0$, provided, over all $\boldsymbol{i}$ …

$$\sum \omega_i \lambda(\omega_i \mid \omega_i) = 0 \quad (\ddagger)$$

Thus we have the following result …

> *If $\lambda(s)$ is a rational density function with countably many* zeroes *$\{\zeta_1, \zeta_2, \zeta_3, \ldots\}$, and simple poles $\{\omega_1, \omega_2, \omega_3, \ldots\}$, then $\Psi(0)$ always exhibits covariance … in any frame of reference, generated by translation, if (‡) is true.*

Mathematical theorems dictate that there be more poles than zeroes, in order to recover the residues using a Bromwich integral, but this is a detail which doesn't affect the overall discussion here. And so, without further ado, we'll lead with the example below …

Suppose we define our dark energy density function to be

$$\lambda(s) = s(s-4)(s+4) / (s-6)(s-2)(s+2)(s+6)$$

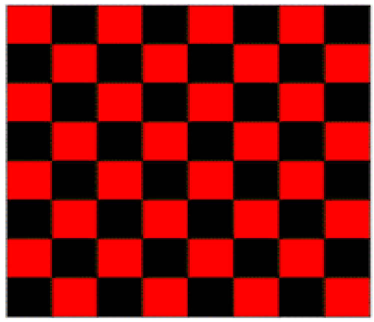
Then the residues $[\lambda(\omega_i \mid \omega_i)]$ for each $\boldsymbol{i}$ are as follows, at $t = 0$ …

$$\text{Res}[-6] = 120/384 \;;\; \text{Res}[-2] = 24/128 \;;\; \text{Res}[+2] = 24/128 \;;\; \text{Res}[+6] = 120/384$$

and the left-hand side of (‡) is indeed 0, since

$$-6\text{Res}[-6] - 2\text{Res}[-2] + 2\text{Res}[2] + 6\text{Res}[6] = 0$$

More generally, we may conclude that (‡) is *always* true, if *both* the zeroes and simple poles of $\lambda(s)$ are symmetric about the origin … in *any* frame of reference [not unlike the red and black squares on a checkerboard of infinite size].



The red squares might represent the zeroes of $\lambda(s)$ ... and the black squares the poles. As you move around on the board … .. your perception of the layout doesn't change.

Thus, if we are searching for a *quantumlike* density function for dark energy, say, our best hope lies in finding one which exhibits this kind of symmetry. If we can, we needn't worry anymore about the covariant nature of $\Psi(t)$, in general, because under translation, the point t in frame S can always be made to 'look like' the origin in another frame S'.

Needless to say, more research needs to be done on this very interesting topic, but hopefully, there is enough here, to give the reader some guidance on how to go forward with these ideas, assuming our approach is feasible ...

Covariance and The Dark Energy Contour Integral, Part III

In Part II, we looked at the covariant nature of the dark energy contour integral, in the case where $g^{u,v}$ is *not* coupled directly to the dark energy density function [$\lambda(s)$], and saw that over all ***i*** …

$$\Sigma\ \omega_i\lambda(\omega_i \mid \omega_i) = 0$$

could well be the condition that we need, in order to establish covariance at the origin, in any frame of reference. In this case, at least with respect to general relativity,

$$G^{u,v} \approx \kappa g^{u,v}\int_\gamma e^{st}\,\lambda(s)\,ds$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, κ is some constant, and γ denotes some form of contour integration, according to Bromwich protocol, that envelops any singularities associated with $\lambda(s)$.

In the case where $g^{u,v}$ *is* coupled directly to $\lambda(s)$, with £' the Laplace inverse,

$$G^{u,v} \approx £' \circ \lambda(s)g^{u,v} = \kappa\int_\gamma e^{st}\,\lambda(s)g^{u,v}\,ds \qquad (\dagger)$$

and we can still say something about covariance, but now, need to realize that $g^{u,v}$ is actually *part* of the residue calculation, and so the right-hand side of (†), at least symbolically and over all ***i***, is to be written as …

$$\Psi(t) \sim \Sigma\ g^{u,v}\,\lambda(\omega_i \mid \omega_i)e^{t\omega_i}$$

Here $g^{u,v}$ is evaluated at the poles of $\lambda(s)$, and thus, is a *fixed* entity, in so much as it is no longer a function of t. Hence, the covariant derivative of $\Psi(t)$ may be written as …

$$\nabla_u\Psi(t) \sim \Sigma\ g^{u,v}\,\omega_i\lambda(\omega_i \mid \omega_i)e^{t\omega_i} \qquad (\ddagger)$$

and so $\nabla_u\Psi(0) = 0$, provided (at t = 0) the right-hand side of (‡) is also zero …

$$\Sigma\ g^{u,v}\,\omega_i\lambda(\omega_i \mid \omega_i) = 0 \qquad (*)$$

To show this is actually possible, for rational densities [$\lambda(s)$] that are *symmetric* about the origin in *any* frame of reference [see the last addendum], consider a 'Schwarzschild-like' solution for a *perfect* star in our universe. Choose the origin [O] to be the center of the star … and solve (†) for $g^{u,v}$, assuming we can.

We expect $g^{u,v}$ (a measure of the gravitational field strength) to be symmetric about O as well, which means this symmetry is actually *retained* in (*) above, at the simple poles [ω_i] of our density function [$\lambda(s$]. In other words, since we *already* have the precondition

$$\sum \omega_i \lambda(\omega_i \mid \omega_i) \;=\; 0$$

it is not much of a leap to suppose (*), by virtue of symmetry, and so, even in the case where $g^{u,v}$ is coupled *directly* to the dark energy density function, one can argue $\Psi(t)$ exhibits covariance at the origin, in *any* frame of reference, generated by translation.

This is good news, because it gives us some confidence that the Bromwich integral is a credible choice when deciding how we want to merge the classical piece of general relativity, say, with the more *quantumlike* density associated with dark energy. The evidence, to use a cliche, is starting to mount up …

We can summarize our findings by appealing to Noether and noting that

symmetries associated with conserved currents in $\lambda(s)$
imply conservation laws in $\Psi(t)$ … … and conversely
symmetries associated with conserved currents in $\Psi(t)$
imply conservation laws in $\lambda(s)$

With O the origin in any frame of reference S, the following diagram illustrates this duality rather nicely …

$$\begin{array}{ccc}
\lambda(s) & & \Psi(t) \\
| & & | \\
\text{reflection about O} & \text{symmetry} & \text{translating O} \\
| & | & | \\
\sum \omega_i \lambda(\omega_i \mid \omega_i) \;=\; 0 & \text{current} & \nabla_u \Psi(0) = 0
\end{array}$$

*****

For now, this concludes our series on *Covariance and The Dark Energy Contour Integral*. Perhaps the reader has benefited from these discussions, and if so, may feel inspired to develop the ideas put forth here, in even greater detail.

But that would be a good thing, in my opinion, because since the time of Einstein, it doesn’t seem as though we’ve made much progress in this area. Maybe it’s finally time to change things …

A Quantumlike Density For The Two-Dimensional Plane

In our previous series on *Covariance and The Dark Energy Contour Integral*, we saw that if the contour integral itself was to exhibit a measure of covariance at the origin [O], in any frame of reference S, then the dark energy density function [$\lambda(s)$] needed to be symmetric about O as well.

Let us start, then, with the diagram below, which is a grid of circles and solid dots, where the circles represent the zeroes of $\lambda(s)$, and the dots represent the poles. The diagram is intended to cover the whole plane, but here we show just a small piece of it.

```
•   o   •   o   •   o   •   y = 4
            |
o   •   o   •   o   •   o
            |
•   o   •   o   •   o   •   y = 0
            |
o   •   o   •   o   •   o
            |
•   o   •   o   •   o   •
            |
          x = 0
```

Notice that if we move the origin to any point *on* the grid, our perception of the layout does *not* change. Thus $\lambda(s)$ is symmetric, relative to O, in any frame of reference generated by translation, when looking at the zeroes and the poles, in particular.

Now let the horizontal *or* vertical distance between any two closest circles be 4 units, and similarly for the dots. Then our *rational* density function at y = 0 can be written, over *all* x, as

$$\lambda(x, 0) = x(x-4)(x+4) \ldots /(x-6)(x-2)(x+2)(x+6) \ldots$$

and at y = 2, over *all* x, as

$$\lambda(x, 2) = (x-6)(x-2)(x+2)(x+6) \ldots /x(x-4)(x+4)(x-8)(x+8) \ldots$$

Observe that $\lambda(x, 0)\cdot\lambda(x, 2) = 1$, and so to connect the points between y = 0 and y = 2, we simply interpolate, *between* the grid, according to the following expression …

$$\lambda(x, y) = \tfrac{1}{2}\,(\lambda(x, 2) - \lambda(x, 0))y + \lambda(x, 0)\,, \quad 0 \le y \le 2$$

Similarly, to connect the points between y = 2 and y = 4, we interpolate again ... and realizing that $\lambda(x, 4)$ is actually a *copy* of $\lambda(x, 0)$, now have …

$$\lambda(x, y) = \tfrac{1}{2}\,(\lambda(x, 0) - \lambda(x, 2))y + 2\lambda(x, 2) - \lambda(x, 0)\,, \quad 2 \le y \le 4$$

More generally, for the upper plane, one has the following expressions, with the integer $n \geq 0$ ...

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$$\lambda(x, y) = \tfrac{1}{2} (\lambda(x, 2) - \lambda(x, 0))y + (2n + 1)\lambda(x, 0) - (2n + 0)\lambda(x, 2) ,$$

$$4n + 0 \leq y \leq 4n + 2, \text{ for } \textit{all} \text{ eligible } x$$

and

$$\lambda(x, y) = \tfrac{1}{2} (\lambda(x, 0) - \lambda(x, 2))y + (2n + 2)\lambda(x, 2) - (2n + 1)\lambda(x, 0) ,$$

$$4n + 2 \leq y \leq 4n + 4, \text{ for } \textit{all} \text{ eligible } x$$

******

A similar thing can be done for the lower plane, but we omit the calculations here, and so … one finally arrives at a formulation for $\lambda(x, y)$ over the *entire* two-dimensional plane, using the methods outlined above.

We can now take this density function [$\lambda(x, y)$] and bolt it into a Bromwich integral of sorts, which is going to be *multidimensional*, and calculate the residues, accordingly. If the poles of $\lambda(x, y)$ were simply along the x-axis or the y-axis, it would be a *no-brainer*, as they say, but here we have poles scattered all over the plane, and so more care needs to be taken … when deciding on an approach. Then, too, one has to move up into higher dimensions in the case of general relativity.

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And finally, if we are looking at a 'Schwarzschild-like' solution, for a *perfect star* centered about the origin O, in a set frame of reference S, then previous studies tell us that $g^{u,v}$ will be symmetric about O as well. Thus, one might only consider residues in the *negative* x, y, z, and t directions.

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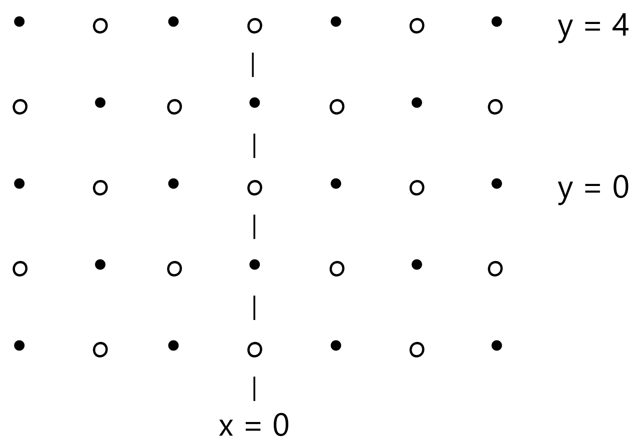
Hopefully this note has been of some assistance to the reader who is interested in this material, and if so, perhaps the individual may feel inspired to develop the ideas put forth here, in even greater detail. Ultimately, our goal is to build a marriage between the classical piece of general relativity [$G^{u,v}$], and the *quantumlike* density associated with the dark energy contour integral, as we've said before.

To me, at least, such a goal is well within our reach, now that most of the conceptual details have been ironed out. But there is more work to do, and maybe in future addenda we'll address some of those issues; yet for now, we should rest content in knowing that quantumlike densities, in higher dimensions, can indeed be constructed, which can then be linked back to the Bromwich integral …

A Quantumlike Density For The Two-Dimensional Plane, Part II

In Part I we put forth a description for a *quantumlike* density [$\lambda(x, y)$] in the plane, but did not deal with the issue of residues in much detail. Here we are going to do just that – a proposal, if you will, using a *diagonalization* argument, for capturing residues over the *entire* plane, as opposed to just the x and y-axes.

Let us start, then, with the diagram below, which is our grid of circles and solid dots, where the circles represent the zeroes of $\lambda(s)$, and the dots represent the poles. The diagram is intended to cover the whole plane, but here we show just a small piece of it.



Define S(n) to be the point (n, n), where n can be any *even* integer, be it positive, negative or zero, and let $\{\omega_i \mid n\}$ be the set of all poles *along* the line to the *left* of S(n), and *along* the line *down* from S(n) [note: n itself is an index, and not a pole in $\{\omega_i \mid n\}$].

For example, if n = 0, we would be on the x-y axes, and so our horizontal and vertical poles would be located at {-2, -6, -10, … | 0}, in keeping with Part I of this series. If n = -2, then the poles would be {-4, -8, -12, … | -2}, and at n = -4, {-6, -10, -14, … | -4}, and so on.

For any point $t = (t_1, t_2)$ in α-space, with κ some constant, the two-dimensional Bromwich integral at S(0) can be written as …

$$\Psi(t \mid 0) \; = \; \kappa \int_{\gamma_0}\int_{\gamma_0} \lambda(x, y)\, e^{xt_1 + yt_2}\, dx\, dy \qquad (\dagger)$$

where γ_0 denotes the imaginary axis at S(0), perpendicular to the plane, and coming out of the page. The line integrals are to be completed with an arc, along which, things vanish as the radius of the arc approaches ∞, and so we may evaluate (†) by calculating residues.

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These two closed contours envelop the poles associated with S(0), in the horizontal and vertical directions, respectively, so first we perform the inner integration leaving us with a function we'll label $\xi(t_1, t_2, y)$, say, and then we perform the outer contour integration against $\xi(t_1, t_2, y)$, yielding $\eta(t_1, t_2)$.

Now we can move on to S(-2), by first relocating our imaginary z-axis to (-2, -2) and calling it γ_{-2}, say. The poles here are {-4, -8, -12, … | -2} in the horizontal and vertical directions, and thus (†) becomes …

$$\Psi(t \mid -2) \;=\; \kappa \int_{\gamma_{-2}} \int_{\gamma_{-2}} \lambda(x, y)\, e^{xt_1 + yt_2}\, dx\, dy$$

The entire lower left region of the plane can be covered in this way, and when it's all said and done, we may write, in the case where $g^{u,v}$ is *not* coupled directly to $\lambda(s)$, and for all *even* $n \leq 0$ …

$$G^{u,v} \;\approx\; g^{u,v} \sum \Psi(t \mid n)$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively.

In the case where the foundational tensor $g^{u,v}$ *is* coupled directly to $\lambda(s)$, it becomes part of the residue calculation, but other than that, the two cases are essentially the same. And, it goes without saying that generalizations to higher dimensions follow a similar line of reasoning …

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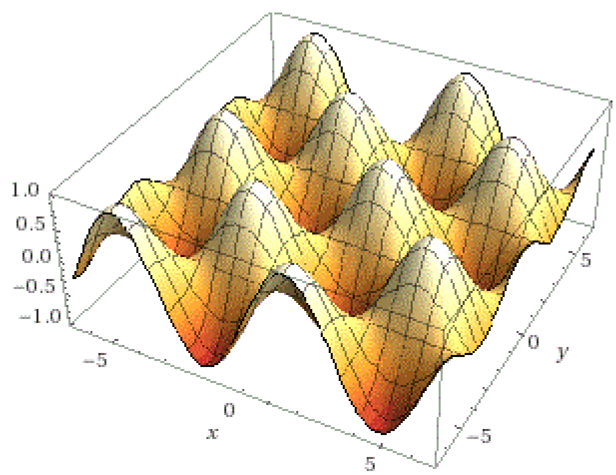
This is our diagonal approach to capturing residues over all or part of the two-dimensional plane. We say *all* here because n is an arbitrary even number. By letting n ---› ∞, we can, at least in theory, do this. Care has to be taken not to double-count at the origin, and where possible, symmetry arguments should be used to reduce the actual number of calculations required. A computer wouldn't be a bad idea either.

But if what we've done is correct here, we are, perhaps for the first time, seeing how one might merge the classical part of general relativity [$G^{u,v}$] with a *quantumlike* density for dark energy, by way of the Bromwich integral. To me, at least, it's an exciting moment …

A Quantumlike Density For The Two-Dimensional Plane, Part III

In the previous two addenda in this series, we looked at a hypothetical *quantumlike* density [λ(s)] in the two-dimensional plane, and saw how it might be replicated everywhere … and then bolted into a Bromwich integral, itself multidimensional. But one could ask the question 'what does this density really look like, if we could graph it ?'

There is no easy answer to the question, but from previous studies we do know the zeroes and the poles of λ(s) are probably symmetric, relative to the origin [O], in any frame of reference [S], which means there is a good chance λ(s) is also symmetric in the same way. If this is so, then perhaps the following picture is a fair representation of the dark energy density function, which fills our universe, in particular …



In the diagram above, the valleys are the dark-shaded areas, where λ(s) ≈ 0, and the peaks are the light-shaded areas, where λ(s) ---› ∞. In other words … the *zeroes* and the *poles* of the density function, respectively.

One can imagine the *intensity* of λ(s) falling off rather quickly … as you move away from a singularity, which means that if we were to try and measure the strength of λ(s) with some kind of instrument, we could easily be deceived into believing it was essentially *zero* everywhere. The instrument, it seems, cannot detect the singularities associated with the *quantumlike* nature of the dark energy density function, itself.

Even so, these singularities exist, in all likelihood, which means we need a mathematical object that traps them, if we are going to merge the classical piece of general relativity [$G^{u,v}$] with λ(s). Our best hope for doing this, in my opinion, is the Bromwich integral, which joins together these two worlds, rather nicely …

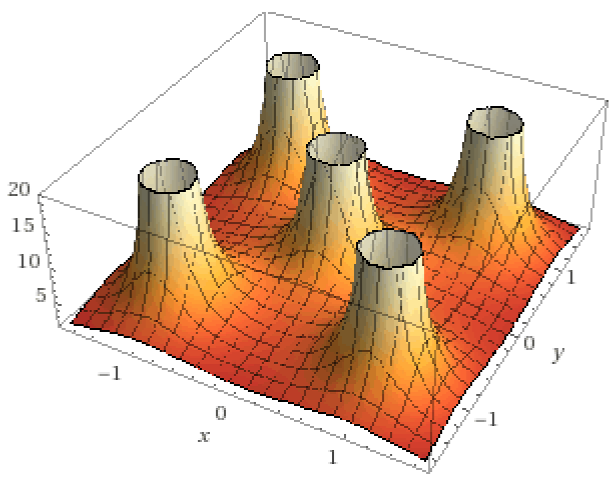
A Quantumlike Density For The Two-Dimensional Plane, Part IV

In Part III we mentioned some of the attributes we'd like to see in a very real *quantumlike* density for dark energy, that might fill our universe, for example. Here such a density [λ(s)] is presented, for the two-dimensional plane, after fiddling around a bit on the WolframAlpha website.

Most notably, we mentioned that the *zeroes* and the *poles* of λ(s) should be symmetric about the origin O in any frame of reference S, and that this property should also apply to λ(s), in general. Without further ado, then, here is the formula and the accompanying picture …

plot $$\frac{1}{x\,x+y\,y}+\frac{1}{(x-1)(x-1)+(y-1)(y-1)}+\frac{1}{(x+1)(x+1)+(y+1)(y+1)}+\frac{1}{(x+1)(x+1)+(y-1)(y-1)}+\frac{1}{(x-1)(x-1)+(y+1)(y+1)}$$

An algebraic representation for part
of λ(x,y) .. .. that meets our criteria



The corresponding 3D plot of λ(x,y)

In future addenda, we may work more with this density function to see how the residues measure out, and it looks like generalizations to higher dimensions should be doable as well. Stay tuned, as they say, but in the meantime, to me at least, this particular density construction is well worth pondering …

A Quantumlike Density For The Two-Dimensional Plane, Part V

In Part IV of this series, we put forth a rather plausible definition for the dark energy density function in the two-dimensional plane. Here, we want to explore functions of this type a little more closely, but before we do, some remarks are in order concerning the inverse of a two-dimensional Laplace transform.

Strictly speaking … if we play by the rules … this inverse is performed according to the following expression [see Part II in this series],

$$\kappa \int_{\gamma_0}\int_{\gamma_0} \lambda(x, y) e^{xt_1 + yt_2}\, dx\, dy \qquad (*)$$

where γ_0 is an imaginary axis perpendicular to the x-y plane, which forms part of a *double* contour integration, that envelops *all* poles associated with $\lambda(s)$ in the x and y directions, respectively.

Thus, in terms of calculating residues, we are *only* interested in the *mathematical poles* of $\lambda(x, y)$, first in the x-direction, associated with the *inner* contour integration, and then in the y-direction, associated with the *outer* contour integration. Almost by definition, this seems to be the correct interpretation of (*) when reading the literature, even though *physically* the singularities affiliated with $\lambda(x, y)$ are scattered everywhere, throughout the plane. As such, we will, going forward, abandon the diagonalization argument put forth in Part II, with a tinge of sadness, and learn to play by the rules …

To see this more clearly suppose, in one dimension, our density function was defined to be …

$$\lambda(x) = 1/(x^2 + 1)$$

There are no *physical* singularities here … … but there are two *imaginary* poles that are used in calculating the Laplace inverse; namely ***i*** and ***-i***, and thus the Bromwich integral becomes …

$$\sin(t) = \kappa \int_{\gamma_0} \lambda(x) e^{xt}\, dx$$

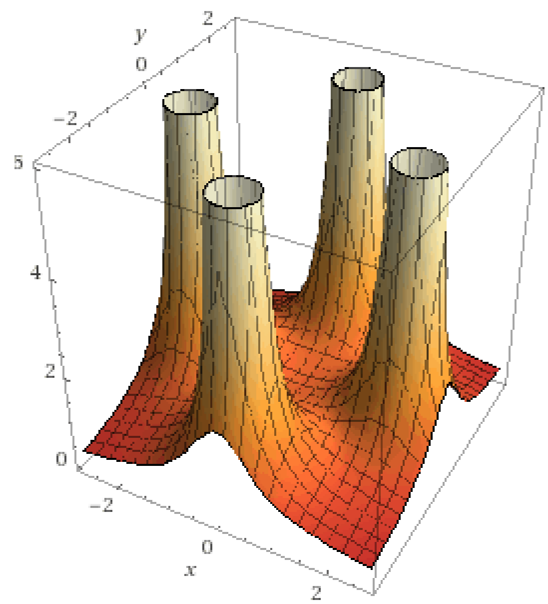
where γ_0 denotes a contour in the complex plane that envelops these two poles.

Hence the *rational* density expression associated with $\lambda(x)$ above becomes a *wave* under inversion, and this should be of some interest to us, since in our previous series titled *Moving Between The Laws In α-Space, Parts I and II (pp 170-71)*, we saw that from a geometric perspective, anyway, all laws could be perceived as *wavelike* patterns. Does this mean, in turn, the dark energy density function that we are really searching for should be a rational construct as well ? Remember, all foundational laws are embedded in a field of dark energy, where both the law and the dark energy are tied together via the Laplace transform and its inverse …

If we believe that $\lambda(s)$ is a rational function of sorts, then previous research tells us that the *zeroes* and the *singularities* of $\lambda(s)$ should be symmetric about the origin O, in any frame of reference S. Additionally, we think that as you move away from a singularity, the intensity of $\lambda(s)$ diminishes *very* rapidly. Thus, expressions of the following kind are probably suitable candidates for the density of dark energy, in the two-dimensional plane, where the sum is to be taken over *all* physical singularities located at (m, n) in π, and k is any integer ≥ 1.

$$\lambda(x, y) = \sum_{m,\, n} 1/((x - m)^{2k} + (y - n)^{2k}) \qquad (\dagger)$$

The picture below shows part of $\lambda(s)$, in the case where k = 1. It thus becomes an inverse power of 2 relationship.



Can functions of type (†) really be used to simulate the density of dark energy, and can we really calculate the Laplace inverse, accordingly ? These are tough questions to answer, but my hunch is that we are on the right track. Our goal, ultimately, is to create a union between the classical piece of general relativity, say, and the *quantumlike* nature of $\lambda(s)$. In the case where the foundational tensor $g^{u,v}$ is *not* coupled directly to $\lambda(s)$, we may write, at least in the case of our two-dimensional plane …

$$G^{u,v} \approx \kappa g^{u,v} \int_{\gamma_0} \int_{\gamma_0} \lambda(x, y) e^{xt_1 + yt_2}\, dx\, dy$$

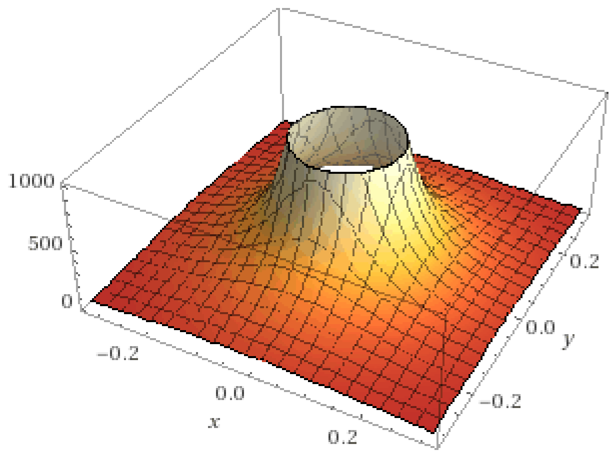
Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, and κ is some constant. Generalizations to higher dimensions are also apparent …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral

Now that we have a better understanding of how things bolt together, I thought it might be instructive to show how one might write down the field equations of general relativity, for a *perfect* star centered about the origin O, in a frame of reference S. We will assume, in this simple illustration, that there is only *one physical* singularity associated with the density of dark energy [$\lambda(s)$], in the vicinity of the star, and that it is located at O. And further to this, we will assume that

$$\lambda(s) \approx \sigma / s^2$$

where σ is some constant, and s is the distance from O.



A picture of $\lambda(s)$ with a singularity
at the center of the perfect star

Since this is a perfect star, we may take a *Schwarzschild* approach to solving the problem near O, and beyond, by assuming that only the *radial* and *timelike* directions are of interest to us. We associate these two parts with $G^{1,1}$ and $G^{4,4}$ respectively, but also know that $g^{u,v}$, a measure of the gravitational field strength, is going to be a function of the radial distance [r] only. Thus, in the case where $g^{u,v}$ is *not* coupled directly to $\lambda(s)$, we may write, for any r …

$$G^{u,v} \approx \kappa g^{u,v} \int_{\gamma} e^{sr} \lambda(s) ds$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, κ is some constant, and γ denotes some form of contour integration that envelops any singularities tied to $\lambda(s)$.

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Evaluating the expression above, we have,

$$G^{u,v} \approx \kappa\sigma g^{u,v} \int_\gamma (e^{sr} / s^2) ds \qquad (*)$$

$$= \sigma r g^{u,v}$$

In the case where $g^{u,v}$ *is* coupled directly to $\lambda(s)$, (*) becomes,

$$G^{u,v} \approx \kappa\sigma \int_\gamma (e^{sr} / s^2) g^{u,v} ds$$

$$= \sigma(r g^{u,v} + [g^{u,v}]')|_0 \qquad (\dagger)$$

Here … assuming it's possible, the $g^{u,v}$ terms in parentheses are to be evaluated at 0, after first differentiating $g^{u,v}$, which shows up in the second term. The variable r is just that – a variable. Since it is $g^{u,v}$ we are trying to solve for, it should be clear that (†) is not without its challenges.

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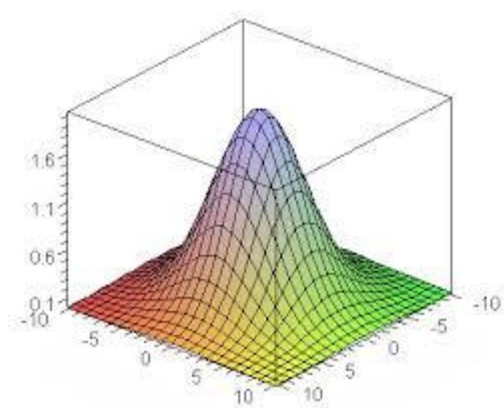
Notice, too, that (*) and (†) will agree at the origin [O] if $g^{u,v}$ *is* well-behaved and $[g^{u,v}]'|_0 = 0$. An example of this type of behavior might be seen in the diagram below, where $g^{u,v}$ follows a bell-shaped pattern, as you move closer to O.



In the case where $g^{u,v}$ is *not* well-behaved at O, one has a choice – use (*) as is, or include any poles associated with $g^{u,v}$ in the residue calculations tied to (†), so long as the physics makes sense.

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Whether we will ever find a complete dark energy density function [$\lambda(s)$] that fills our universe, say, is somewhat up for grabs, in my opinion. It may be worse than searching for the proverbial 'needle in a haystack'. But this should not deter us from developing approximate solutions to problems in cosmology and astronomy that seek to understand how dark energy factors into the bigger picture. The example above is just one simple illustration of how we might tackle dark energy in a *local* context, before fanning out to the universe and beyond …

OTHER CONSIDERATIONS

If we let $\lambda(s) \approx \sigma / s$, then the density function has a *simple* pole at the origin of the star, and (*) and (†) become, (for any r) respectively …

$$G^{u,v} \approx \sigma g^{u,v} \qquad (\S)$$

$$G^{u,v} \approx \sigma g^{u,v}|_0 \qquad (\ddagger)$$

Notice that both equations are fully *covariant* because we are dealing with a *simple* pole here, and that (§) is actually Einstein's original formulation of the field equations with a constant dark energy density permeating the whole of our universe, say. An interesting confluence, to say the least !

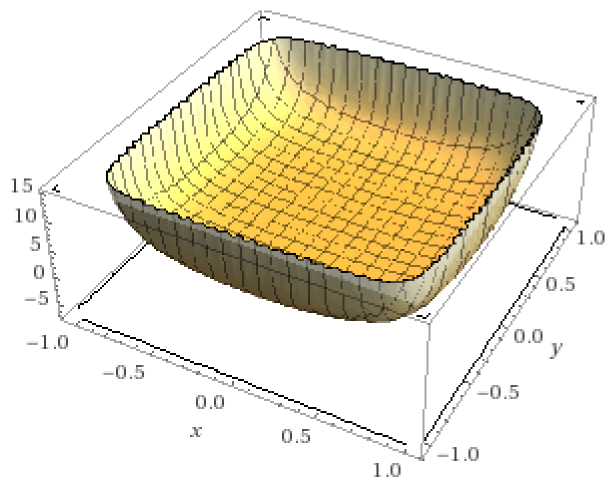
Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part II

In Part I, we offered an approach for solving the field equations of general relativity, where the dark energy singularity existed at the center of the star. But what happens when there is more than one such singularity ? In this note, we offer a proposal for solving the equations, in this case, by first examining things in two dimensions, and then moving up into higher spaces by inference.

Let us begin, then, by looking at a *perfect* star, centered about the origin O, in a frame of reference S, and a density function for dark energy, which can be written as …

$$\lambda(x, y) \;=\; \sigma / ((x - 1)(x + 1)(y - 1)(y + 1))$$

where σ is some constant. Notice that $\lambda(x, y)$ is symmetric about O.



The picture above shows part of this density, which, for the most part, hovers around 0 in the basin before rising up to ∞ along the four edges. Needless to say, it is a hypothetical construct, but it will serve a purpose when calculating the Laplace inverse.

To this end, we first note that the Laplace inverse can be written as

$$\kappa \int_{\gamma_0}\int_{\gamma_0} \lambda(x, y)e^{xt_1 + yt_2}\, dx\, dy \qquad (*)$$

where γ_o denotes a contour path that envelops any singularities associated with $\lambda(x, y)$, first in the x-direction and then in the y-direction, and κ is some constant.

In the x-direction, excluding σ for the moment, the residues sum to

$$e^{t_1} / \{2(y - 1)(y + 1)\} \;-\; e^{-t_1} / \{2(y - 1)(y + 1)\}$$

and plugging this into (*), we get, where κ is another constant …

$$\kappa\sigma(e^{t_1} - e^{-t_1})\int_{\gamma_0} e^{yt_2}/\{2(y-1)(y+1)\}\,dy \qquad (\dagger)$$

Finally, (†) computes to

$$\sigma(e^{t_1} - e^{-t_1})(e^{t_2} - e^{-t_2})/4$$

and thus, for any point $t = (t_1, t_2)$ in α-space, where $g^{u,v}$ is *not* coupled directly to λ(s), the field equations of general relativity, in two dimensions, may be written as follows, where *sinh* means *hyperbolic sine* [it should be clear that both $G^{u,v}$ and $g^{u,v}$ are to be evaluated at t] …

$$G^{u,v} \approx \sigma[\mathit{sinh}(t_1)][\mathit{sinh}(t_2)]g^{u,v} \qquad (\ddagger)$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, and $g^{u,v}$ is the *foundational* tensor, itself a measure of the gravitational field strength. Generalizations to higher dimensions are quickly apparent.

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Hopefully, this simple illustration is of some use to the reader who is interested in this type of material. If we believe the density of dark energy contains singularities, then necessarily we need a mechanism to trap them. By using the Bromwich integral to do this, it is possible to bridge the classical piece of general relativity, say, with a more *quantumlike* function for λ(s).

The key, here, is to see that we can move freely between two domains; the first being the fabric of *space-time* and the second being the *realm* of the density function itself … and that this freedom of movement is accomplished via the Laplace transform and its inverse …

OTHER CONSIDERATIONS

In the case where $g^{u,v}$ *is* coupled directly to λ(s), we have the following equation, where $G^{u,v}$ is to be evaluated at $t = (t_1, t_2)$ …

$$G^{u,v} \approx [\sigma/4](e^{t_1 + t_2}g^{u,v}(1,1) - e^{-t_1 + t_2}g^{u,v}(-1,1) - e^{t_1 - t_2}g^{u,v}(1,-1) + e^{-t_1 - t_2}g^{u,v}(-1,-1))$$

It is instructive to compare this to (‡) at the origin [(0,0)], and more generally, it is worth pondering the difference between the two equations. What does it really mean to say that $g^{u,v}$ *is* or *isn't* coupled directly to λ(s), especially if the singularities associated with λ(s) are everywhere ? Is (‡) a *smoothed out* version of the more *discrete* representation above, or are there other interpretations that are worth considering …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part III

In Part II, we outlined an approach for solving the field equations of general relativity, in the case where the dark energy density function [$\lambda(s)$] was multidimensional. Here, we wish to continue that discussion by modifying $\lambda(x, y)$ somewhat, and then, using an 'in the limit' argument, draw some interesting conclusions regarding $g^{u,v}$, when it *is* coupled directly to $\lambda(s)$.

So let's begin by defining our two-dimensional dark energy density function to be

$$\lambda(x, y) = \sigma / ((x - \varepsilon)(x + \varepsilon)(y - \varepsilon)(y + \varepsilon))$$

where ε is an arbitrarily small number which is greater than 0. Then in the case where $g^{u,v}$ is *not* coupled directly to $\lambda(s)$, we have, for any point $t = (t_1, t_2)$ in α-space …

$$G^{u,v} \approx (\sigma / \varepsilon^2)[\mathit{sinh}(\varepsilon t_1)][\mathit{sinh}(\varepsilon t_2)]g^{u,v} \qquad (*)$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, and $g^{u,v}$ is the *foundational* tensor, itself a measure of the gravitational field strength.

In the case where $g^{u,v}$ *is* coupled directly to $\lambda(s)$, we have … for any point $t = (t_1, t_2)$, the following equation, which we'll label (†) …

$$G^{u,v} \approx [\sigma/4\varepsilon^2](e^{\varepsilon t_1 + \varepsilon t_2}g^{u,v}(\varepsilon,\varepsilon) - e^{-\varepsilon t_1 + \varepsilon t_2}g^{u,v}(-\varepsilon,\varepsilon) - e^{\varepsilon t_1 - \varepsilon t_2}g^{u,v}(\varepsilon,-\varepsilon) + e^{-\varepsilon t_1 - \varepsilon t_2}g^{u,v}(-\varepsilon,-\varepsilon))$$

Now let ε ---› 0, so that (*) becomes (where $G^{u,v}$ and $g^{u,v}$ are to be evaluated at t, unless otherwise shown) …

$$G^{u,v} \approx \sigma t_1 t_2 g^{u,v} \qquad (\S)$$

Repeating the exercise for (†), we see that as ε ---› 0,

$$G^{u,v} \approx \sigma t_1 t_2 g^{u,v}(0, 0) \qquad (\ddagger)$$

Thus, for the case where $\lambda(x, y) = \sigma/(x^2y^2)$, the two expressions above [(§) and (‡)] are the correct forms for the field equations, where $g^{u,v}$ *isn't* and *is* coupled directly to $\lambda(s)$, respectively. But, you may ask, why do we care ?

Well suppose we had to do the computations directly, according to the following expression, where $\lambda(x, y) = \sigma/(x^2y^2)$ …

$$G^{u,v} \approx \kappa g^{u,v} \int_{\gamma_0}\int_{\gamma_0} \lambda(x, y)e^{xt_1 + yt_2}\, dx\, dy$$

It would be a pretty simple thing to calculate (§), using the above formula, since here $g^{u,v}$ is *uncoupled* from $\lambda(s)$. But what happens when there *is* a coupling, as shown below …

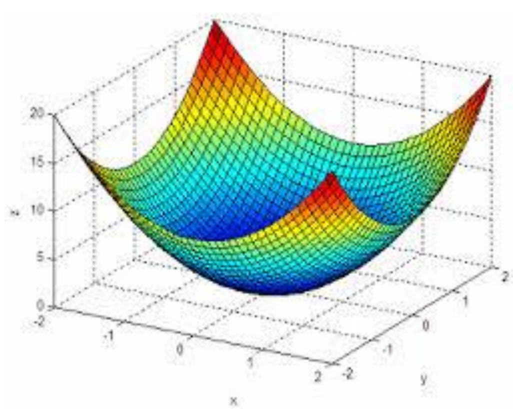
$$G^{u,v} \approx \kappa \int_{\gamma_0}\int_{\gamma_0} \lambda(x, y) e^{xt_1 + yt_2} g^{u,v}\, dx\, dy \qquad (\#)$$

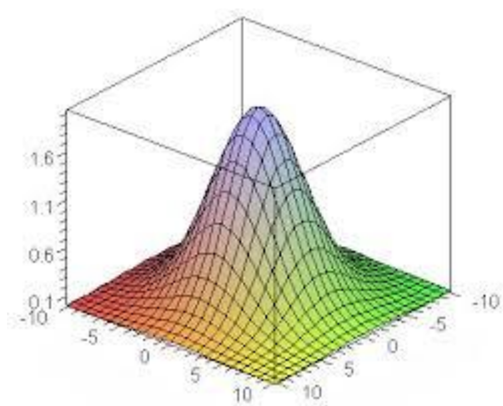
In this case it is no longer a trivial exercise to compute the double contour integral, because it involves partial derivatives in x and y, when looking at the foundational tensor $g^{u,v}$. In fact, in order to get the contour integral above to agree with (‡), these partial derivatives with respect to $g^{u,v}$ must vanish at the origin [O], in our chosen reference frame S !!

For the reader who is interested … it can be shown that … for our limiting density $[\sigma/(x^2y^2)]$, (#) computes to

$$G^{u,v} \approx \sigma[t_1 t_2 g^{u,v} + t_2 \partial g^{u,v}/\partial x + t_1 \partial g^{u,v}/\partial y + \partial^2 g^{u,v}/\partial x \partial y]$$

where $g^{u,v}$ and its derivatives, on the right-hand side, are to be evaluated at (0,0). Smooth surfaces near the origin, where the partial derivatives above are all zero at O, might include things like bell-shaped patterns, parabolic sheets, and so on, when describing the geometry of $g^{u,v}$.





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Thus, just as in Part I of this series, for a well-behaved $g^{u,v}$, vanishing derivatives at O seem to be a fundamental requirement, and here we were able to demonstrate this by way of an ‘in the limit’ argument, as ε ---› 0, in the case where $g^{u,v}$ *is* coupled directly to $\lambda(s)$. The key here is to see the connection between the *well-behavedness* of $g^{u,v}$ and the coupling itself … which then begs the question … does $g^{u,v}$ really have any singularities if it is joined at the hip with $\lambda(s)$?

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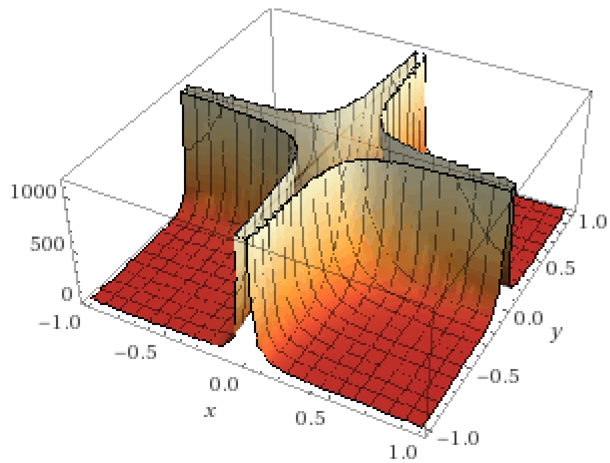
A picture of the limiting density $[\sigma/(x^2y^2)]$ is shown below, for a small portion of the x-y plane, and at the origin [O] in this picture, in particular, we should ponder the following question … can $g^{u,v}$ really have a singularity here … or is a well-behaved $g^{u,v}$ simply being *funnelled* through the singularity associated with the dark energy density function $[\lambda(s)]$, itself, when they are coupled together ?



Even though we are ultimately searching for a dark energy density function that permeates the whole of our universe, say, nothing stops us, in the meantime, from doing various functional analyses on the field equations of general relativity, in both the coupled and uncoupled cases.

To me, at the present time, the idea that the gravitational tensor $g^{u,v}$ *may* actually be coupled to $\lambda(s)$ is a very deep mystery, indeed, and more research needs to be done in this area, to say the least. But even if we don't know whether the coupling exists or doesn't exist, nothing stops us from deepening our knowledge of $g^{u,v}$ by assuming that it does. This addendum is but one example of how we might use the coupling to learn more about the foundational tensor itself.

In the end, we may learn that $g^{u,v}$ *is* coupled to $\lambda(s)$, and that gravitational singularities really don't exist after all … only the singularities associated with $\lambda(s)$, through which gravity is funnelled …

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Finally, it should be pointed out that nothing new was recovered from this analysis in the case where $g^{u,v}$ is *not* coupled to $\lambda(s)$. As such, we can't speak meaningfully about the well-behavedness of $g^{u,v}$ in this regard. Maybe it's there … and maybe it's not …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part IV

In Part III, we saw how to develop forms for the field equations of general relativity … in two dimensions, using an 'in the limit' argument, which led to some interesting observations concerning vanishing derivatives for $g^{u,v}$, in the case where it *was* coupled directly to the dark energy density function [$\lambda(s)$]. Here we'd like to extend the last addendum to four dimensions.

Let us begin, then, by recalling that for two dimensions, our *limiting* density

$$\lambda(x, y) = \sigma/(x^2y^2)$$

produced the following equations, for any point $t = (t_1, t_2)$ … … in the *uncoupled* and *coupled* cases, respectively …

$$G^{u,v} \approx \sigma t_1 t_2 g^{u,v} \qquad (\S)$$

$$G^{u,v} \approx \sigma t_1 t_2 g^{u,v}(0, 0) \qquad (\ddagger)$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, and $g^{u,v}$ is the *foundational* tensor, itself a measure of the gravitational field strength.

How, then, do we take this limiting density in two dimensions, and scale up to four dimensions, without getting too bogged down in a bunch of algebra, as we calculate residues ?

To answer the question, first note that in four dimensions, we may define the density function to be

$$\lambda(x, y, z, w) = \sigma/(x^2y^2z^2w^2)$$

and realize immediately that in the *uncoupled* case, residue calculations in 2-space are essentially *no* different than in 4-space, and so the expression below

$$G^{u,v} \approx \kappa g^{u,v} \int_{\gamma_0}\int_{\gamma_0}\int_{\gamma_0}\int_{\gamma_0} \lambda(x, y, z, w)\, e^{xt_1 + yt_2 + zt_3 + wt_4}\, dx\, dy\, dz\, dw$$

naturally extends (§) to

$$G^{u,v} \approx \sigma t_1 t_2 t_3 t_4 g^{u,v}$$

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But what about the *coupled* case where the following expression applies …

$$G^{u,v} \approx \kappa \int_{\gamma_0}\int_{\gamma_0}\int_{\gamma_0}\int_{\gamma_0} \lambda(x, y, z, w) e^{xt_1 + yt_2 + zt_3 + wt_4} g^{u,v}\, dx\,dy\,dz\,dw$$

To say the least, this is an intolerable mess, but the good news is … there is a way out of it. To see that this is so, recall from Part III in this series, that in *two* dimensions, the above expression led to the following result …

$$G^{u,v} \approx \sigma[t_1 t_2 g^{u,v} + t_2 \partial g^{u,v}/\partial x + t_1 \partial g^{u,v}/\partial y + \partial^2 g^{u,v}/\partial x \partial y] \quad (*)$$

where $g^{u,v}$ and its derivatives, on the right-hand side, were to be evaluated at (0,0). Rewriting (*) in *operator* format, it now becomes …

$$G^{u,v} \approx \sigma(t_1 + \partial/\partial x)(t_2 + \partial/\partial y)g^{u,v} \quad (\dagger)$$

Notice that in *one* dimension [say x], we recover from (†) the *coupled* expression in Part I of this series, for the density $\sigma/(x^2)$, and so it is natural to suppose that (†) will extend to 4-space in the following way, where again $g^{u,v}$ and its derivatives are to be evaluated at the origin [O] …

$$G^{u,v} \approx \sigma(t_1 + \partial/\partial x)(t_2 + \partial/\partial y)(t_3 + \partial/\partial z)(t_4 + \partial/\partial w)g^{u,v} \quad (\#)$$

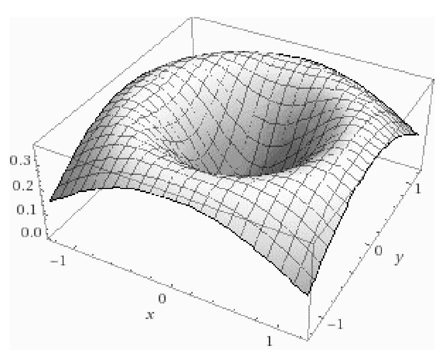
Assuming our extrapolation is correct, this, then, becomes the correct form for the field equations of general relativity … for the limiting density

$$\lambda(x, y, z, w) = \sigma/(x^2 y^2 z^2 w^2)$$

in the case where $g^{u,v}$ *is* coupled directly to $\lambda(s)$. If the derivatives in (#) *all* vanish at the origin [O], which we expect, then (#) becomes the following result, for any point $t = (t_1, t_2, t_3, t_4)$ …

$$G^{u,v} \approx \sigma t_1 t_2 t_3 t_4 g^{u,v}(0, 0, 0, 0)$$

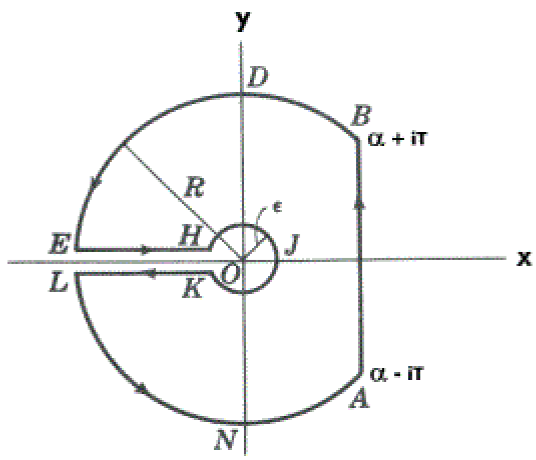
Just as in Part III, smooth surfaces near the origin, where the partial derivatives in (#) are all zero at O, might include things like bell-shaped patterns, parabolic sheets, and so on, when describing the geometry of $g^{u,v}$. But now, of course, we are in 4-space …



Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part V

In this addendum, we'd like to explore a dark energy density function [$\lambda(s)$] of type $\sigma/\sqrt{s}$. So far we haven't looked at these kinds of densities, mainly because the contour integration is more complex to compute, but here we are going to take the plunge, as they say, and see what we can learn, in this case.

In order to work with square root densities, a contour of the following kind is required, because we are dealing with a *branching* point at the origin [O] of the star.



The Laplace inverse is actually the line integral along AB, and since the contour integral itself vanishes along the large arcs BDE and LNA, as R ---› ∞, all we are left with is the *two* arms EH and KL, as well as the small circle HJK, where in the latter case, the contour integral will *again* vanish as ε ---› 0, provided $g^{u,v}$ is well-behaved near the origin [O] of our perfect star. The well-behavedness of $g^{u,v}$ applies within the context of a *coupling* to the density function $\lambda(s)$.

In the case where $g^{u,v}$ is *not* coupled to $\lambda(s) = \sigma/\sqrt{s}$, we have, for any radial distance r [see Part I in this series] …

$$G^{u,v} \approx \kappa g^{u,v} \int_{\gamma} e^{sr} \lambda(s) ds$$

$$= (\sigma/\sqrt{\pi r})g^{u,v}$$

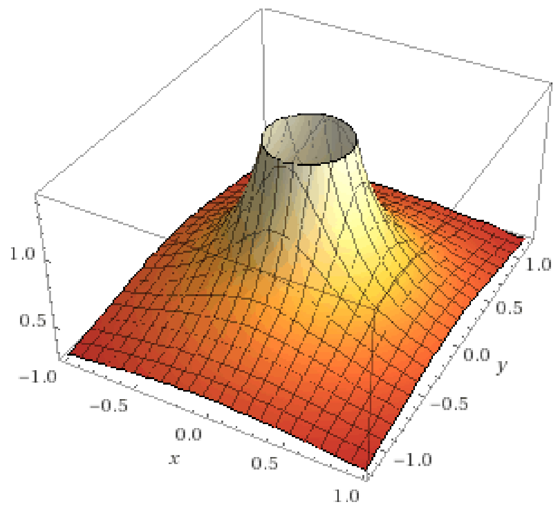
Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, κ is some constant, and γ denotes the line integral along AB in the picture above. We are, therefore, essentially calculating the Laplace inverse by computing its value along the arms EH and KL, and then letting R ---› ∞.

Now for the harder, but more interesting part. Suppose $g^{u,v}$ *is* indeed coupled to $\lambda(s) = \sigma/\sqrt{s}$. Then the same approach applies, as above, but now we calculate according to the following expression, where r is the usual *radial* distance from the origin [O], in our chosen frame of reference S …

$$G^{u,v} \approx \kappa \int_{\gamma} e^{sr}\, \lambda(s) g^{u,v} ds$$

$$= (\sigma / \pi) \int_{0}^{\infty} [e^{-sr} g^{u,v}(-s) / \sqrt{s}\,]ds \qquad (*)$$

The above integral is indeed a peculiar one, because it implies the *existence* of a gravitational field at a *negative* radial distance [s] from the origin of the star. There is no getting around this, if we are to follow the rules for computing the Laplace inverse where it involves a branching point, as it does here. The gravitational tensor *must* exist for negative s, which means the gravitational field itself is being *funnelled* into and out of our reality, via the singularity associated with λ(s), as shown below …



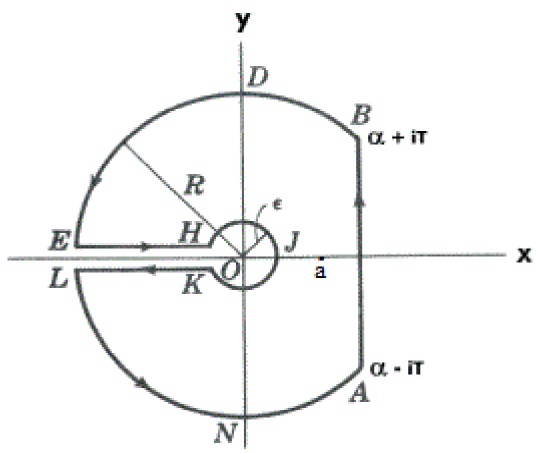
Thus, you could say that we are, perhaps, getting our first glimpse of the *funnelling effect* via (*), which up until now, is not something we could have seen, in any previous addenda in this series.

And since we have to consider $\lambda(s) = \sigma/\sqrt{s}$ to be a legitimate density of sorts, the notion or even the *existence* of a gravitational field in the *negative* radial direction should not be dismissed lightly, in the case where $g^{u,v}$ *is* coupled directly to λ(s). It could very well exist, and indeed … if (*) has anything to say about it … it really does exist …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part VI

In Part V, we looked at the density $\lambda(s) = \sigma/\sqrt{s}$, which had a *branching* point at the center of the star [O], and discovered some interesting things concerning the gravitational tensor $g^{u,v}$, in the case where it *was* coupled directly to the dark energy density function [$\lambda(s)$]. Here we'd like to continue that exercise by examining the function $\lambda(s) = \sigma/s\sqrt{s}$, which has both a *simple* pole and a *branching* point at O, and see what we can learn about the field equations of general relativity, in this case.

Let us begin, then, with the picture below, which depicts the contour we shall use, for an *offsetting* density defined to be $\lambda(s) = \sigma/(s-a)\sqrt{s}$, where a is an arbitrarily small *radial* distance from the center of the star [O]. Later on, we will let a ---› 0 to achieve the desired result.



Just as in Part V, the Laplace inverse is the line integral along AB, and since things vanish along the large arcs BDE and LNA, as well as the small circle HJK, as R ---› ∞ and ε ---› 0, all we are left with is the arms EH and KL, plus the residue at a. And again, we assume that in the case where $g^{u,v}$ *is* coupled directly to $\lambda(s)$, it is well-behaved in a neighborhood of O. Indeed, we expect the derivatives of $g^{u,v}$ to *vanish* here !!

In the case where $g^{u,v}$ is *not* coupled to $\lambda(s)$, the field equations of general relativity may be written (for any radial distance r) as follows …

$$G^{u,v} \approx \kappa g^{u,v}\int_{\gamma} e^{sr}\,\lambda(s)ds \qquad (*)$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, κ is some constant, and γ denotes the line integral along AB in the picture above.

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Realizing the *entire* contour envelops a *simple* pole at a … we find that (*) computes to …

$$G^{u,v} \approx \sigma g^{u,v}\{e^{ar}/\sqrt{a} \;-\; (1/\pi)\int_0^\infty [e^{-sr}/((s+a)\sqrt{s})]ds\} \qquad (\dagger)$$

Since the integral on the right-hand side is actually …

$$(\pi/\sqrt{a})e^{ar}\mathrm{erfc}(\sqrt{ar})$$

where erfc() is the complement to the *error function* erf(), (†) reduces to …

$$G^{u,v} \approx \sigma g^{u,v}\{e^{ar}/\sqrt{a}\}\mathrm{erf}(\sqrt{ar})$$

And in the limit, as a ---› 0, the expression above becomes the following, for the field equations of general relativity, in the case where $g^{u,v}$ is *not* coupled to $\lambda(s) = \sigma/s\sqrt{s}$ …

$$G^{u,v} \approx 2\sigma\sqrt{r/\pi}\,g^{u,v}$$

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Undaunted, we'll now repeat the exercise, in the case where $g^{u,v}$ *is* coupled to λ(s), by starting with the *offsetting* density $\lambda(s) = \sigma/(s-a)\sqrt{s}$. In this scenario (†) becomes …

$$G^{u,v} \approx \sigma\{e^{ar}g^{u,v}(a)/\sqrt{a} \;-\; (1/\pi)\int_0^\infty [e^{-sr}g^{u,v}(-s)/((s+a)\sqrt{s})]ds\} \qquad (\ddagger)$$

Rewriting (‡) as follows,

$$G^{u,v} \approx \sigma\{e^{ar}g^{u,v}(a)/\sqrt{a} \;-\; (1/\pi)\int_0^\infty [e^{-sr}(g^{u,v}(-s) + g^{u,v}(a) - g^{u,v}(a))/((s+a)\sqrt{s})]ds\}$$

and, after combining terms as we did above in the *uncoupled* case, and first letting a ---› 0 *outside* the integral sign, one has …

$$G^{u,v} \approx \sigma\{2\sqrt{r/\pi}\,g^{u,v}(0) \;-\; (1/\pi)\int_0^\infty [e^{-sr}(g^{u,v}(-s) - g^{u,v}(a))/((s+a)\sqrt{s})]ds\}$$

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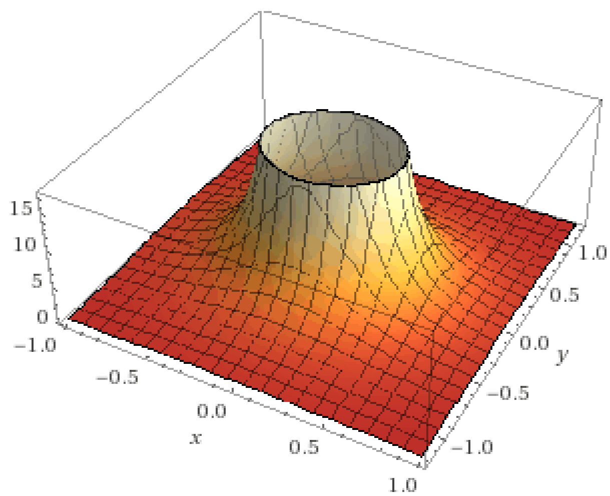
Now repeat, but this time let a ---› 0 *under* the integral sign, and we finally arrive at our destination for the field equations of general relativity, in the case where $g^{u,v}$ *is* coupled directly to the density function $\lambda(s) = \sigma/s\sqrt{s}$ …

$$G^{u,v} \approx \sigma\{2\sqrt{r/\pi}\,g^{u,v}(0) \;-\; (1/\pi)\int_0^\infty [e^{-sr}(g^{u,v}(-s) - g^{u,v}(0))/(s\sqrt{s}\,)]ds\} \qquad (\S)$$

Just as in Part V, we see again that a gravitational field exists at a *negative* radial distance [s] from the origin of the star [O], which speaks to the *funnelling effect* through the singularity associated with $\lambda(s)$ at O.



The gravitational tensor $g^{u,v}$ is being funnelled into and out of our reality via the singularity at the origin of the star associated with $\lambda(s)$ …

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And finally, when looking at the integral in [§] above, one can make the case for *convergence* near the origin, because we expect both $g^{u,v}$ *and* its derivatives to be well-behaved here. For example, if $g^{u,v}$ behaved in *parabolic* fashion near O (because of vanishing derivatives at O), then a Taylor series expansion gives us, for small s …

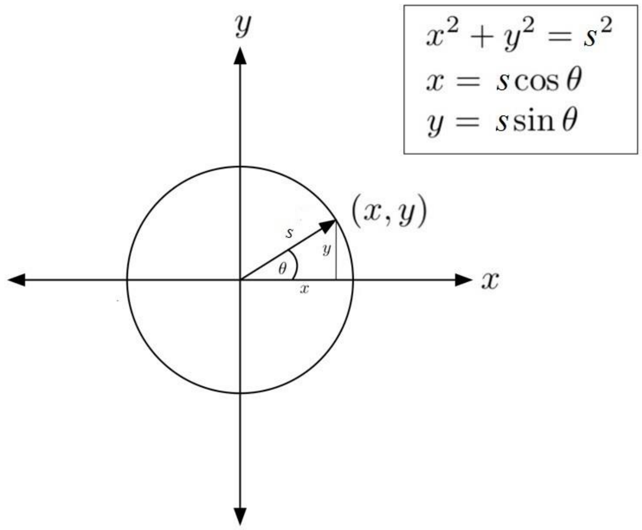
$$g^{u,v}(-s) \;-\; g^{u,v}(0) \;\approx\; -g^{u,v}(0)'s \;+\; \tfrac{1}{2}g^{u,v}(0)''s^2 \;+\; \ldots$$

$$\approx\; \tfrac{1}{2}g^{u,v}(0)''s^2$$

Here a tick mark means differentiate once and then evaluate at O; two tick marks mean differentiate twice and then evaluate at O, and so on …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part VII

In previous addenda in this series, we really haven't dealt with a *finite* number of singularities associated with the dark energy density function [λ(s)], other than the one at the *center* of our perfect star, in a chosen frame of reference S. So here we are going to do just that, and we'll begin, then, by referring to the diagram below …



The perfect star is to be of radius 1, centered at the origin [O] in a frame S … where the density function [λ(s)] has *physical singularities* at O … *and* at the points (1, 0) and (-1, 0). Furthermore, we will assume an 'inverse square' relationship for λ(s), and that the *overall* dark energy density function is obtained through *addition*.

At the origin, $\lambda(s) = \sigma / s^2$, where σ is some constant, and at the other two points, we have the following distance constructions, for any point (x, y) in the two-dimensional plane, where we use ξ and η as labels …

$$\xi(s) = (x - 1)^2 + y^2 = s^2 - 2s\cos(\theta) + 1 \quad \ldots \quad \text{from } (x, y) \text{ to } (+1, 0)$$

$$\eta(s) = (x + 1)^2 + y^2 = s^2 + 2s\cos(\theta) + 1 \quad \ldots \quad \text{from } (x, y) \text{ to } (-1, 0)$$

Thus, our overall density function is the following sum …

$$\lambda(s) = \sigma[1/s^2 + 1/\xi(s) + 1/\eta(s)]$$

Now the *simple* zeroes of ξ(s) are $e^{i\theta}$ and $e^{-i\theta}$, and for η(s), are $-e^{i\theta}$ and $-e^{-i\theta}$. We can, therefore, for any *given* θ, compute the Laplace inverse, according to the following expression, in the case where λ(s) is *not* coupled to $g^{u,v}$ (see Part I in this series) …

$$G^{u,v} \approx \kappa g^{u,v} \int_{\gamma} e^{sr} \lambda(s) ds \qquad (*)$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, κ is some constant, and γ denotes some form of contour integration that envelops any singularities tied to $\lambda(s)$.

Excluding the leading $\sigma g^{u,v}$ term for the moment … for $\xi(s)$ and $\eta(s)$, (*) computes to

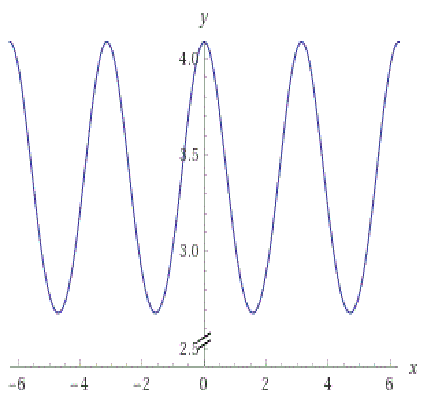
$$e^{r\cos(\theta)}\sin(r\sin(\theta))/\sin(\theta) \text{ and } e^{-r\cos(\theta)}\sin(r\sin(\theta))/\sin(\theta)$$

respectively, and of course, $1/s^2$ inverts to r. Combining all three, we have, for *any* radial distance r, given an angle θ between 0 and 2π, say, our two-dimensional solution for the plane, where $g^{u,v}$ is *not* coupled directly to $\lambda(s)$ …

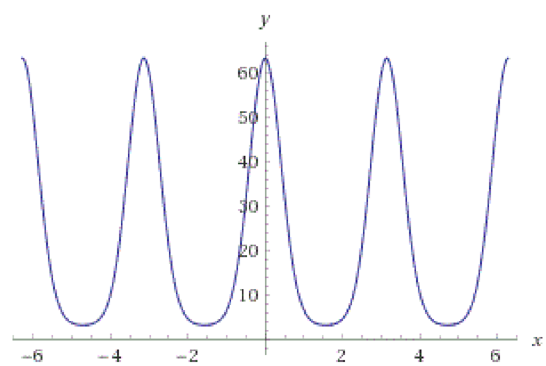
$$G^{u,v} \approx \sigma[r + e^{r\cos(\theta)}\sin(r\sin(\theta))/\sin(\theta) + e^{-r\cos(\theta)}\sin(r\sin(\theta))/\sin(\theta)]g^{u,v}$$

$$= \sigma[r + 2\sin(r\sin(\theta))\cosh(r\cos(\theta))/\sin(\theta)]g^{u,v} \quad (\dagger)$$

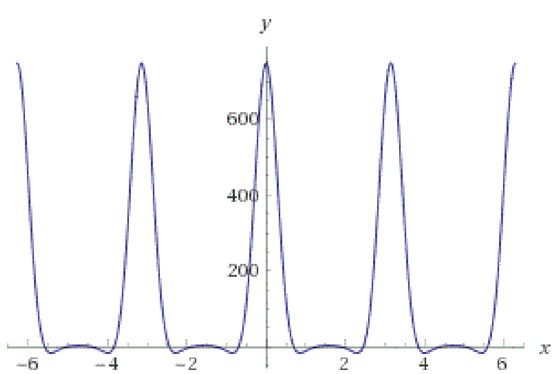
******



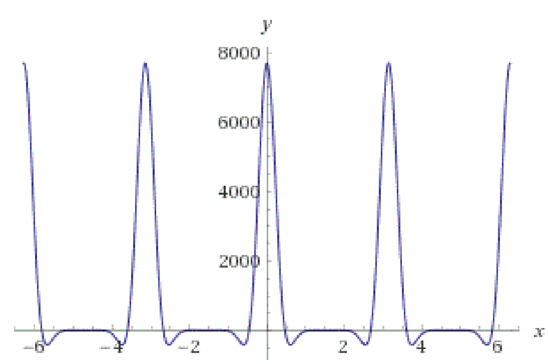
r = 1 , θ = x



r = 3 , θ = x



r = 5 , θ = x



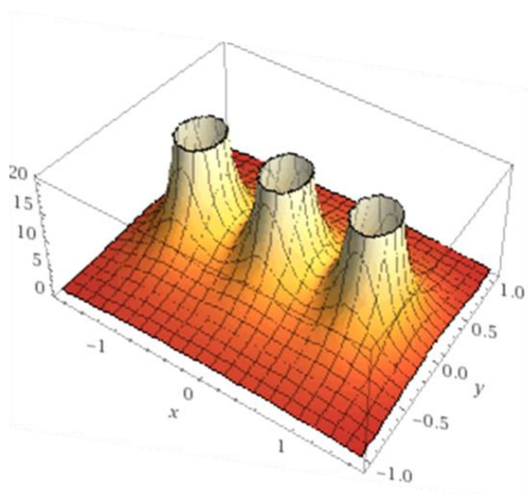
r = 7 , θ = x

******

The plots above are of the bracketed expression in (†), for the case where r = 1, 3, 5, and 7. They are actually our *conscious* or *subconscious* perception of dark energy as a whole, at different radial distances from the origin [O], along different polar angles, relative to the x-axis. This is because the bracketed expression is the Laplace *inverse* of the dark energy density function [$\lambda(s)$] … which brings us back into the domain of dark energy itself. That is to say, $G^{u,v} = C^{u,v} - kT^{u,v}$ … the remainder after subtracting out *tangible* matter from the space-time fabric. It is the singularities associated with $\lambda(s)$ that power this substance we call dark energy.

Notice that as r increases, the dark energy, for the most part, hovers around 0 for most values of θ, and then periodically spikes over a short range. It is these spikes in the dark energy that the astronomers cannot see with their instruments, at the present time, and it is doubtful they will ever be able to detect singularities associated with $\lambda(s)$, since these anomalies are *quantumlike* in nature.

With only *one* singularity associated with $\lambda(s) = \sigma/s^2$, and centered at the origin [O] of the star, it is not possible to see the real behavior of dark energy, because it behaves *uniformly* in all directions, at least relative to O. With three singularities tied together in additive fashion, the picture now becomes clearer …



A picture of $\lambda(s)$, with three singularities following an inverse square relationship, that was used in this analysis.

******

In the case where $g^{u,v}$ *is* coupled to $\lambda(s)$, we need to better understand how the foundational tensor is analytically continued to the *entire* unit circle in the complex plane C. I'm currently working on some ideas here, and plan to release an addendum on the subject in the near future. Stay tuned, as they say …

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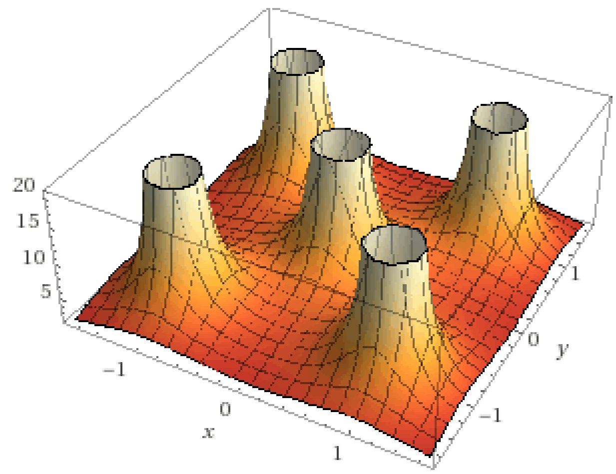
OTHER CONSIDERATIONS

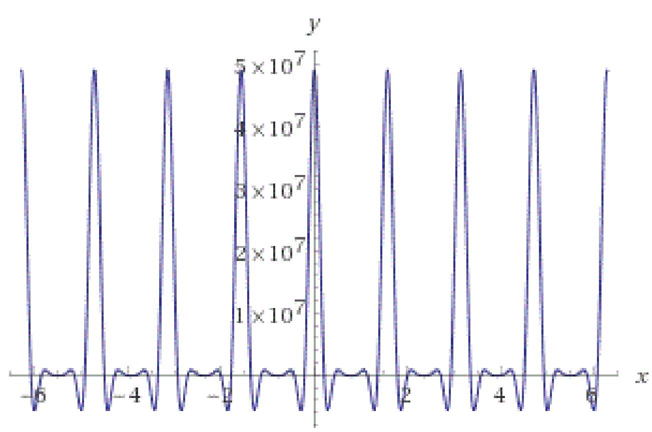
If we include the points (0, 1) and (0, -1) in the analysis, so that, in total, we have five singularities associated with the dark energy density function [λ(s)], then (†) becomes …

$$G^{u,v} \approx \sigma[r + 2\sin(r\sin(\theta))\cosh(r\cos(\theta))/\sin(\theta) + 2\sin(r\cos(\theta))\cosh(r\sin(\theta))/\cos(\theta)]g^{u,v} \quad (\S)$$



A picture of the density function for λ(s) in the case where there are five singularities .. .. that are all inverse square, and interacting together

A plot of the bracketed expression in (§) above … reflecting our perception of dark energy, for a radius r = 15 and varying polar angles, relative to the x-axis …



r = 15 , θ = x

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part VIII

In this addendum, we'd like to continue the discussion laid out in Part VII, and find a description for the field equations of general relativity … in the case where $g^{u,v}$ *is* coupled directly to the dark energy density function [$\lambda(s)$]. Here, we will assume $\lambda(s)$ has *five* singularities, and I will refer the reader to the details in the last addendum for more information.

Recall that the poles of $\lambda(s)$ fall on the unit circle [$\mathcal{C}$] in the complex plane C, *and* also at the center [O] of this unit circle, in our chosen frame of reference S. In the case of a coupling, we need to evaluate the following expression …

$$G^{u,v} \approx \kappa \int_{\gamma} e^{sr} \lambda(s) g^{u,v} ds \qquad (*)$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, κ is some constant, and γ denotes some form of contour integration that envelops any singularities tied to $\lambda(s)$.

Since there are poles on $\mathcal{C}$ and they could be *anywhere* here, it is not possible to evaluate $g^{u,v}$ in (*), unless, for example, we first *average* $g^{u,v}$ over the *entire* unit circle. To do this, let us start with a function g(s), which is well-behaved in C, and note from residue theory that

$$g(0) = (1/2\pi i)\int_{\gamma} (g(s)/s)\, ds$$

where γ is the contour $\mathcal{C}$. Now let $s = e^{i\theta}$, so that $ds = ie^{i\theta}d\theta$. Then the integral above becomes .. ..

$$g(0) = (1/2\pi i)\int_{\gamma} (g(s)/s)\, ds = (1/2\pi)\int_{0}^{2\pi} g(e^{i\theta})\, d\theta$$

Let $0 = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_n = 2\pi$, be a set of angles, so that the *radial* line ℓ_i runs from the origin [O] to $\mathcal{C}$, at an incline of θ_i to the x-axis, say. Then the integral just above, on the right-hand side, is the *limiting* case ($n \longrightarrow \infty$) of the following expression, where $\Delta_i = \theta_i - \theta_{i-1}$ …

$$(1/2\pi)[g(e^{i\theta_1})\Delta_1 + g(e^{i\theta_2})\Delta_2 + \ldots + g(e^{i\theta_n})\Delta_n]$$

Now make *all* of the Δ_i equal so that $\Delta_i = 2\pi/n$. Then in the limit, as $n \longrightarrow \infty$, we have …

$$g_{avg} = [g(e^{i\theta_1}) + g(e^{i\theta_2}) + \ldots + g(e^{i\theta_n})]/n = g(0)$$

Thus, g(0) *really* is the average of g(s) over the unit circle $\mathcal{C}$ in the complex plane, and so we have the following theorem, concerning $g^{u,v}$, in the case where it *is* coupled directly to λ(s), and thus well-behaved …

> If $g^{u,v}$ is the gravitational tensor extended to the complex plane $\mathbb{C}$, then the *average* value of $g^{u,v}$ over $\mathcal{C}$ is $g^{u,v}(0)$, where $\mathcal{C}$ is the unit circle in $\mathbb{C}$, centered at O.

We can use this powerful result to rewrite (*) as follows …

$$G^{u,v} \approx \kappa g^{u,v}(0)\int_{\gamma} e^{sr}\,\lambda(s)ds \qquad (\dagger)$$

By doing so … the field equations of general relativity … in the case of *five* singularities, now become, for the *coupled* scenario …

$$G^{u,v} \approx \sigma[r + 2\sin(r\sin(\theta))\cosh(r\cos(\theta))/\sin(\theta) + 2\sin(r\cos(\theta))\cosh(r\sin(\theta))/\cos(\theta)]g^{u,v}(0)$$

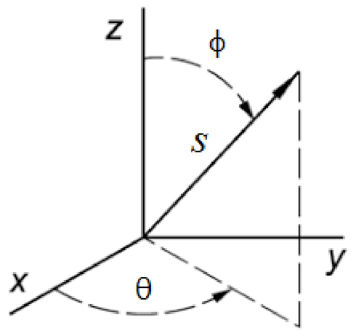
Note that here we have dropped the term $[g^{u,v}]'|_0$ … which is the derivative of $g^{u,v}$, evaluated at the origin. Previous studies in this series indicate that vanishing derivatives at O … seem to be a fundamental requirement, when examining the well-behavedness of $g^{u,v}$, and so, we'll give it the boot, as they say [this remark applies in the case of a coupling].

More generally, if the singularities associated with λ(s) are *symmetrically* distributed, relative to O, then the ideas put forth in this note should apply, as we scale up in count. In other words, aside from a little more algebra, it shouldn't matter whether the number of singularities is just a few or perhaps a dozen.

And by using averaging techniques, like the one shown here, we can simplify greatly the residue calculations, and thus, the forms for the field equations, themselves. I suppose there is no end to what's possible, when you think about it …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part IX

In this note, we're going to take our lessons learned in the last few addenda and scale up to three dimensions. In our quest to write down a more *complete* form for the field equations of general relativity, it only makes sense to do this, so we'll outline the approach, and keep the details to a minimum. Let us begin, then, with the diagram below, which shows a typical layout for spherical coordinates …



$$x^2 + y^2 + z^2 = s^2$$

$$x = s\sin(\phi)\cos(\theta) \;\; ; \;\; y = s\sin(\phi)\sin(\theta) \;\; ; \;\; z = s\cos(\phi)$$

Our perfect star shall be a unit sphere, centered at the origin [O] in a frame of reference S, and we'll scale up to seven *physical* singularities this time … where each follows an inverse *square* relationship. These singularities will be located at the origin, *and* at the points

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1)$$

The distance construction for any of these points is routine, and so we'll only show one here; viz …

$$\xi(s) = (x-1)^2 + y^2 + z^2 = s^2 - 2s\sin(\phi)\cos(\theta) + 1 \quad \ldots \quad \text{from } (x, y, z) \text{ to } (1, 0, 0)$$

Now define the angle α so that

$$\cos(\alpha) = \sin(\phi)\cos(\theta)$$

Then we can recast the distance above as

$$\xi(s) = s^2 - 2s\cos(\alpha) + 1$$

and here we know from Part VII … that the *zeroes* of $\xi(s)$ are going to be located at $e^{i\alpha}$ and $e^{-i\alpha}$, respectively.

Thus, we can take the Laplace *inverse* of the following expression …

$$G^{u,v} \approx \kappa\sigma g^{u,v}\int_{\gamma}(e^{sr} / \xi(s))ds \qquad (*)$$

and so begins our journey into the three-dimensional world of general relativity and dark energy, in the case where $g^{u,v}$ is *not* coupled directly to the dark energy density function $[\sigma/\xi(s)]$, itself.

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, κ and σ are constants, and γ denotes some form of contour integration that envelops any singularities or *poles* tied to $\sigma/\xi(s)$.

The exercise can now be repeated for each point where there is a *physical* singularity, and when it's all said and done, we have the following expressions, which we'll label (†) …

$$G^{u,v} \approx \sigma[r + 2\sin(r\sin(\alpha))\cosh(r\cos(\alpha))/\sin(\alpha) + 2\sin(r\cos(\beta))\cosh(r\sin(\beta))/\cos(\beta) + \mu(\phi)]g^{u,v}$$

$$\cos(\alpha) = \sin(\phi)\cos(\theta) \;;\; \sin(\beta) = \sin(\phi)\sin(\theta)$$

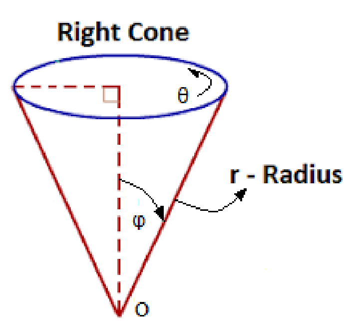
$$\mu(\phi) = 2\sin(r\sin(\phi))\cosh(r\cos(\phi))/\sin(\phi)$$

Notice that the *last* term, in the bracketed expression above, is actually the contribution from the singularities at the north and south points of the star, and that at $\phi = \pi/2$, α and β become θ. We are in the *equatorial* plane here, but now have an additional term $\mu(\phi)$, which computes to $2\sin(r)$. This term would *not* appear, if our analysis was restricted to *two* dimensions initially, just as it was in Part VII, where there were *five* physical singularities.

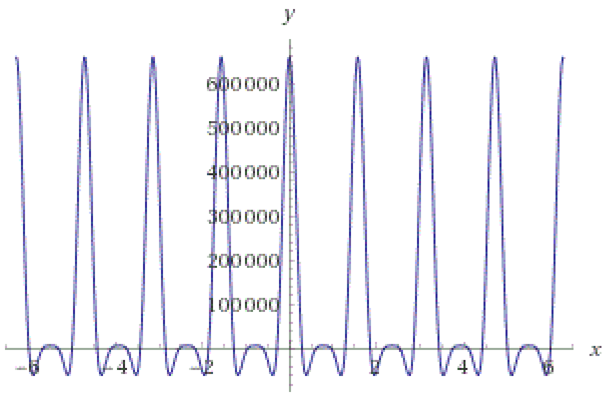
In the case where $g^{u,v}$ *is* coupled directly to the *overall* dark energy density function $[\lambda(s)]$, the only change to the equation above is $g^{u,v}$ mapping over to $g^{u,v}(0)$. Please see Part VIII for more details on how this comes about …

OTHER CONSIDERATIONS

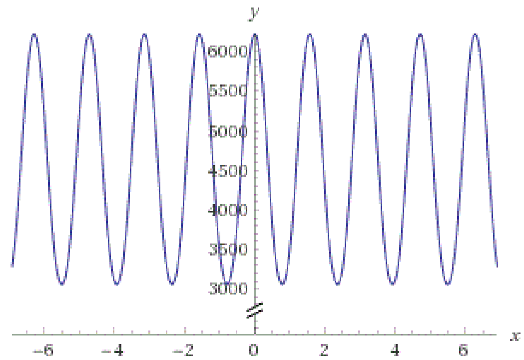
Here we'd like to do a few plot comparisons to see how theory lines up, by comparing results in the *equatorial* plane ($\phi = \pi/2$), with those at an angle $\phi = \pi/4$, to the z-axis.



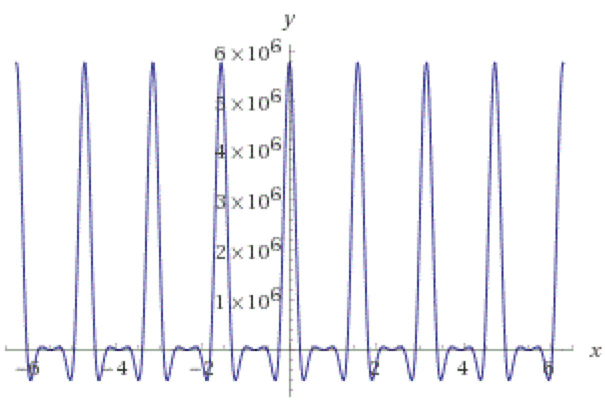
In the diagram above, the cone contains a blue circle at an angle ϕ, and at a radius r, from the origin [O]. We want to know, for certain values of r, what the bracketed expression in (†) looks like, as we traverse the circle itself, and then compare to the *corresponding* circle in the equatorial plane, where ϕ = π/2. Our perspective is always going to be relative to O, so essentially, we are comparing our *perception* of dark energy in the two-dimensional setup of Part VII, with the three-dimensional setup in this addendum. Here are the plots …



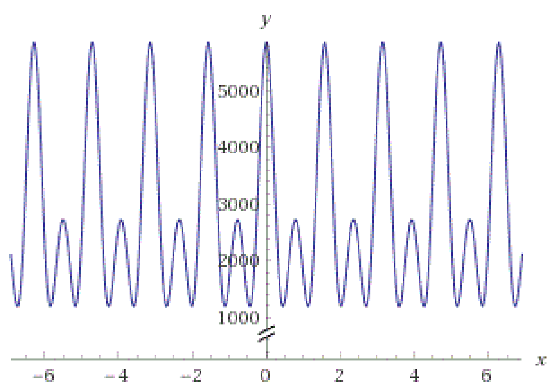
r = 11 , θ = x , ϕ = π/2 , 2D star



r = 11 , θ = x , ϕ = π/4 , 3D star



r = 13 , θ = x , ϕ = π/2 , 2D star



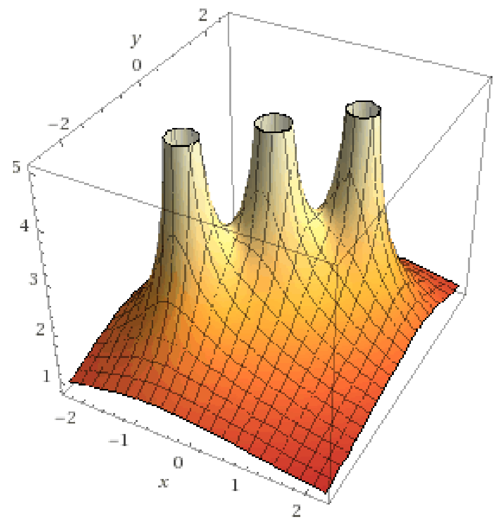
r = 13 , θ = x , ϕ = π/4 , 3D star

The first thing to note about these pictures is that, while they are *similar*, there are some notable differences. For a 2D star in the equatorial plane, not much changes for r = 11 versus r = 13. But with a 3D star the patterns do look different, and no doubt, this is due to two things: first, we are dealing with *seven* physical dark energy singularities now, instead of *five* in the 2D model, and secondly, our perspective for the 3D star is *relative* to the origin [O] of our coordinate system. We are, in the 3D case, looking *up* at an angle of 45 degrees, toward the north pole, and traversing the blue circle in the cone, as we make measurements of the dark energy, itself.

And there are other strange, and yet fascinating outcomes, at different radii and zeniths [ϕ], which, perhaps, we'll explore in future addenda …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part X

In this note, mainly out of curiosity, we are going to explore densities of type $\lambda(s) = \sigma/s$ again, but this time we'll include *two physical* singularities at the points (-1, 0) and (1, 0), for a *perfect* star in *two* dimensions, centered at the origin [O], in our chosen frame of reference S. As in previous addenda, the star shall be a *unit* circle, and at the origin there shall be a *third* physical singularity as well. In all, three physical singularities following a *simple* inverse relationship, as shown in the diagram below …



At the origin, $\lambda(s) = \sigma/s$, where σ is some constant, and at the other two points, we have the following distance constructions, for any point (x, y) in the two-dimensional plane, where we use ξ and η as labels (see Part VII for more details on the general setup for polar coordinates) …

$$\xi(s) = \sqrt{(x-1)^2 + y^2} = \sqrt{s^2 - 2s\cos(\theta) + 1} \quad \ldots \text{ from } (x, y) \text{ to } (+1, 0)$$

$$\eta(s) = \sqrt{(x+1)^2 + y^2} = \sqrt{s^2 + 2s\cos(\theta) + 1} \quad \ldots \text{ from } (x, y) \text{ to } (-1, 0)$$

Thus, our overall density function is the following sum …

$$\lambda(s) = \sigma[1/s + 1/\xi(s) + 1/\eta(s)]$$

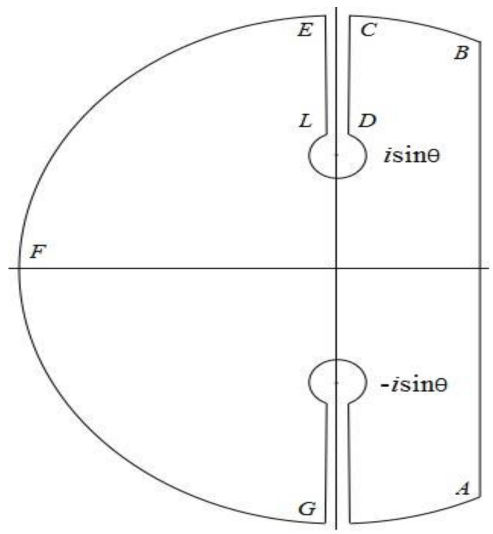
Now the zeroes of $\xi(s)$ are $e^{i\theta}$ and $e^{-i\theta}$, and for $\eta(s)$, are $-e^{i\theta}$ and $-e^{-i\theta}$. We can, therefore, for any *given* θ, compute the Laplace inverse, according to the following expression, in the case where $\lambda(s)$ is *not* coupled to $g^{u,v}$ (see Part I in this series) …

$$G^{u,v} \approx \kappa g^{u,v} \int_\gamma e^{sr} \lambda(s) ds \qquad (*)$$

…

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, κ is some constant, and γ denotes some form of contour integration that envelops any singularities tied to λ(s).

Let us now introduce the following *Bessel* contour, which we will use to calculate the Laplace inverse for λ(s), which, itself, is actually the line integral along AB in the diagram below …



By first translating s ---› s – cos(θ), we can deal with the inverse of 1/ξ(s), realizing now that the zeroes of ξ(s) fall along the *imaginary* axis at ***i***sin(θ) and **-*i***sin(θ), respectively. All we need to do is evaluate the integration along the arms CD and LE, noting that ***i***sin(θ) is a *branching* point, and then, of course, do the same thing for the two arms associated with **-*i***sin(θ). By letting the large arc BCEFGA tend to ∞ and the small circles ---› 0, we will achieve the desired result. To wit …

$$-2e^{r\cos(\theta)}\int_{\sin(\theta)}^{\infty} e^{iyr} / \sqrt{y^2 - \sin^2\theta}\, dy \qquad 2e^{r\cos(\theta)}\int_{\sin(\theta)}^{\infty} e^{-iyr} / \sqrt{y^2 - \sin^2\theta}\, dy$$

isin(θ) arms **-*i***sin(θ) arms

Combining the two, and factoring in the constants κ and σ, yields, for our Laplace inverse …

$$(2\sigma/\pi)e^{r\cos(\theta)}\int_{\sin(\theta)}^{\infty} \sin(yr) / \sqrt{y^2 - \sin^2\theta}\, dy$$

******

The same approach can be used for η(s), but now we translate according to s ---› s + cos(θ). The result is …

$$(2\sigma/\pi)e^{-r\cos(\theta)}\int_{\sin(\theta)}^{\infty} \sin(yr)\,/\sqrt{y^2 - \sin^2\theta}\; dy$$

Adding the two pieces together and noting that the Laplace inverse of 1/s is 1 … we have the following expression for the field equations of general relativity [for *any* r, and any angle θ between 0 and π, say], in the case where $g^{u,v}$ is *not* coupled directly to λ(s) …

$$G^{u,v} \approx \sigma[1 + (4/\pi)\cosh(r\cos(\theta))\int_{\sin(\theta)}^{\infty} \sin(yr)\,/\sqrt{y^2 - \sin^2\theta}\; dy]g^{u,v}$$

******

There are a couple of things worth noting here. First, when θ = 0 or π, we are looking to the east or to the west, relative to the origin [O] of the star. This is where we should expect to see dark energy at its strongest, for any radial distance r, due to the location of the physical singularities associated with the density function λ(s). And indeed, the bracketed expression above computes to

$$1 + 2\cosh(r)$$

here.

At θ = π/2, we are looking at the north pole of the star, relative to O, and the bracketed expression now computes to

$$1 + 2J_0(r)$$

where $J_0(r)$ is a Bessel function of the first kind. In between 0 and π/2, say, it looks like you are going to get a mix of exponential harmonics and Bessel-like behavior, at differing radii [r]. Thus, Bessel functions seem to play a critical role in the behavior of dark energy and general relativity, when the underlying dark energy density function [λ(s)] follows a simple inverse relationship. This could be a significant finding in terms of understanding how dark energy is woven into the space-time fabric, but at the same time, we *canno*t disregard the patterns we discovered when studying inverse *square* densities in the last few addenda. Perhaps a combination of the two results lies closest to the truth … ??

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XI

In the last addendum, we saw that for a *two-dimensional* star, centered about the origin O, in a chosen frame of reference S [with *physical* singularities at (-1, 0) and (1, 0), respectively], the field equations of general relativity, for *simple* inverse densities, could be written as [for *any* r, and any angle θ between 0 and π, say] …

$$G^{u,v} \approx \sigma[1 + (4/\pi)\cosh(r\cos(\theta))\int_{\sin(\theta)}^{\infty} \sin(yr)\,/\sqrt{y^2 - \sin^2\theta}\,dy]g^{u,v}$$

Further to this, we observed that at θ = 0 or π, the *bracketed* expression above [which we'll label (†)], reduced to

$$1 + 2\cosh(r)$$

and at θ = π/2, became

$$1 + 2J_0(r)$$

where $J_0(r)$ is a Bessel function of the first kind (derived from the Mehler-Sonine formula). Now from a *physical* standpoint, we *must* have agreement in (†) at r = 0, *no* matter the angle θ. Clearly, in the two cases above, both agree at the origin [O], and compute to 3, but this must be so for *any* θ between 0 and π, and thus we have the following intriguing result, which may be of some interest to number theorists …

$$\lim_{r \to 0} (4/\pi)\cosh(r\cos(\theta))\int_{\sin(\theta)}^{\infty} \sin(yr)\,/\sqrt{y^2 - \sin^2\theta}\,dy = 2$$

The expression can be simplified, just a bit, by noting that the cosh() term tends to 1 as r ---› 0, and so we have, for *all* 0 ≤ θ ≤ π …

$$\lim_{r \to 0} \int_{\sin(\theta)}^{\infty} \sin(yr)\,/\sqrt{y^2 - \sin^2\theta}\,dy = \pi/2$$

Whether there is any number-theoretic significance to this result, I'm not sure, but it is interesting to, at least, see how we might derive a result of the kind above, arguing from physical principles. Indeed, if we had to show it was true, just using mathematics, could we … ??

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XII

In the last addendum, we commented on limiting expressions for our *two*-dimensional solution in the plane, with *three* physical singularities, where the density function was of type $\lambda(s) = \sigma/s$. The solution is reproduced below …

$$G^{u,v} \approx \sigma[1 + (4/\pi)\cosh(r\cos(\theta))\int_{\sin(\theta)}^{\infty} \sin(yr)/\sqrt{y^2 - \sin^2\theta}\, dy]g^{u,v}$$

After consulting various handbooks on Laplace transforms and their inverses, it turns out that we can simplify the above equation even more, and rewrite it as …

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))]g^{u,v}$$

where $J_0()$ is Bessel function of the first kind. Thus, as r ---› 0, the bracketed expression does indeed compute to 3, for *any* $0 \leq \theta \leq \pi$, say, just as we conjectured in Part XI.

And, if we decide to scale up to *five* physical singularities … by adding the north and south points (0, 1) and (0, -1), all we need to do is let $\sin(\theta)$ and $\cos(\theta)$ trade places, giving us the following …

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta)) + 2\cosh(r\sin(\theta))J_0(r\cos(\theta))]g^{u,v}$$

Moving into the *three*-dimensional world of general relativity, with *seven* physical singularities located at the origin [O], in our chosen frame of reference S, as well as at

(1, 0, 0) (-1, 0, 0) (0, 1, 0) (0, -1, 0) (0, 0, 1) (0, 0, -1)

yields the following expressions [see Part IX for more information on the general setup regarding spherical coordinates] …

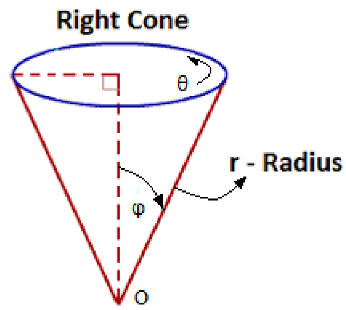
$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha)) + 2\cosh(r\sin(\beta))J_0(r\cos(\beta)) + \mu(\phi)]g^{u,v} \quad (\dagger)$$

$$\cos(\alpha) = \sin(\phi)\cos(\theta) \; ; \; \sin(\beta) = \sin(\phi)\sin(\theta)$$

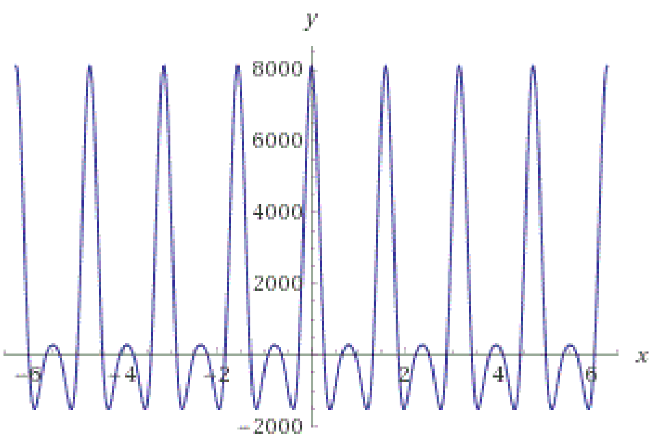
$$\mu(\phi) = 2\cosh(r\cos(\phi))J_0(r\sin(\phi))$$

Notice that the *last* term, in the bracketed expression above, is actually the contribution from the singularities at the north and south points of the star, and that at $\phi = \pi/2$, α and β become θ. We are in the *equatorial* plane here, but now have an additional term $\mu(\phi)$, which computes to $2J_0(r)$. This term would *not* appear, if our analysis was restricted to *two* dimensions initially, where there are *five* physical singularities, say.

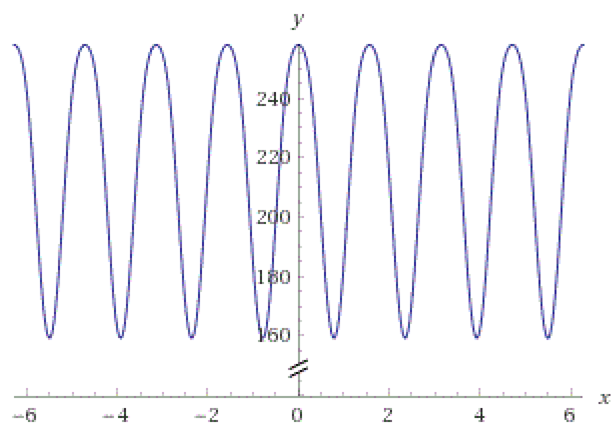
Now we'd like to do a few plot comparisons to see how theory lines up, by comparing results in the *equatorial* plane ($\phi = \pi/2$), with those at an angle $\phi = \pi/4$, to the z-axis.



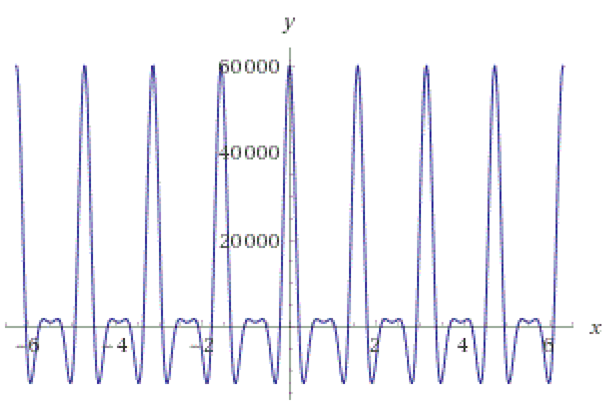
In the diagram above, the cone contains a blue circle at an angle ϕ, and at a radius r, from the origin [O]. We want to know, for certain values of r, what the bracketed expression in (†) looks like, as we traverse the circle itself, and then compare to the *corresponding* circle in the equatorial plane, where ϕ = π/2. Our perspective is always going to be relative to O, so essentially, we are comparing our *perception* of dark energy in the two-dimensional setup … with *five* physical singularities, versus the three-dimensional setup with *seven*. Here are the plots, to mull over ...



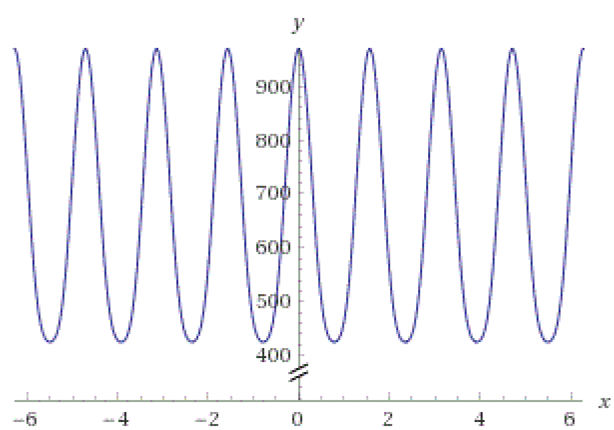
r = 9 , θ = x , ϕ = π/2 , 2D star



r = 9 , θ = x , ϕ = π/4 , 3D star



r = 11 , θ = x , ϕ = π/2 , 2D star



r = 11 , θ = x , ϕ = π/4 , 3D star

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XIII

In this short note we offer a quick proof that

$$J_0(r\sin(\theta)) \;=\; (2/\pi)\int_{\sin(\theta)}^{\infty} \sin(yr)\,/\sqrt{y^2 - \sin^2\theta}\;dy$$

where $J_0()$ is a Bessel function of the first kind. This identity forms the backbone of our compact forms for the field equations in general relativity, for densities of type $\lambda(s) = \sigma/s$, and ultimately allows us to plot *impressions* of dark energy, itself. Please see Part XII for more of the details here.

When $\theta = \pi/2$, the right-hand side of the expression above becomes $J_0(r)$, according to the Mehler-Sonine formula, and so we may write, replacing the variable r with $r\sin(\theta)$ …

$$J_0(r\sin(\theta)) \;=\; (2/\pi)\int_{1}^{\infty} \sin(y\sin(\theta))\,/\sqrt{y^2 - 1}\;dy \qquad (\dagger)$$

Now let $u = y\sin(\theta)$. Then the right-hand side of (†) becomes, for θ between 0 and π, say …

$$(2/\pi)\int_{\sin(\theta)}^{\infty} \sin(ur)\,/\sqrt{u^2 - \sin^2\theta}\;du$$

which is what we wanted to show.

It's not a terribly rigorous proof, but it seems to get the job done, and in my opinion, is better than looking things up in a handbook, which is the approach that was taken in the last addendum. And with this, we've come full circle in our study of dark energy density functions, that are both inverse *square*, and inverse *simple*. Hopefully the reader has benefited from these discussions, and may feel inspired to research the ideas further …

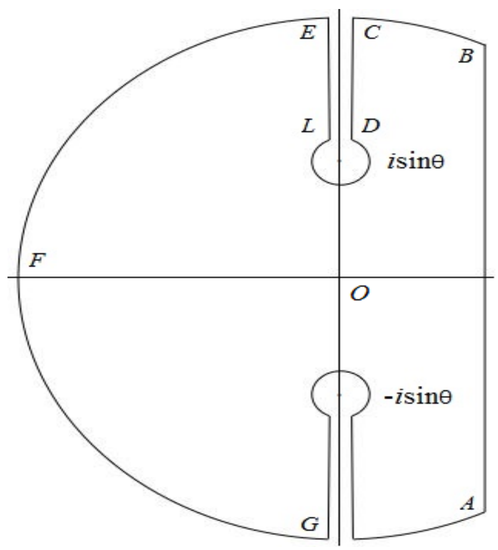
******

At the same time, stay tuned. I'm always working on something, and who knows what's coming down the pike next. Something interesting … I hope …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XIV

In this note, we are going to discuss densities of type $\lambda(s) = \sigma/s$ again, just as we have in the last few addenda, but now address the case where the gravitational tensor $g^{u,v}$ *is* coupled *directly* to the dark energy density function [$\lambda(s)$]. So far, we've avoided the issue, largely because it's a complex one, but alas, the time has come to deal with it …

Let us begin, then, by recalling the Bessel contour that was used to produce a solution in the first place [see Part X], and shown below for convenience …



In evaluating the contour for $\xi(s)$ [again, see Part X for the general setup], we are really evaluating the following expression, after first translating s ---› s – cos(θ), in the case of a coupling …

$$\kappa\sigma e^{r\cos(\theta)}\int_{\gamma} e^{sr} g^{u,v}(s) \,/\sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\; ds$$

Here, κ and σ are constants and γ denotes the contour above. But now imagine ***i***sin(θ) drifting *down* to the origin [O], as θ ---› 0 in the picture above, and –***i***sin(θ) drifting up to O. Then the expression above becomes …

$$\kappa\sigma e^{r}\int_{\gamma} e^{sr} g^{u,v}(s) \,/\sqrt{(s)(s)}\; ds \qquad (\dagger)$$

where γ is *now* the large contour BCEFGAB. In other words, the arms CD and LE *no* longer exist, and *nor* do their counterparts in the lower half-plane, and so (†) computes to

$$\sigma e^{r} g^{u,v}(0)$$

since we are dealing with a *simple pole* here at O. The exercise can now be repeated for $\eta(s)$, after first translating s ---› s + cos(θ), and in the limit as θ ---› 0, we obtain

$$\sigma e^{-r} g^{u,v}(0)$$

This takes care of the *physical* singularities at the points (1, 0) and (-1, 0) … and since there was a *third* physical singularity at O to begin with, defined by $\lambda(s) = \sigma / s$, we take its Laplace inverse [which is $\sigma g^{u,v}(0)$], and so recover the field equations of general relativity in the case of a *coupling*, when θ is small …

$$G^{u,v} \approx \sigma[1 + 2\cosh(r)]g^{u,v}(0) \qquad (\S)$$

Comparing to the general solution in the *uncoupled* case, where $g^{u,v}$ is *always* a function of r, given *any* angle θ, viz …

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))]g^{u,v} \qquad (*)$$

we see that as θ ---› 0, (*) becomes

$$G^{u,v} \approx \sigma[1 + 2\cosh(r)]g^{u,v}$$

and so (§) and (*) agree *symbolically*, when θ is small. This is *very good* news, because it allows to make a choice *philosophically*, and that choice is this: do we take this partial result and live with it for *all* $0 \le \theta \le \pi$, in the case of a coupling, or do we struggle to find another expression for (§) as θ drifts northward to $\pi/2$, say.

I say we take what we can get in this game, and live with what we've got. If we do that, then in the case where $g^{u,v}$ *is* coupled *directly* to $\lambda(s)$, the field equations of general relativity may be written as

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))]g^{u,v}(0)$$

no matter the angle θ … a result which is consistent with inverse *square* densities, discussed in previous addenda, and other densities too, throughout this series. And with that, we'll now say with some confidence, that inverse *simple* densities have been dealt with, to the best of our ability …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XV

In this addendum, we are going to revisit densities of type $\lambda(s) = \sigma/s\sqrt{s}$ again, and derive a result in number theory, from physical principles. It isn't always the case that we need to be working with general relativity per se, or any other *foundational* duality, for that matter, so here we'll use what we've learned in previous addenda, and show that

$$\pi = 2\sqrt{2}\,\Gamma(3/4)\,\Gamma(5/4) \quad (\S)$$

where Γ() is the gamma function. Note that since π is transcendental, at least one of Γ(3/4) and Γ(5/4) must be as well. Most likely they both are, but we'll take what we can get in this case ...

Now because we are dealing with more complex densities here, we'll defer to handbooks on Laplace transforms and their inverses, at least initially, and so begin by noting the following …

$$\Gamma(\nu + ½)u^{-2\nu - 1} \text{ ---› Laplace Inverse ---› } \sqrt{\pi}\,(2\alpha)^{-\nu}r^{\nu}J_{\nu}(\alpha r)\,,\ u = \sqrt{s^2 + \alpha^2}\,,\ \mathrm{Re}(\nu) > -1/2$$

Here Γ() denotes the gamma function, as we said, and $J_\nu()$ is a Bessel function of order ν. Now when ν = 1/4 we are essentially dealing with $\lambda(s) = \sigma/s\sqrt{s}$ … and so … for a *two* dimensional star, with a *physical* singularity at the origin [O], we may write the field equations of general relativity as follows, in the case where $g^{u,v}$ is *not* coupled to λ(s) …

$$G^{u,v} \approx \sigma[(\sqrt{\pi} / \Gamma(3/4)(2\alpha)^{1/4})r^{1/4}J_{1/4}(\alpha r)]g^{u,v} \quad (*)$$

See Part VI for more on the Laplace inverse of $\sigma/s\sqrt{s}$ itself, and also see some of the more recent addenda on how Bessel contours were used in the case of *simple* inverse densities. In this setup, we define α to be an *arbitrarily* small number, and eventually we'll let α ---› 0.

Now as z ---› 0, it can be shown that (see, for example, the online handbook titled *Digital Library of Mathematical Functions*)

$$J_{\nu}(z) \text{ ---› } (z/2)^{\nu} / \Gamma(\nu + 1)$$

and so (*) becomes, for *fixed* r, and α ---› 0 …

$$G^{u,v} \approx \sigma[(\sqrt{\pi}/(\Gamma(3/4)\Gamma(5/4)\sqrt{2}))\sqrt{r}]g^{u,v} \quad (\dagger)$$

Comparing to our result in Part VI, and reproduced here, for the *uncoupled* case …

$$G^{u,v} \approx 2\sigma\sqrt{r/\pi}\,g^{u,v}$$

we see that the bracketed expression in (†) *must* agree with $2\sqrt{r/\pi}$ as α ---› 0, and thus we have no choice but to conclude (§) is indeed true …

We can now deal with the case where $\lambda(s) = \sigma/s\sqrt{s}$ has *three* physical singularities, where one is at the origin O of our frame of reference S, and the other two are at (1, 0) and (-1, 0), respectively. Here we are dealing with a *two* dimensional star, which is a *unit* circle, centered at O.

In this scenario, where $g^{u,v}$ is *not* coupled to $\lambda(s)$, (*) becomes, after first translating s ---› s – cos(θ) and then to s ---› s + cos(θ) …

$$G^{u,v} \approx 2\sigma[\sqrt{r/\pi} + (\sqrt{\pi} / \Gamma(3/4)(2\alpha)^{1/4})r^{1/4}\cosh(r\beta)J_{1/4}(r\alpha)]g^{u,v}$$

with $\alpha = \sin(\theta)$ and $\beta = \cos(\theta)$. And now letting θ ---› 0, the above equation reduces to …

$$G^{u,v} \approx 2\sigma\sqrt{r/\pi}\,[1 + 2\cosh(r)]g^{u,v} \quad (\#)$$

In the case where $g^{u,v}$ *is* coupled to $\lambda(s)$, we repeat the exercise in Part XIV, where the roots $\boldsymbol{i}\sin(\theta)$ and $-\boldsymbol{i}\sin(\theta)$ merge at the origin of our *Bessel* contour. As θ ---› 0, from Part VI, the integral

$$\kappa\sigma e^{r\cos(\theta)}\int_{\gamma} e^{sr}g^{u,v}(s)\,/\big((s - \boldsymbol{i}\sin(\theta))(s + \boldsymbol{i}\sin(\theta))\big)^{3/4}\,ds$$

evaluates to $2\sigma e^{r}\sqrt{r/\pi}\,g^{u,v}(0)$ for *large* r, after first translating s ---› s – cos(θ). Repeating, in the case where s ---› s + cos(θ) … and combining with the *physical* singularity that was already at O to begin with, gives us the following, for the coupled scenario …

$$G^{u,v} \approx 2\sigma\sqrt{r/\pi}\,[1 + 2\cosh(r)]g^{u,v}(0) \quad (\ddagger)$$

Thus, (#) and (‡) agree *symbolically* for *small* θ and *large* r, when $\lambda(s) = \sigma/s\sqrt{s}$. If we're content with this agreement, under these constraints, then more generally, we may write

$$G^{u,v} \approx 2\sigma[\sqrt{r/\pi} + (\sqrt{\pi} / \Gamma(3/4)(2\alpha)^{1/4})r^{1/4}\cosh(r\beta)J_{1/4}(r\alpha)]g^{u,v}(0)$$

when $g^{u,v}$ *is* coupled directly to $\lambda(s)$ …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XVI

In this piece, we are going to formulate a general theorem for the types of densities we have been studying, based on our previous research. Let us begin then, with the following statement, which was first introduced in the last addendum, and which we'll label (†) …

$$\Gamma(\nu + ½)u^{-2\nu - 1} \text{ ---› Laplace Inverse ---› } \sqrt{\pi}\,(2\alpha)^{-\nu}r^{\nu}J_{\nu}(\alpha r)\,,\ u = \sqrt{s^2 + \alpha^2}\,,\ \mathrm{Re}(\nu) > -1/2$$

Here $\Gamma()$ denotes the gamma function and $J_{\nu}()$ is a Bessel function of order ν. Now suppose our dark energy density function is defined, in principle, to be of type

$$\lambda(s) = \sigma/s^{\mu}$$

where $\mu = 2\nu + 1$. Then *any* formulation of the field equations of general relativity, that incorporates dark energy via $\lambda(s)$, will *always* include Bessel functions. That is to say,

$$G^{u,v} \approx \Psi(.)g^{u,v} \quad (*)$$

where $\Psi(.)$ denotes the Laplace inverse of $\lambda(s)$, obtained from (†). Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, and $g^{u,v}$ is the gravitational tensor.

For example, if $\nu = 1/2$, we would be looking at inverse *square* densities, discussed early on, and if $\nu = 0$, we'd be looking at inverse *simple* densities, and so forth. No matter our choice, so long as we have $\mu > 0$, so that $\lambda(s)$ ---› 0 with *increasing* s, Bessel functions will *always* be an *inherent* part of the solution, as per (*).

It is important, I feel, to stop and think about this for a minute … because Bessel functions, themselves … appear everywhere in nature, from electromagnetic waves, to heat conduction and acoustics, fluid flow and even Schrodinger's equation … there you'll find them. So, we could ask, why *wouldn't* they appear in general relativity or *any* other foundational duality [such as energy-mass or wave-particle], where we are interested in the overall connection to dark energy ?

******

In the case where $g^{u,v}$ *is* coupled directly to $\lambda(s)$, it is not that easy to produce an equivalent to (*), because $g^{u,v}$ becomes part of the residue calculation. However, we have enough evidence now to offer up the following conjecture, viz …

$$G^{u,v} \approx \Psi(.)g^{u,v}(0) \quad (\S)$$

where here, $g^{u,v}$ is *well-behaved* at the origin [O] of our frame of reference S, and $[g^{u,v}]'|_0 = 0$. The tick mark means differentiate once, and evaluate at O …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XVII

We're getting to the end in this series, so in this note a few closing and clarification remarks are in order. First, we'd like to go back and talk, for a moment, about the translated integral that was used in Part XIV, to establish a *coupling* in the case of small θ … for inverse *simple* densities. It is reproduced here …

$$\kappa\sigma e^{r\cos(\theta)}\int_\gamma e^{sr}g^{u,v}(s)\,/\sqrt{(s-i\sin(\theta))(s+i\sin(\theta))}\,ds \qquad (*)$$

By translating s ---› s – cos(θ), what we are really doing is setting u = s – cos(θ), say, and rewriting the original integral as above, where the original integral is (see Part X for the details) …

$$\kappa\sigma\int_\gamma e^{sr}g^{u,v}(s)\,/\sqrt{s^2 - 2s\cos(\theta) + 1}\,ds$$

The operation is completed by mapping *u* back to *s* in (*). But doesn't this mean $g^{u,v}$ is really a function of s + cos(θ) in the translated integral [(*)] ? The answer is actually yes, but here we are using an *implicit* averaging technique to *simplify* the outcome, by noting that the *average* value of *u* is really *s*, as θ ranges between 0 and π, say. And so, (*) takes on the form that it does, accordingly.

******

Now let's return to Part VII, where we first studied inverse *square* densities, for a *two* dimensional star, with *physical* singularities at the origin [O] of the star, and at (1, 0) and (-1, 0). The solution is reproduced below, in the *uncoupled* case …

$$G^{u,v} \approx \sigma[r + 2\sin(r\sin(\theta))\cosh(r\cos(\theta))/\sin(\theta)]g^{u,v} \qquad (\S)$$

From the last addendum [Part XVI] … we know this problem involves Bessel functions of order ½, according to the following expression, which we'll label (†) …

$$\Gamma(\nu + ½)u^{-2\nu-1} \text{ ---› Laplace Inverse ---› } \sqrt{\pi}\,(2\alpha)^{-\nu}r^{\nu}J_\nu(\alpha r)\,,\ u = \sqrt{s^2+\alpha^2}\,,\ \mathrm{Re}(\nu) > -1/2$$

And so, at the origin [O], as α ---› 0, the Laplace inverse becomes, for ν = ½ ...

$$(\sqrt{\pi}\,r)\,/\,2\Gamma(3/2)$$

according to our limiting formula

$$J_\nu(z) \text{ ---› } (z/2)^{\nu}\,/\,\Gamma(\nu+1) \text{ as } z \text{ ---› } 0$$

where Γ() is the gamma function. Comparing to the first term in the *bracketed* expression for (§), this means we *must* have $\Gamma(3/2) = \sqrt{\pi}\,/\,2$.

******

When $\alpha = \sin(\theta)$ and $\beta = \cos(\theta)$, we are dealing with the two *physical* singularities situated at (1, 0) and (-1, 0), respectively. Here, (†) resolves as …

$$2(\sqrt{\pi} / (2\alpha)^{1/2})r^{1/2}\cosh(r\beta)J_{1/2}(r\alpha)$$

after first translating s ---› s – cos(θ) and then to s ---› s + cos(θ). Since this must agree with the *second* term in the *bracketed* expression for (§), it means, among other things, that

$$J_{1/2}(r) \;=\; \sqrt{2/\pi r}\,\sin(r)$$

and indeed, this turns out to be the case, when consulting the various handbooks.

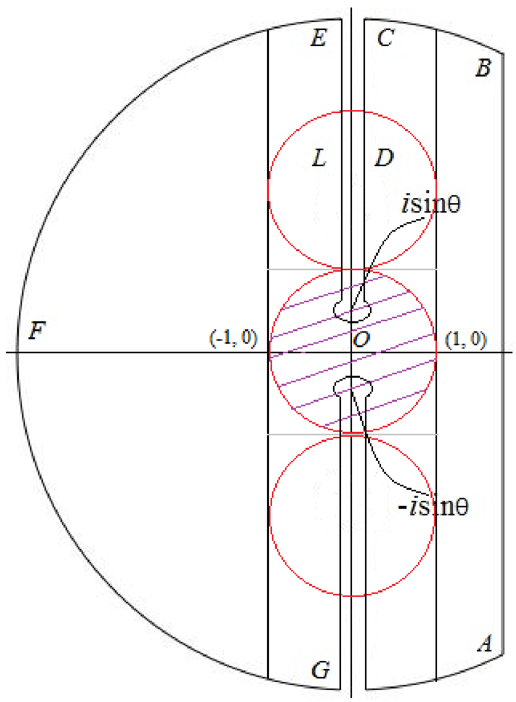
******

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XVIII

How accurate is our *implicit* averaging technique, discussed in the last addendum ? Recall, the integral under discussion is

$$\kappa\sigma e^{r\cos(\theta)} \int_\gamma e^{sr} g^{u,v}(s) \,/ \sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\, ds \qquad (*)$$

where γ is the Bessel contour, as shown below (but not drawn to scale), *and* the argument to $g^{u,v}$, which is really s + cos(θ), has been replaced by *s*.



Notice in the diagram above, that the Laplace inverse … defined as the line integral along AB, really amounts to calculating things along the *arms* associated with the *branching* points at ***i***sin(θ) and –***i***sin(θ), respectively. These arms, in the upper half-plane, are labelled CD and LE.

Thus, as θ ranges between 0 and π, say, s + cos(θ) ranges between *s* – 1 and *s* + 1, and so, when we are integrating along the arms, in the limit, as the large arc ---› ∞, and the small circles associated with the *branching* points tend to 0, we are, in effect, integrating along the imaginary axis, which we can define as *s* = ***i****y*. And thus, our question of accuracy amounts to studying s + cos(θ) in the (infinitely long) complex strip S, defined by -1 ≤ Re(z) ≤ +1, where z itself, is complex.

…

Let us now build an infinitely long set of *red* circles, as shown above, which is really a set of *unit* disks, where the *origin* of each disk lies on the imaginary axis ***i**y*. The unit disk centered at O is thatched in purple, where O is the origin of the complex plane C.

Now choose *two* circles, the first being the *red* one centered at O, and the second one [not shown], also centered at O, but say of radius = ½. We'll call these $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, and we'll start by distributing *n* points *evenly* on both circles, so that in all, the total number of points is 2*n*. Next, let us *evaluate* $g^{u,v}$ on $\mathcal{C}_1$ and $\mathcal{C}_2$, at these points, and define our *overall* average to be …

$$g_{avg} = \Big(\sum_{n \text{ pts } \mathcal{C}_1} g^{u,v} + \sum_{n \text{ pts } \mathcal{C}_2} g^{u,v} \Big) \,/\, 2n$$

$$= \tfrac{1}{2} \Big(\sum_{n \text{ pts } \mathcal{C}_1} g^{u,v} \,/\, n + \sum_{n \text{ pts } \mathcal{C}_2} g^{u,v} \,/\, n \Big)$$

Since we know from residue theory, and our research in Part VIII, that the *average* value of $g^{u,v}$ over a circle centered at O is always $g^{u,v}(0)$, it follows that g_{avg} approaches $g^{u,v}(0)$, as n ---› ∞. And, since the disk [$\mathcal{D}$], bounded by the *red* circle at O is really just a *union* of circles, of differing radii, but *all* centered at O, we can extend the argument, rather naturally … and conclude that g_{avg} over $\mathcal{D}$ is actually $g^{u,v}(0)$.

By extension, this will *also* be true for *any* disk [$\mathcal{D}'$] bounded by a *red* circle in the diagram. Thus, the *average* value of $g^{u,v}$ over that disk will be $g^{u,v}(\xi)$, where ξ is the *center* of $\mathcal{D}'$, and lying on the imaginary axis $s = iy$. This, according to our *union of circles* argument …

To determine accuracy, all we need to do is ask ourselves how *often* $u = s + \cos(\theta)$ passes *through* a disk, as θ ranges between 0 and π, when $s = iy$, for any *y*. When *u* does, we can replace $g^{u,v}(u)$ with $g^{u,v}(\xi)$, because $g^{u,v}(u)$ is now part of the *averaging* process, and so, a more accurate portrayal of things emerges. In *all* other cases, we let $u = s$, and, either way, carry out the integration along ***i**y*.

Now since the area of any disk is π, and the area of the square containing that disk is 4, the answer in regards to the question of accuracy, is π/4, or about 80% of the time. This means that in terms of using the integral defined by (*), in the case of a *coupling*, we should be satisfied that it will do what we want it to do, most of the time. And, in fact, as θ ---› 0, we can actually rewrite (*) as

$$\kappa\sigma e^{r\cos(\theta)} \int_\gamma e^{sr} g^{u,v}(0) \,/ \sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\; ds \qquad (\dagger)$$

since now the roots $i\sin(\theta)$ and $-i\sin(\theta)$ have merged at O, putting us inside the disk [$\mathcal{D}$], centered at O. And here we know the average value of $g^{u,v}$ over $\mathcal{D}$ is, in fact, $g^{u,v}(0)$ [we are inside $\mathcal{D}$ because we are now dealing with a *simple* pole at O, and thus, without loss of generality, can treat γ as $\mathcal{C}_1$, or, for that matter, as *any* circle within $\mathcal{D}$, centered at O].

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XIX

In this note, we are going to veer off course a bit, and look at 'Yukawa-style' densities, named after the Japanese physicist Hideki Yukawa. These potentials (or densities) arise when studying scalar fields associated with particles like the boson, and so here we want to *adapt* the Yukawa potential as a candidate for the dark energy density function, thusly …

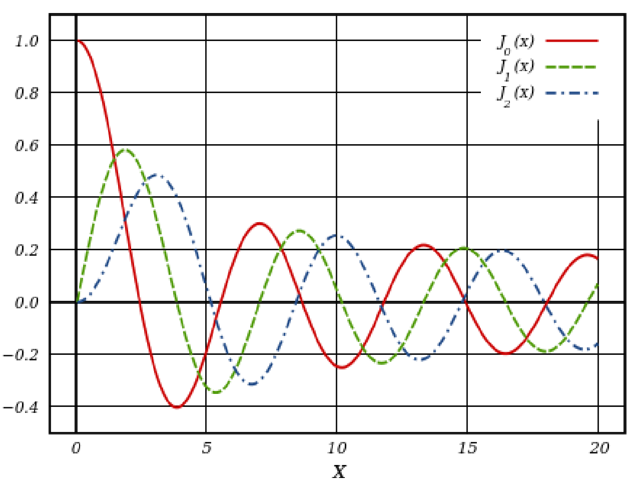
$$\lambda(s) = \sigma e^{-\mu s}/s \qquad (*)$$

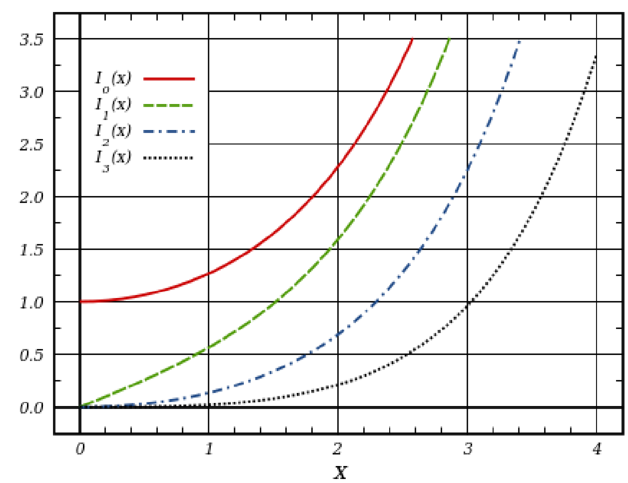
Here, σ and μ are positive constants, and we see that λ(s) falls off *very quickly* as s increases, especially if μ is *large*. Notice, too, that if μ = 0, we are back to inverse *simple* densities, discussed earlier in this series.

So let's begin by noting that for r > μ the following holds, where J_0 is a Bessel function of order 0 ..

$$e^{-\mu\nu}/\nu \text{ ---› Laplace Inverse ---› } J_0(\omega\alpha)\,,\ \nu = \sqrt{s^2 + \alpha^2}\,,\ \omega = \sqrt{r^2 - \mu^2} \qquad (\dagger)$$

When r < μ, we'll simply interpolate, by noting that $I_0(x) = J_0(\boldsymbol{i}x)$, where I_0 is a *modified* Bessel function of order 0, and ***i*** is $\sqrt{-1}$. Thus, $\omega = \sqrt{\mu^2 - r^2}$ if r < μ, and we are dealing with I_0.





In the plots above, the red lines represent J_0 and I_0, respectively, and we see that *both* agree at 0. However, when r < μ in (†), $J_0(\omega\alpha)$ is a function of some imaginary number, and so we defer to I_0.

As an example, suppose for the time being, we let α = 1 and μ = 10. Then when r > 10, we use $J_0(\omega)$ in our Laplace inverse, and when r < 10, we use $I_0(\omega)$. At r = 10, both J_0 and I_0 compute to 1, since ω is now 0. Thus, with I_0, we 'work our way back' to ω = 0 *from* ω = 10, as r goes from 0 *to* 10, and then we let J_0 take over, for r > 10.

You can see that I_0 falls pretty quickly with *increasing* r, and this makes sense, since the Yukawa density itself, falls off very quickly. The dividing point is μ … in so far as Laplace transforms and

their inverses are concerned, and here we can track this descent by splitting things up, using I_0 to the *left* of μ, and J_0 to the *right* of μ.

******

For a *two* dimensional star, with *physical* singularities at the origin [O] of the star, and at (1, 0) and (-1, 0), we can now formulate the field equations of general relativity as follows. As α ---› 0, for *fixed* r, $J_0(\omega\alpha)$ ---› 1, whether or not ω is *real* or *imaginary*. Thus, the Laplace inverse of our *physical* singularity at O is always going to be 1, using interpolation.

For the two *physical* singularities at (1, 0) and (-1, 0), with α = sin(θ) and β = cos(θ), the Laplace inverse computes to [labelling as (§)] …

$$2\cosh(r\beta)J_0(\omega\alpha)\ ,\ r > \mu\ ,\ \omega = \sqrt{r^2 - \mu^2}$$

$$2\cosh(r\beta)I_0(\omega\alpha)\ ,\ r < \mu\ ,\ \omega = \sqrt{\mu^2 - r^2}$$

after first translating s ---› s – cos(θ) and then to s ---› s + cos(θ). Thus,

$$G^{u,v} \approx \sigma[1 + \psi(r)]g^{u,v} \quad (‡)$$

where ψ(r) is one of the two expressions above.

This, then, is our solution for the formulation of the field equations of general relativity, in the case of a Yukawa density [λ(s)], which is *not* coupled directly to $g^{u,v}$. Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, and $g^{u,v}$ is the gravitational tensor. In the case of a coupling, simply replace $g^{u,v}$ with $g^{u,v}(0)$.

Even though the Yukawa potential [density] originally finds its roots in quantum mechanics, this addendum shows us how we *might* adapt it to general relativity and dark energy. This is good news, because it tells us these two disciplines may have deeper connections than perhaps we first thought possible. Certainly the results here seem encouraging …

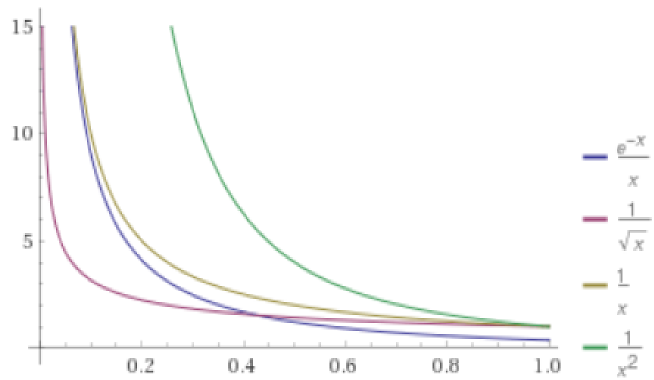
******

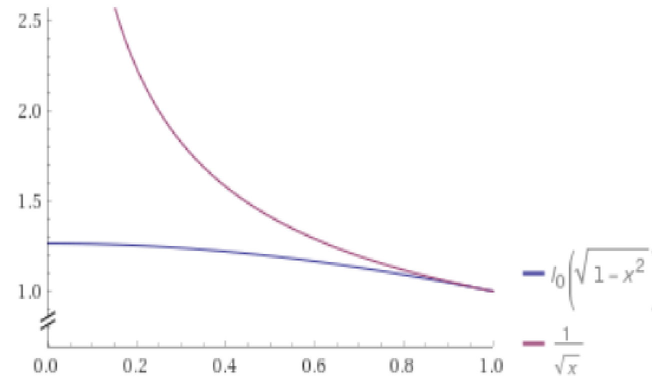
Some sanity checks are in order. When ω = 0, both expressions in (§) above yield the same thing, so this is reassuring. However, when r ---› 0, the second equation in (§) applies, giving us $2I_0(\mu\alpha)$. If α was strictly 0, because we were dealing with the *physical* singularity at O, all would be well. But there are *also* the *physical* singularities at (1, 0) and (-1, 0), and here α = sin(θ). Thus, $2I_0(\mu\alpha)$ varies with α as θ ranges between 0 and π, say. Not exactly the most desirable of outcomes, but we've arrived at this point because we decided to interpolate when r < μ, by flipping to I_0.

Had we not done this … there would really be no way to understand, at all, what was going on when r < μ, using Laplace. So I guess something is better than nothing, and for now, we'll simply be content to ponder the meaning of (‡), for *all* possible values of r …

OTHER CONSIDERATIONS

The following plots compare various densities and Laplace inverses in the range [0, 1]. Notice that the Yukawa density, with μ and σ = 1, falls off somewhat faster than 1/s. Thus, when looking for a Laplace inverse for Yukawa, we need to find something that is 'stronger' than the inverse of 1/s [which is 1], but 'not as strong' as the Laplace inverse of $1/\sqrt{s}$, which also varies as $1/\sqrt{r}$.





Dark energy densities [λ(s)]. The Yukawa density is shown in blue … just below 1/x

Laplace inverse of Yukawa and $1/\sqrt{x}$, where Yukawa is interpolated using I_o

Observe in the rightmost diagram, that the modified Bessel function $I_0(\sqrt{\mu^2 - r^2})$ ranges between 1 and about 1.3, as *x* (which is really *r*) moves from 1 to 0. Thus $I_0 \geq 1$ here. Also, I_0 is always below $1/\sqrt{r}$ in this range, and so, is 'weaker' than the latter. Hence, if we are simply trying to *fit* the Yukawa density somewhere *in between* 1 and $1/\sqrt{r}$, in terms of Laplace inverses, then the *modified* Bessel function I_0 is actually a good choice … and is also the natural choice when the argument to J_0 is *imaginary*.

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Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XX

Now that we're more comfortable with the dark energy density function [$\lambda(s)$], and how it bolts into general relativity, we are going to revisit Einstein's famous equation,

$$E = \mathfrak{m}c^2$$

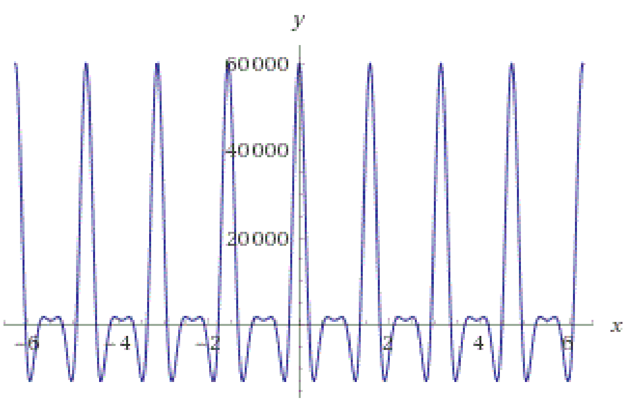
and see how this *instance* of the *energy-mass* duality can be embedded within dark energy, itself. [The reader is encouraged to revisit the series titled *The Case For Negative Mass In α-Space* (Parts I and II, pp 175-177) before proceeding].

We will assume a *two* dimensional star centered at the origin [O], in our chosen frame of reference S, with *physical* singularities at O, and at (1, 0) and (-1, 0), respectively. And further to this, we shall assume that $\lambda(s) = \sigma / s$, so that it is an inverse *simple* density function.

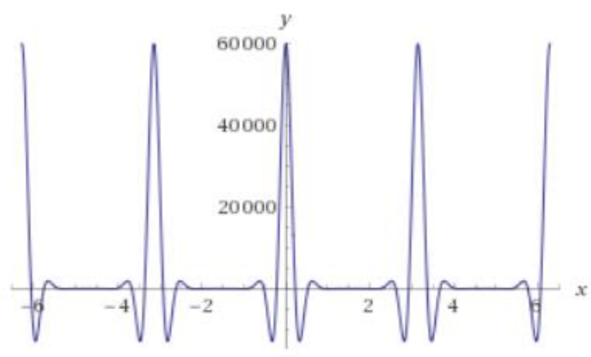
Now recall, from Part XII in this series, that the field equations of general relativity are, in the case where $g^{u,v}$ is *not* coupled directly to $\lambda(s)$ …

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))]g^{u,v} \quad (*)$$

and that our perception of dark energy is, in fact, the bracketed expression above. A plot of the dark energy is shown below, when the radius r = 11, and θ traverses the circle $\mathcal{C}$ at this radius, relative to O …



r = 11 , θ = x , ϕ = π/2 , 2D star (5 sing)



r = 11 , θ = x , ϕ = π/2 , 2D star (3 sing)

Now imagine a photon travelling along $\mathcal{C}$. As it does … it's going to encounter dark energy [ξ] in varying amounts, depending on the angle θ. Every now and then, you get a *spike*, followed by retracement, and thus, according to previous research, the *frequency* [ν] of the photon should change as it moves along $\mathcal{C}$. This change could be *very* small indeed, since we expect σ to be a very small constant … even imperceptibly so.

From our earlier studies on *negative mass* [pp 175-77], we believe that

$$E = h\nu \approx (\mathfrak{m} + \mathfrak{m}_\xi)c^2 \quad (\S)$$

which is the extended form of de Broglie's law, that incorporates dark energy, where h is Planck's constant, ɱ is the 'mass' of the photon, ν its frequency and for any (r, θ) …

$$[ɱ_{\xi}]c^2 = \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))] \quad (\dagger)$$

Here, $ɱ_{\xi}$ can be thought of as a varying 'special mass' associated with dark energy … be it positive *or* negative. And, of course, if σ happened to be 0, we would be back to classical de Broglie.

In the case where $g^{u,v}$ *is* coupled directly to λ(s), (*) can be *estimated* as …

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))]g^{u,v}(0)$$

and so, the bracketed expression remains the same, and (†) applies.

******

No doubt, the reader will say that the energy E in (§) could grow without bound, or, on the other hand, turn negative … and the reader would be correct. However, there are some mitigating factors here, the first of which is that σ is likely to be very small. Secondly, we don't know physically, what the spikes mean. Are they *real* or more on the *imaginary* side. And thirdly, there is nothing wrong with the idea of *negative* mass or energy. What the photon does, in such a case, is somewhat up for grabs, in my opinion, when looking at (§) [perhaps it is *suppressed*, whatever that finally means, if $E < 0$].

But these are the issues that need to be worked out, if we are going to merge the laws of physics with dark energy, where there is an underlying density function [λ(s)], permeating the whole of our universe, say.

******

Although there is no way, at the present time, anyway, to verify this approach, we can, at least, outline a method that we might use, when incorporating dark energy into *other* foundational dualities, such as *energy-mass* or *wave-particle*. Indeed, Schrodinger's equation might be a good candidate here.

In my opinion, if physics is to advance along these lines, this kind of pure research is worthwhile, and in the end, could well prove to be rather fruitful. Time will tell, I suppose, if this is the right road to go down …

Schwarzschild, Perfect Stars and The Dark Energy Contour Integral, Part XXI

In this note, we are going to explore, a little more deeply, the connection between the gravitational tensor $g^{u,v}$, and its *influence* on our *perception* of dark energy. There are two cases to consider: first, the scenario in which $g^{u,v}$ is *not* coupled directly to the dark energy density function [$\lambda(s)$], and then, of course, the case where there *is* a coupling.

So let's start by rolling back to Part I in this series, and look at the density function $\lambda(s) = \sigma/s$, for a single *physical* singularity at the center [O] of our perfect star. Now recall the formulations for the field equations of general relativity are, in the *uncoupled* and *coupled* cases …

$$G^{u,v} \approx \sigma g^{u,v} \qquad (\S) \quad \text{uncoupled}$$

$$G^{u,v} \approx \sigma g^{u,v}|_0 \qquad (\ddagger) \quad \text{coupled}$$

Here $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$ and $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively, and $g^{u,v}$ is the foundational tensor, itself a measure of the gravitational field strength.

Notice that since there is only *one* such physical singularity, located at O, our perception of dark energy is, in fact, σ … regardless of which direction we are looking in, and for any radius r. This is true for both equations above [we believe], because here there is a clear separation between σ and $g^{u,v}$.

Now let's move on to a more complex density; namely $\lambda(s) = \sigma/s\sqrt{s}$, which has a *branching point* at the center of the star. Again, we'll assume only *one* physical singularity at O. In this case (see Part VI for the details), the formulations for the field equations are [uncoupled and coupled, respectively] …

$$G^{u,v} \approx 2\sigma\sqrt{r/\pi}\, g^{u,v} \quad (*)$$

$$G^{u,v} \approx \sigma\{2\sqrt{r/\pi}\, g^{u,v}(0) \; - \; (1/\pi)\int_0^\infty [e^{-sr}(g^{u,v}(-s) - g^{u,v}(0))/(s\sqrt{s}\,)]ds\} \quad (\dagger)$$

For *large* r, we expect the integral in (†) to fade away, and so *symbolically*, (*) and (†) agree in this case, and our perception of dark energy, coupled *or* uncoupled, is, in fact …

$$2\sigma\sqrt{r/\pi} \quad (\&)$$

in *any* direction, and for any *large* radius r.

…

But what happens when the radius is not so large ? In the *uncoupled* case (*) … our perception of dark energy, at any point, does not change. It is still (&). But what about the *coupled* case [(†)] ? What part of the expression inside the curly braces in (†) can now be attributed to dark energy …. ?

If we say *all* of it, then we have no choice but to conclude that the gravitational tensor itself [$g^{u,v}$], does, in fact, have an influence on our perception of dark energy, where the latter is obtained from the Laplace inverse of the density function $\lambda(s)$, when there *is* a coupling between the two …

$$G^{u,v} \approx \kappa \int_{\gamma} e^{sr} \lambda(s) g^{u,v} ds \qquad (**)$$

Not only that, but the influence in the *radial* direction [$g^{1,1}$], could well be perceived differently than the influence in the *timelike* direction [$g^{4,4}$], if (†) has anything to say about it.

******

Throughout this series, we have endeavoured to align (symbolically) the formulations of the field equations, in the uncoupled and coupled cases. We did this in the last addendum, when considering the *energy-mass* duality and de Broglie's law, and in many of the previous addenda as well.

One of the reasons for doing this was simplification, so that we would, indeed, have a *consistent* view of dark energy on the one hand, and a better chance at solving the field equations, on the other, particularly in the case of a coupling. However, it is more than likely that $g^{u,v}$ does, indeed, affect our perception of dark energy, when it is coupled directly to $\lambda(s)$, so that (**) holds in this case.

Only in the *simplest* of cases, like, for example, a single *physical* singularity at O, with $\lambda(s) = \sigma / s$, might we argue in favor of *no* influence [$g^{u,v}$ against dark energy], where there is a coupling, and that is because things separate out quite nicely, at least mathematically speaking. They also separate out quite nicely in (†) for *large* r … so that we achieve the kind of *symbolic* confluence we are looking for, in the uncoupled and coupled scenarios.

But these are special cases, and so long as we are willing to live with what the estimations give us, as we have all along in this series, we should feel content in knowing that we are one step closer to understanding the mysterious connections between general relativity, the foundational tensor $g^{u,v}$, dark energy as a whole … and, of course, the underlying density function [$\lambda(s)$] … that powers this peculiar substance we call dark energy …

On The Nature of Autonomous and Ordinary Souls

For a long while now, I have wondered if 'God the Father' and 'God the Son' actually have souls, and if so, what happens to them at purification. In this note we'll make the assumption that they do, and refer to these essences as *autonomous* souls [S] versus the *ordinary* soul [T] that exists in α-space.

Let us begin, then, by noting that in any *perfectly* random space β, the distance between points is always infinite

$$\Delta(p, q) = \infty \quad \text{for all } p, q \ \varepsilon \ \beta$$

and thus, for an observer here, it isn't possible to measure the energy of any particle because it's *too far* away. Ergo, we have the following theorem, which we've seen before ...

If β is any perfectly random space of points, then
for a local observer the energy E_β is always 0

Since S is autonomous, it [or any member] can move back and forth between α and β seamlessly, and so the energy [E_s] must be 0, relative to β, in either domain. But the real question is ... what happens to S at purification ?

$$E_{s,\alpha} = \Lambda \circ E_{s,\beta}$$

Clearly S is uniquely equivalent to the Cross, and therefore the action [Ⱥ], and so if ~F holds in γ, then S must *vanish*. Such an outcome doesn't contradict any conservation theorems because $E_s = 0$. On the other hand, ordinary souls are *also* equivalent to Ⱥ, but *preserved* in γ, and hence the following equivalency chain implies S ought to be preserved as well ...

$$S \;<\!\equiv\!>\; Ⱥ \;<\!\equiv\!>\; T$$

Accordingly, if S finds itself in β then it vanishes under recombination, but if S finds itself in α then its existence in γ is guaranteed.

$$[\sim S\ ,\ S] \;<\!\equiv\!>\; \gamma$$

Thus we have a contradiction, making it impossible to decide the fate of an autonomous soul in Σ, and so the following theorem (†) applies ...

If S is any autonomous soul in Σ, then the fate
of S in transitioning from Σ to γ cannot be
determined

I had always thought it was more of a slam dunk -- since the symmetries in the Godhead would vanish in γ, so would the 'Father and Son' ... but now I'm not so sure. If, indeed, they have given themselves souls of the autonomous kind, maybe they will be with us in the unified mosaic ... and maybe they won't .... we'll just have to wait and see, as the saying goes ...

ERRATUM

In Part IV on the Covariance series I stated that souls probably didn't migrate from β to α initially, for if they had, it would actually violate conservation theorems, at least from the β-perspective. This is not a true statement, in light of what we now know.

Ordinary souls could have migrated, by design, but once in α, would remain there, otherwise there is no *material* difference between them and the autonomous soul, whose fate in γ is *undecidable*.

Alternatively, ordinary souls could have been situated in α initially, in which case there is no way to know just what happened originally. It will always be an open question, as they say ...

OTHER CONSIDERATIONS

It is possible the autonomous soul [S] could take the path α ---› β ---› γ when considering the Σ to γ transition [versus α ---› γ], however in so doing,

$$\pi \circ E_{s,\beta} \;=\; \omega \circ E_{s,\alpha}$$

where π and ω are the unified transforms referenced in previous writings. Thus, the energy of S, relative to an observer in *either* α or β, would *always* be 0 and so we have the following theorem …

> *If S has non-zero measurable energy in α-space, then S cannot migrate from β to γ when considering the Σ to γ transition.*

The theorem is quite interesting in that it implies S must necessarily *vanish* under recombination in β, provided it has non-zero measurable energy in α, assuming the longer route. Alternatively, if the energy of S happened to be zero, relative to an observer in α, then S could conceivably migrate from β to γ, assuming the longer path, provided it didn't vanish under recombination here.

In all likelihood, the autonomous soul does indeed have non-zero measurable energy in α-space, based on the NDE research, which means its fate in γ is undecidable, as per (†). But if, for some reason, this isn't true, and E_s is zero, relative to α, in either domain, then the fate of S in γ is *still* undecidable because we will never know if it did or didn't vanish under recombination in β …

Incompleteness and The Simultaneous Now

Although a lot of what we're going to talk about has already been discussed in previous research, I thought it would be instructive to write a brief note on our limitations when trying to understand the *simultaneous now* [S]. Let us start, then, with the following thought experiment ...

Suppose an observer in β-space [S] perceives α-space to be purified, at the outset, with

$$\Delta_\beta = \xi = 0 \qquad (\ddagger)$$

Accordingly, α is now in an *untenable* state, and so the transition to γ-space occurs relative to the following chain ...

$$\sim A\ ,\ A \quad <\equiv> \quad \sim F \quad <\equiv> \quad \gamma = \alpha + \beta$$

Thus α and β no longer exist, under this scenario, which implies, for the observer in β, that

$$\beta \implies \sim\beta \qquad (*)$$

In other words, if $O_{s,\alpha}$ denotes the *perception* of S from the *perspective* of an observer in α-space, then (*) implies

$$[\sim\beta\ ,\ \beta] \quad <\equiv> \quad O_{s,\alpha}$$

and so $O_{s,\alpha}$ is *undecidable* up to β, making it an attribute of Σ. Hence, by equivalency, we have the following theorem (†) ...

> *For any observer in α-space, the simultaneous now S is both an incomplete and inconsistent construct*

Thus, we can never say with full assurance just how an observer in β *actually* perceives S, and so our studies and conclusions on this topic will always be hampered by (†).

Although, for example, we'd like to believe (‡) is true, there is, in the final analysis, *no* way to know just what exactly $O_{s,\beta}$ *really* means. As such, about all we can do is make an educated guess as to how things work ... and hope our logic is correct ...

$$[\ \sim O_{s,\alpha}\ ,\ O_{s,\alpha}\ ] \quad <\equiv> \quad \Sigma \qquad (\S)$$

Finally, (§) implies $O_{s,\beta} = \sim O_{s,\alpha}$ and so if we are on the right track here, any observer in β does indeed have a fully consistent and complete view of Creation ... by way of S ... and that has to be reassuring ...

On The Twin Prime Conjecture and Godel's Theorem, Part I

In a previous note concerning Godel's Theorem and The Green Tao Theorem, Part II, we showed that for any random system R

~[Godel] ≡ ~[Green Tao, R] ,

meaning R is only an attribute of the Godhead if it is *perfectly* random. In this note we are going to take a closer look at the Twin Prime Conjecture, which asks if there are infinitely many twin primes separated by a distance of 2. Apparently the problem has never been solved ...

Let us start by defining S to be the set {p, p + k, p + 2*k, p + 3*k, ...}, an arbitrarily long arithmetic sequence of primes guaranteed by Green Tao, and define q = p + 2, say. Define as well the set T to be the sequence {q, q + k, q + 2*k, q + 3*k, ...}, and assume for the moment the conjecture is *not* true. In this case, then, T can only contain a finite number of primes, which means there is an odd number q, somewhere on the number line, and an even number k, for which {q + ℓ*k} in T is never prime again, where ℓ >= n and q is larger than 2.

```
           |--------------- subset of  ~P, the odd numbers
           |                       excluding primes
           |
q          q + n*k      q + (n+1)*k      q + (n+2)*k    ...
              |              |                 |
             no             no                no         ...
```

Hence, we may conclude ~[Twin Prime] ==> [Green Tao, ~P], and equivalently, ~[Green Tao, ~P] ==> [Twin Prime].

In other words, if the Twin Prime Conjecture is false, ~P is not perfectly random, and if ~P is perfectly random, then the Twin Prime Conjecture is true. Both statements are identical, actually, but neither is very helpful, since it is almost a certainty that ~P is not perfectly random anyway [in fact, since the *average* distance between primes becomes arbitrarily large as you move toward infinity, there ought to be infinitely many arithmetic sequences of arbitrary length with period 2 in ~P, thus settling the issue once and for all …]

However, if by some independent means one were able to show ~P was indeed perfectly random, it would be tantamount to proving the Twin Prime Conjecture, and if nothing else, give us some feel for just how *unlikely* such a finding might be. In all probability, then, perfectly random systems beyond the membrane are a figment of the imagination, for if we were to find such a thing it would be akin to finding God ...

On The Twin Prime Conjecture and Godel's Theorem, Part II

In the first note on this subject, we discussed, in general terms, The Twin Prime Conjecture and some of the consequences of it being true or false. Here we are going to offer a few more ideas on the conjecture, and so we'll begin by noting for any random system R,

$$\sim[\text{Godel}] \quad \equiv \quad \sim[\text{Green Tao, R}] \qquad (*)$$

Thus, on the number line, any random system R will *always* have predictable regions of arbitrary size, whether these regions can be enumerated by arithmetic sequences, quadratic sequences, cubic sequences, ..., or whatever the formula might happen to be.

Recent work in this area tells us there *is* a large, even spacing N for which [Twin Prime, N] is true, and so we'll start by assuming [Twin Prime, u] to be *false* if $u < N$. Now let's generalize things a bit by assuming for some $N' > 2N$ the conjecture is true, but otherwise not, and for some $N'' > 2N'$ true again, but otherwise not, and so on. In other words, for the sequence {v} = {$0, N, N', N'', ...$} we have [Twin Prime, v] true, else *false*, subject to the string of intermediate inequalities.

Let us now define S to be {q, r, s, ...} and T to be {q + N, r + N, s + N, ...}, where the elements of S and T form the infinite collection of prime pairs with proper spacing. Since S is random, (*) tells us there are predictable regions of arbitrary size here, and so without loss of generality, we may say there is an arbitrarily long arithmetic sequence {p + ℓ*k} in S, where k is arbitrarily large and even, and ℓ >= 0.

$$\begin{array}{lcccccccc}
T \sim\sim> & p + N & \ldots & p + n*k + N & p + (n+1)*k + N & p + (n+2)*k + N & \ldots \\
 & | & & | & | & | & \\
S \sim\sim> & p & \ldots & p + n*k & p + (n+1)*k & p + (n+2)*k & \ldots \\
 & | & & | & | & | & \\
 & w(0) & & w(n) & w(n+1) & w(n+2) & \ldots
\end{array}$$

As we advance along the intervals in S above, from left to right, it should be clear there are only a *finite* number of these blocks for which $w(l) + 2$ is actually prime, otherwise … … our original assumption would no longer be true. So let's dispense with these intervals and focus on the remaining ones. Repeating the exercise, we see again there are only a finite number of intervals in S where $w(l) + 4$ is actually prime, and after dispensing with these, we can examine the remaining blocks for each of $u = 6, 8, 10, ..., N - 2$.

Finally, after exhausting all possible values of $u < N$, we arrive at a point where for all values of $l >= some\ n$, say, no interval [$w(l), w(l + 1)$] in S contains *any* primes in the range

$$w(l) + 2, w(l) + 4, ..., w(l) + N - 2$$

which, for $l = n$, is the same as saying

$$p + n*k + 2, p + n*k + 4, ..., p + n*k + N - 2$$

In other words, comparing S and T now, the interior interval [$p + n*k, \; p + n*k + N$] contains absolutely *no* primes whatsoever, and this is somewhat interesting because N is *not* cast in stone, according to our assumptions.

Repeating the entire exercise for the next $N' > 2N$ for which [Twin Prime, N'] is true, we can come up with another interior interval [$p' + n*k' + N, \; p' + n*k' + N'$] containing no primes, by examining all numbers u in the range $N + 2, N + 4, ..., N' - 2$, and discarding all irrelevant blocks. And so it goes, with each successive interval growing in length by more than $N, 2N, 4N$ ... as we move from N to N' to N'' to N''' ..., etc.

Finally, then, we are going to reach a point where the interval becomes arbitrarily large but contains *no* prime numbers, and this most certainly will lead to a contradiction, given our current assumptions.

To see this more clearly, note that [Green Tao, R] actually tells us there are *infinitely* many arithmetic sequences, of arbitrary length, running through S, and so there is an infinite set I of non-overlapping intervals of length N containing *no* primes, save for the endpoints. Similarly, there is an infinite set I' of non-overlapping intervals of length > N containing *no* primes, save for the endpoints ..... and an infinite set I'' of non-overlapping intervals of length > $2N$ containing no primes, save for the endpoints, and so on ...

$$\begin{array}{ccccccccc} I & \cup & I' & \cup & I'' & \cup & I''' & & ... \\ =N & & >N & & >2N & & >4N & & ... \end{array}$$

Clearly the sets { I', I'', I''' ... } need to be 'woven into' the spaces between elements of I, without overlap, and thus there must be an infinite subset $I^* \subset I$ with

$$\liminf \text{dist}(\alpha, \beta) = \infty \text{ for any } \alpha, \beta \in I^*$$

In other words, there exists in I an infinite subset of elements I^* ... where the spacing between *neighboring* intervals, in particular, is also infinite ... which is impossible.

Therefore, either [Twin Prime, u] is *never* true if $u > N$, true for only *finitely* many $u > N$, or true *more often* than our current inequality string would allow for. Most likely, the last of these three outcomes is correct ...

So let's choose a constant $1 < c < 2$ for which $N' > cN$, $N'' > cN'$, ... and go through the entire exercise again, top to bottom. In this case the intervals containing *no* primes grow by more than

$$(c - 1)N, \; c(c - 1)N, \; c\text{^}2(c - 1)N, ... \qquad (\dagger)$$

as we move from N to N' to N'' to N''' ... and eventually we reach a point again where the interval becomes arbitrarily large, leading to a contradiction. Thus we are forced into believing $c = 1$, meaning the sequence $0 < N < N' < N''$ ... must grow *arithmetically* if it is, in fact, infinite. Indeed,

because the conjecture is *already* true for the first two values *0* and *N*, one could argue by symmetry the sequence really is {*0* + *k*N* } for *k* = 1, 2, 3, ...

Although this is not a proof of the Twin Prime Conjecture, we can at least say that if there is an *infinite* set of even numbers {v} for which [Twin Prime, v] is true, it follows some kind of arithmetic progression. Taken to the limit, this remark would lead us to believe the conjecture is true for *all* even numbers ... something to think about for the future ...

CONNECTIONS TO NOETHER'S THEOREM

Noether's Theorem tells us symmetries imply laws and vice-versa by way of a *least* action which encodes both. If here the law [Θ] is 'Twin Prime Conjecture *true* for any element of {v}', then Noether states that some symmetry [Ψ] in {v} *must* exist which implies Θ and conversely.

$$\Psi \;\; <\equiv> \;\; \text{Ꭿ} \;\; <\equiv> \;\; \Theta$$

The simplest and most compact symmetry is indeed the arithmetic sequence {*0* + *k*N* }, and in fact, by choosing this arrangement, the length of any interval in *any* of the sets { I', I'', I''' ... } is always *N*. Thus, weaving these sets into the spaces between elements of I, is *equivalent* to saying there are infinitely many, non-overlapping intervals of length *N* containing *no* primes, save for the endpoints, or said another way, the weaving process [Ꮿ] essentially returns us to I. This is the best outcome we could ever hope for, as a matter of fact.

$$\text{Ꮿ} \circ (\mathrm{I} \ll \mathrm{I}', \mathrm{I}'', \mathrm{I}''' \ldots) \dashrightarrow \mathrm{I}$$

As well, the value of any element in the sequence depicted by (†) is always *zero* if *c* = 1, meaning, in all likelihood, the lengths of the intervals containing no primes do *not* grow as we move through the various elements of {v}. In turn, this can only bolster our view that Ψ is indeed arithmetic with period *N*.

OTHER CONSIDERATIONS

You don't actually need to run arithmetic sequences through S = {q, r, s, ...} in order to generate the intervals in { I, I', I'', I''' ... } at each iteration. In fact, at each stage one need only discard the appropriate elements of S, as we did initially, to arrive at the same conclusion.

If Θ finds its origins in β-space, according to the following duality, then previous theorems tell us the covariant derivative of both S and R in α-space is 0. In particular, $\nabla \circ R = 0$, implying d{v}/dx is also 0, since R is ostensibly a representation of {v}. Thus {v} is moving along with *constant velocity*, meaning it is arithmetic, and indeed, we surmised as much originally by appealing to Noether and *least action* principles, after showing that {v} could *not grow* without bound [$\nabla \circ S = 0$ is most easily interpreted by realizing that S is an infinite strand of prime pairs whose differences are always the same value, and so is equivalent to an arithmetic sequence].

For any strand S there exists a root R such that R induces S, and conversely.
Properties of S are inherited by R, and vice versa, because of duality.

On The Twin Prime Conjecture and Godel's Theorem, Part III

In Part II we saw that if {v} is an infinite set of even numbers ≥ 0 for which [Twin Prime, u] is true, where u is any element in {v}, then {v} *must* be an arithmetic sequence. Here we offer a `proof ` of the generalized Twin Prime Conjecture, by assuming a *weaker* version of *Maillet's conjecture;* namely, that every even number is the difference of two primes.

In 2013, an unorthodox mathematician by the name of Zhang discovered the existence of an even number N, below 70 million, for which [Twin Prime, N] holds. No one knows what this number is; only that it exists. If anything, then, numbers *larger* than N should *also* exist where the conjecture is true, because the primes become sparser as you move toward infinity, with a limiting density. So we'll run with this very reasonable assumption as well, for the time being, and worry about the details later.

If {v} = {*0, N, N', N'', ...*}, then by Part II, {v} is actually {*0, N, 2N, 3N, ...*}, where *N* is Zhang's number. Let us define {w} to be the set

$$\{w\} = \{u, N + u, 2N + u, 3N + u, ...\}$$

with $0 < u < N$, and even.

Define now the set R = {p, q, r, s, ... } to be the collection of all unique primes for which Δ(p, q) Ɛ {w}, and assume for the moment R is *infinite*. Because R is random, and

$$\sim[\text{Godel}] \quad \equiv \quad \sim[\text{Green Tao, R}] \,,$$

there are, without loss of generality, *infinitely* many ... *arbitrarily* long arithmetic sequences of type

$$S = \{ p + nk \mid n = 0, 1, 2, 3, ...\}$$

running through R with *k* Ɛ {w}. And so, there is a corresponding $0 < u < N$ associated with *k* for which Π(*u,k),* the number of elements in S, is arbitrary. In fact, $u = k \bmod N$, the remainder after dividing *k* by *N,* relocated to a specific interval, as shown below.

```
0                N                2N                3N        ...
|                |                |                 |
|                |                |                 |
                     u                        u'
|---------- κ --------------->
|------------------------ κ' ------------------------------->          ...
```

If Θ(*u*) is now defined to be the *frequency* with which *u* appears, as sets of type S are created, then for *at least* one such *u,* we must have … … where Ω is the number of arithmetic sets S, properly constructed,

$$\lim\ \Theta(u) \dashrightarrow \infty \text{ as } \Omega \dashrightarrow \infty$$

Thus $\Theta(u) = \infty$, for some relocatable $u = k \bmod N$ with $k\ \varepsilon\ \{w\}$, which we'll assume to be unique.

To see this more clearly, suppose in parallel we 'throw down', onto the number line, the infinite number of arithmetic sequences S, to begin with. Clearly the spacing k can land anywhere in {w} for each S, but in terms of the expression $u = k \bmod N$, there are only *finitely* many choices. Thus, either $\Theta(u)$ is finite or infinite -- there is *no* middle ground here -- and since we've assumed $\Theta(u)$ is infinite for only *one* choice of relocatable u, it must, perforce, be bounded *everywhere* else.

Either $\Theta(u) = \infty$ is true over a finite or infinite number of intervals, relative to {v}. In the first scenario this would imply $\Theta(u)$ is infinite over *at least* one specific interval in {v}, in which case [Twin Prime, u] follows. In the second case, there must *always* be an *infinite* set of distinct intervals, given any arbitrary M, for which

$$M < \Pi(u,k) < \Pi(u,k') < \Pi(u,k'') < \ldots \quad (\dagger)$$

Why might this be ? Well, if $\Pi(u,k) < M$ in *every* interval of {v}, this would contradict the fact that there are infinitely many arithmetic sequences S, of *arbitrary* length here … … for the case $\Theta(u) = \infty$. Hence, some S exists for which $\Pi(u,k) > M$, and if this was the only such S, we could simply repeat the argument where M is now defined to be $\Pi(u,k)$.

```
0          N          2N   .....   αN  u   (α + 1)N  .....  βN  u   (β + 1)N  ......
|          |          |            |          |             |          |
|          |          |            |          |             |          |
| ------------------ κ ----------------------›
| ----------------------------------- κ' --------------------------------›              ......
```

Thus, [Twin Prime, u] is true for γN, where $\gamma = 0, 1, 2, \ldots$ and [Twin Prime, u] is *almost* true whenever $u = k, k', k'' \ldots. \bmod N$, but otherwise not. By 'almost true' we mean the conjecture holds for any *arbitrary* number of prime pairs determined by arithmetic sequences, and by 'otherwise not' we mean *not* true, which one can *assume*, without loss of generality.

******

As a result, we can use the techniques employed in Part II to infer the series in the diagram above *must* form an arithmetic sequence with period u, and since it is arithmetic, we have no *choice* but to conclude the Twin Prime Conjecture is true for every element in $\{v\} = \{0, u, 2u, 3u, \ldots\}$.

To see this more clearly, let $\{\vartheta\}$ be the series in the diagram above and *pro tempore*, define Θ to be the law 'Twin Prime Conjecture *true* for any arbitrary number of prime pairs over the set $\{\vartheta\}$'. Because of (†) there are always infinitely many u in $\{\vartheta\}$ for which Θ holds, and so by Part II in this series, a symmetry Ψ in $\{\vartheta\}$ *must* exist which implies Θ, and conversely, according to Noether and *least* action principles. That symmetry is, in fact, the arithmetic sequence

$$\{\vartheta\} = \{0, u, 2u, 3u, \ldots\},$$

for any arbitrary M, since Θ is always true for γN, where $\gamma = 0, 1, 2, \ldots$ (essentially, N morphs into u, which is possible because of its fluid nature).

Now let $M \longrightarrow \infty$ and observe that $\{\vartheta\}$ *remains* arithmetic with period u, implying that in the limit, (at infinity), Θ now *becomes* the law 'Twin Prime Conjecture *true* for any element of $\{\vartheta\}$', or said another way, the Twin Prime Conjecture is true for *every* element in $\{v\} = \{0, u, 2u, 3u, \ldots\}$.

******

If, on the other hand, R is finite, there will come a point *beyond* which *no* element in {w} can be resolved as the difference of two primes, contradicting our original assumption regarding a weaker version of *Maillet's Conjecture*. In fact, *almost all* even numbers *can* be resolved as the difference of two primes, according to the literature, and since {w} has *positive density* in **N**, the set of integers, it is almost a given that R cannot be finite.

The entire process can now be repeated, top to bottom, dividing up {v} ever more finely until we reach a point where, at last, at last … … ... the Twin Prime Conjecture is true for *all* even numbers in **N** ... a result, which, when you think about it, makes a lot of sense ...

OTHER CONSIDERATIONS

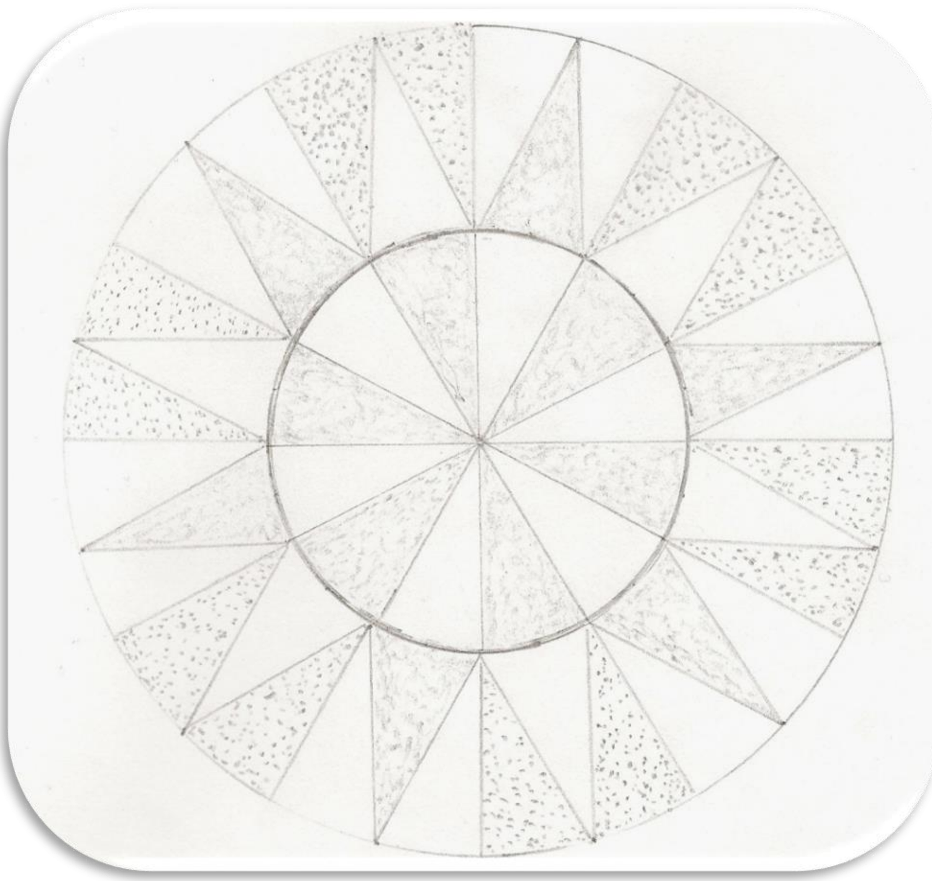
The construction of R is actually a precursor to the Green Tao Theorem. Essentially, we walk along the primes {P}, and whenever we find a neighboring pair (p, q) whose difference lies in {w}, the pair is automatically moved into R and crossed off our list.

If there never was a Zhang's number to begin with ($N \longrightarrow \infty$), then {v} becomes $\{0, \infty\}$ and {w} becomes the entire set of even numbers $\mathbf{N} = \{2, 4, 6, 8, \ldots\}$. The set R, in turn, becomes P, and of course, there are infinitely many arithmetic sequences of arbitrary length running through P, with period k in **N**, according to Green Tao.

As an alternative to constructing R one can simply use the set of primes {P}, and realize there are infinitely many arithmetic sequences of *arbitrary* length running through P, where now the period k belongs to $\{w\} = \{u, N + u, 2N + u, 3N + u, \ldots\}$, with $0 < u < N$, and even. If this were not so, k would always belong to the set $\{v\} = \{0, N, 2N, 3N, \ldots\}$ if the arithmetic sequence was sufficiently large, thereby placing an *artificial* constraint on Green Tao that otherwise would not exist, if Zhang's number itself didn't exist.

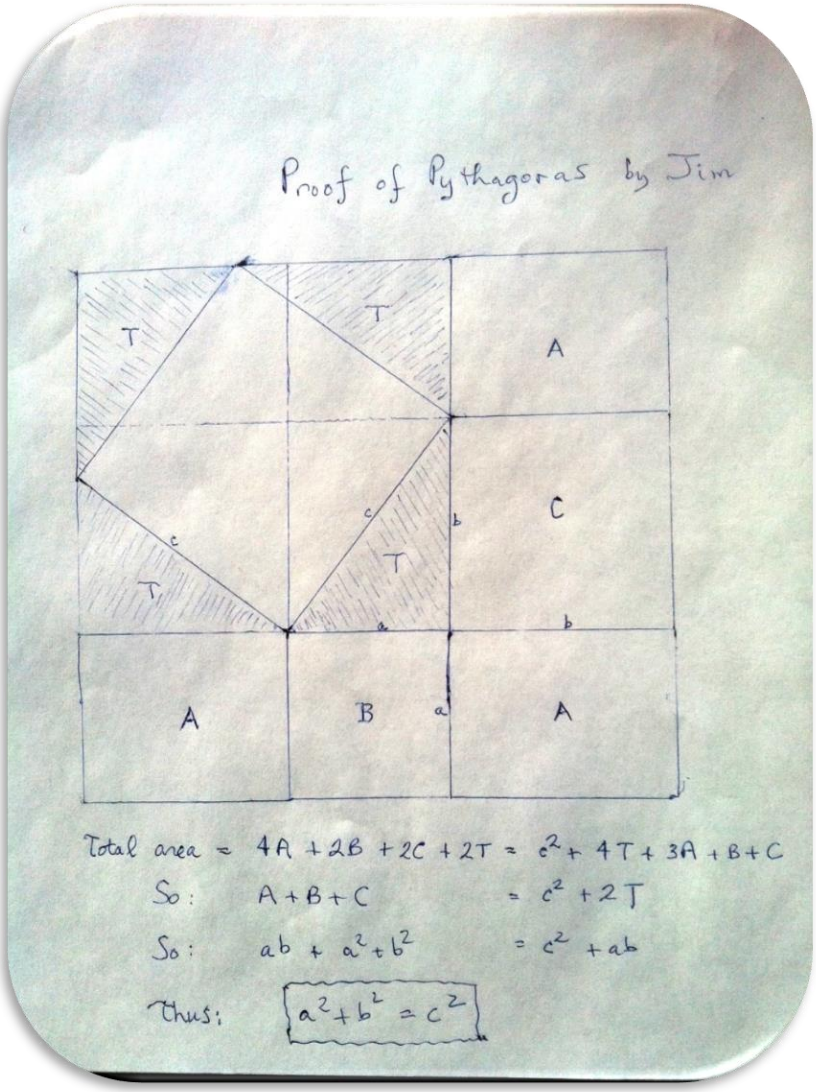
The Mosaic of Creation

This is a simple, hand-drawn version of the mosaic of creation, using ruler and compass. The dark circle is the *action*, which divides the inner *symmetries* from the outer *laws,* and lives in one dimension below the two. Essentially, Noether's Principle, upon which The Fundamental Theorem of Creation is based …



An Original Proof of Pythagoras' Theorem

This is an original proof of the Pythagorean Theorem I did some years back. It is included to demonstrate the idea of conceptual thinking, where we embed smaller ideas inside larger ones so that the bigger picture emerges. Much of the research relies on this train of thought …



A Modest Eulogy

I have decided it is time to write my own, albeit modest, eulogy. For a long time now I have been fascinated with numbers, and in the last few years of my life, how they might relate to me from a spiritual perspective.

In the essay I learned that my life was connected to the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

and seen as pairs (11, 23, 58 ...), the first element corresponds to the level of spiritual understanding I have reached, the second to the number of incarnations I have had, and just recently, the third to my age, or so it would seem …

And wouldn't you know it -- I go to a quaint little coffee shop in downtown Penticton by the name of Fibonacci, almost daily, and it was there the other day I saw a shiny nickel on the cushion of my favorite chair. Almost as though it had been placed in that spot, especially for me.

The number 5 in Cayce's world speaks of a change which led me to guess that the third pairing can't be anything other than my age. I suspect 58 years is the end of the road for me in this world, and indeed, a welcome relief if it is true.

****

Seen as the hour, minute and second of a digital clock, say, these pairs form a time on the clock (11:23:58) at which things grind to a halt. Just by a hair, the clock stops ticking 'two seconds' before 11:24, meaning a 24th incarnation will never come to pass. The spiritual lessons, it would appear, have been learned in this incarnation in the allotted span of time.

****

It should be mentioned I was born in Penticton and recall a vision or a dream I had long ago where I saw Skaha Beach as it really looks on the other side, in a droplet of water running down a window pane. People were very happy there -- the beach seemed like a hub of sorts for many athletic competitions and across the road I saw what looked to be a rather ornate beach house draped in vines, which literally wrapped around the house and ran inside through the open front door.

Standing in the doorway was a beautiful young brunette, very shapely, and dressed in a short skirt and top. I was struck by both her beauty and her countenance, for she gazed at me rather intently with a sadness in her face. No doubt it was my soul mate looking back at me through whatever portal joined us together for a brief moment in time.

Ever since then I have been waiting for the day when I would walk through that portal and be back home once again. No doubt Skaha Beach, on the other side, is where I was before deciding to do

another incarnation, perhaps to right the wrongs in a previous descent, or maybe for other reasons that remain a mystery. Whatever the case, the time has come to move on, as they say.

By the way, and rather unexpectedly, I moved back to Penticton just a few short weeks ago. It was never part of the plan originally, but looking back now, maybe it was ...

And just tonight, after writing this piece, on my way to the car I saw a dime on the pavement, again placed in that spot as though it was especially for me. The number 10 in Cayce's world means a return to Unity or God, if you will ...

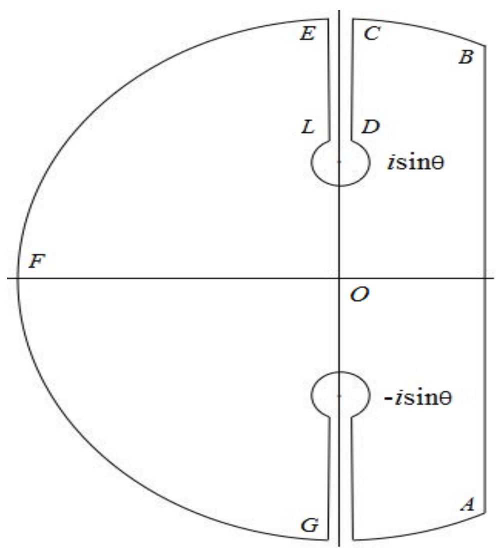
Supplementary Material

This research note is dedicated to a deeper understanding of the Riemann zeta function [ζ(s)], using some of the material in the original essay. Specifically, we will develop *harmonic* integrable representations for variants of ζ(s), both in the critical strip S, where 0 < Re(z) < 1 and z is complex, *and* to the right of S as well. The technical details will be kept to a minimum, and along the way, we shall also verify our work, so that theory and experiment line up, as they should. The hope is that this note will lead to a better understanding of the Riemann Hypothesis [p 181], within the context of harmonic representations.

Let us begin, then, with the following contour integral, which was used in the original essay when discussing *couplings* between the gravitational tensor $g^{u,v}$ and the underlying *density* function [λ(s)] associated with dark energy. It is reproduced here [pp 228-9] …

$$\kappa\sigma e^{r\cos(\theta)}\int_{\gamma} e^{sr} g^{u,v}(s) \,/\sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\, ds \qquad (*)$$

according to the following *Bessel* contour γ, which denotes the Laplace inverse (line integral) along AB. And here we are



going to modify (*) to suit our purposes, by deleting the $\sigma e^{r\cos(\theta)}$ term, and replacing $g^{u,v}(s)$ with the function g(s) = ζ(α – s), where α is any *real* number > 1. The constant κ remains and is equal to 1/2πi, where $\boldsymbol{i}$ is imaginary and equal to $\sqrt{-1}$.

$$\kappa\int_{\gamma} e^{sr} g(s) \,/\sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\, ds \qquad (\dagger)$$

.
.

We choose g(s) in the way that we do, because ζ(s) is 'well enough behaved' to the *right* of any vertical line in the complex plane ℂ, and so, along the large arc BCEFGA, we are assured of convergence to 0 under Laplace inversion, as the radius of this arc tends to ∞. Similarly with the small circles associated with the branching points ***i***sin(θ) and **-*i***sin(θ), as their radii shrink to 0.

All we need to do now, is evaluate the integration in (†) along the arms CD and LE, noting that ***i***sin(θ) is a *branching* point, and then, of course, do the same thing for the two arms associated with **-*i***sin(θ). By letting the large arc BCEFGA tend to ∞ and the small circles ---› 0, we will achieve the desired result; namely, Laplace inversion along AB.

Now since α > 1 and ζ(s) has a *simple* pole at s = 1, ζ(α – s) has a *simple* pole at s = α – 1 > 0, and so we *exclude* it by choosing 0 < Re(AB) < α – 1. In doing so, (†) evaluates to 0 as we traverse γ. Thus,

$$(\dagger) \;=\; \kappa\int_{AB} \;+\; \kappa\int_{\boldsymbol{i}\sin(\theta)\text{ arms}} \;+\; \kappa\int_{-\boldsymbol{i}\sin(\theta)\text{ arms}} \;=\; 0$$

where it is understood, that we are carrying the integrand in (†) in all terms above [see pp 221-3 for more on this construction].

Now let θ ---› 0, so that (†) *becomes* the Laplace inverse in this limiting case, and we may write ….

$$\lim_{\theta \to 0\text{ (along AB)}} \kappa\int e^{sr}g(s)\,/\sqrt{(s - isin(\theta))(s + isin(\theta))}\,ds \;+\; \kappa\int_{\boldsymbol{i}\sin(\theta)\text{ arms}} \;+\; \kappa\int_{-\boldsymbol{i}\sin(\theta)\text{ arms}} \;=\; 0$$

where *now* … the integration along the arms *extends* from 0 to ∞ along the imaginary axis in both the upper and lower-half planes … according to the following description below …

$$\underbrace{-2\int_0^\infty e^{iyr}g(iy)\,/\sqrt{(y)(y)}\,dy}_{\boldsymbol{i}\sin(\theta)\text{ arms}} \qquad \underbrace{2\int_0^\infty e^{-iyr}g(-iy)\,/\sqrt{(y)(y)}\,dy}_{-\boldsymbol{i}\sin(\theta)\text{ arms}}$$

In the limit, as θ ---› 0, (†) becomes

$$\kappa\int_\gamma e^{sr}g(s)\,/\sqrt{(s)(s)}\,ds \qquad (\S)$$

where γ is now the large contour BCEFGAB, *because* we are dealing with a simple pole here. The residue is ζ(α), and so from our work above …

.
.

$$\zeta(\alpha) = -\kappa\int_{\boldsymbol{i}\sin(\theta)\text{ arms}} + -\kappa\int_{-\boldsymbol{i}\sin(\theta)\text{ arms}}$$

Now the right-hand side of the expression above may be written as …

$$2\kappa\int_0^\infty e^{iyr}g(iy)\,/\sqrt{(y)(y)}\,dy \quad - \quad 2\kappa\int_0^\infty e^{-iyr}g(-iy)\,/\sqrt{(y)(y)}\,dy$$

$$= 2\kappa\int_0^\infty \{e^{iyr}g(iy) \quad - \quad e^{-iyr}g(-iy)\}dy\,/\,y$$

And since g(iy) = ζ(α – iy), whilst g(-iy) = ζ(α + iy), we see that we are dealing with conjugates here in the argument list, and so may define ζ(α – iy) to be A – iB, and ζ(α + iy) to be A + iB. We can do this because ζ(s*) = ζ(s)*, where * means 'take the conjugate of'.

If we substitute these definitions into the expression above, we finally arrive at our destination, after collecting terms …

$$\int_0^\infty \{A\sin(yr) - B\cos(yr)\}dy\,/\,y = (\pi/2)\zeta(\alpha) \qquad (\ddagger)$$

This result is true *for all* r > 0 and for all real α > 1. Here

$$A = \{\zeta(\alpha + iy) + \zeta(\alpha - iy)\}/\,2$$

$$B = \{\zeta(\alpha + iy) - \zeta(\alpha - iy)\}/\,2i$$

******

Now we'd like to see how theory lines up, by choosing different values for α and r, and compare the results in (‡) with integrations carried out on the WolframAlpha website. We'll choose α to be 3/2 and 5/2, and we'll let r = 1, 3, 5. Here are the snapshots from Wolfram …

.
.
.

$$\int_0^{300}\left(\frac{i\cos(t)\left(-\zeta\left(\frac{3}{2}-i\,t\right)+\zeta\left(\frac{3}{2}+i\,t\right)\right)}{2\,t}+\frac{\sin(t)\left(\zeta\left(\frac{3}{2}-i\,t\right)+\zeta\left(\frac{3}{2}+i\,t\right)\right)}{2\,t}\right)dt=4.10311$$

$$\int_0^{300}\left(\frac{i\cos(t)\left(-\zeta\left(\frac{5}{2}-i\,t\right)+\zeta\left(\frac{5}{2}+i\,t\right)\right)}{2\,t}+\frac{\sin(t)\left(\zeta\left(\frac{5}{2}-i\,t\right)+\zeta\left(\frac{5}{2}+i\,t\right)\right)}{2\,t}\right)dt=2.10706$$

$$\int_0^{300}\left(\frac{i\cos(3\,t)\left(-\zeta\left(\frac{3}{2}-i\,t\right)+\zeta\left(\frac{3}{2}+i\,t\right)\right)}{2\,t}+\frac{\sin(3\,t)\left(\zeta\left(\frac{3}{2}-i\,t\right)+\zeta\left(\frac{3}{2}+i\,t\right)\right)}{2\,t}\right)dt=4.10365$$

$$\int_0^{300}\left(\frac{i\cos(5\,t)\left(-\zeta\left(\frac{5}{2}-i\,t\right)+\zeta\left(\frac{5}{2}+i\,t\right)\right)}{2\,t}+\frac{\sin(5\,t)\left(\zeta\left(\frac{5}{2}-i\,t\right)+\zeta\left(\frac{5}{2}+i\,t\right)\right)}{2\,t}\right)dt=2.10721$$

The theoretical values, out to a few significant digits are …

$$(\pi/2)\zeta(3/2) \approx 4.1035$$

$$(\pi/2)\zeta(5/2) \approx 2.1072$$

The agreement is stunning, in my opinion. Our theory lines up exactly with experiment, and gives us the confidence we need to go forward with more complex harmonic representations for $\zeta(s)$, and that is something we will do in future notes in this section.

To wit, we will examine similar results in the *critical* strip, where $0 < \alpha < 1$, and … we will also examine functions of type g(s) that incorporate roots of $\zeta(s)$. Stay tuned, as they say, but in the meantime, it is well worth pondering the results above, and the approach used to obtain them, as this methodology forms the basis for all future work in this area …

******

Now we'd like to move into the *critical* strip where $0 < \alpha < 1$. Our function $g(s) = \zeta(\alpha - s)$ remains the same, but since $\zeta(\alpha - s)$ has a *simple* pole at $s = \alpha - 1 < 0$, we can no longer exclude it when traversing the *Bessel* contour γ, as shown on page 258. The vertical line AB in this contour *must* be to the *right* of the imaginary axis so that we trap the *branching* points $\boldsymbol{i}\sin(\theta)$ and $-\boldsymbol{i}\sin(\theta)$, and thus, by default, the pole associated with $\zeta(\alpha - s)$ is automatically included. Here, $0 < \mathrm{Re}(AB) < \alpha$.

Traversing γ, we find that since the residue of $\zeta(\alpha - s)$ is 1 at $s = \alpha - 1$ ... then with $\beta = \alpha - 1$, one has

$$(\dagger) = \kappa\int_{AB} + \kappa\int_{i\sin(\theta)\text{ arms}} + \kappa\int_{-i\sin(\theta)\text{ arms}} = e^{r\beta} / \sqrt{(\beta - isin(\theta))(\beta + isin(\theta))}$$

where (†) is defined on page 258, and reproduced here …

$$\kappa\int_{\gamma} e^{sr} g(s) / \sqrt{(s - isin(\theta))(s + isin(\theta))}\, ds \qquad (\dagger)$$

Now let $\theta \dashrightarrow 0$, so that (†) *becomes* the Laplace inverse in this limiting case, and we may write ….

$$\lim_{\theta \to 0\ (\text{along AB})} \kappa\int e^{sr} g(s) / \sqrt{(s - isin(\theta))(s + isin(\theta))}\, ds + \kappa\int_{i\sin(\theta)\text{ arms}} + \kappa\int_{-i\sin(\theta)\text{ arms}} = e^{r\beta} / \sqrt{(\beta)(\beta)} \quad (\#)$$

where *now* … the integration along the arms extends from 0 to ∞ along the imaginary axis in both the upper and lower-half planes … according to the following description below …

$$-2\int_0^\infty e^{iyr} g(iy) / \sqrt{(y)(y)}\, dy \qquad\qquad 2\int_0^\infty e^{-iyr} g(-iy) / \sqrt{(y)(y)}\, dy$$

$\boldsymbol{i}\sin(\theta)$ arms $\qquad\qquad$ $-\boldsymbol{i}\sin(\theta)$ arms

In the limit, as $\theta \dashrightarrow 0$, (†) becomes

$$\kappa\int_{\gamma} e^{sr} g(s) / \sqrt{(s)(s)}\, ds \qquad (\S)$$

where γ is *now* the large contour BCEFGAB, *because* we are dealing with *two* simple poles here. These are at the points 0 and β. The residue at 0 is $\zeta(\alpha)$, and the residue at β is, *up to* sign, just as it was before …

$$e^{r\beta} / \sqrt{(\beta)(\beta)} \qquad (\sim)$$

However, in order that theory *and* experiment agree, the residue in (#) *must* be *negative* and its counterpart in (~) *positive*. We can do this because we have a *choice* when taking square roots here.

Since $\beta = \alpha - 1 < 0$, this means

$$\zeta(\alpha) - 2e^{r\beta}/\beta \quad = -\kappa\int_{\boldsymbol{i}\sin(\theta)\text{ arms}} + -\kappa\int_{\boldsymbol{-i}\sin(\theta)\text{ arms}}$$

Now we can finish off the exercise, just as we did when we were to the *right* of the critical strip with $\alpha > 1$, and we find that

$$\int_0^\infty \{A\sin(yr) - B\cos(yr)\}dy / y = (\pi/2)\zeta(\alpha) + \pi e^{-r(1-\alpha)}/(1-\alpha)$$

This result is true *for all* $r > 0$ and for all real $0 < \alpha < 1$. Here

$$A = \{\zeta(\alpha + iy) + \zeta(\alpha - iy)\}/2$$

$$B = \{\zeta(\alpha + iy) - \zeta(\alpha - iy)\}/2i$$

It is interesting to me, at least, that this result contains a *decaying* exponential term on the right-hand side; something we do *not* see when we are *outside* the critical strip with $\alpha > 1$. Only as r tends to ∞, do we see that the two equations agree [cf. (‡), p. 260]. In turn, this means the Riemann zeta function behaves in a peculiarly different manner *inside* the strip, than it does *outside* the strip with $\alpha > 1$.

But perhaps more importantly, we are getting our first glimpse of Fourier theory at work here, in so much as the harmonic integration above can be thought of as a *sum* over all frequencies in the *frequency* [y] domain, leading to a corresponding signal in the *time* [r] domain. But to see this effect, we have to be in the critical strip itself, oddly enough …

Now it's time for some verification of our work, so we'll do a few integrations on the Wolfram site, and see how things play out. Here are the snapshots for $\alpha = 1/4$, $1/2$, and $3/4$, with $r = 1$. Note that we always try to integrate from 0 to 300 on this website, but on occasion, due to limitations in execution time, it isn't possible to go quite that far. Nevertheless, the results are encouraging, to say the least …

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$$\int_0^{260}\left(\frac{i\cos(t)\left(-\zeta\left(\frac{1}{4}-i\,t\right)+\zeta\left(\frac{1}{4}+i\,t\right)\right)}{2\,t}+\frac{\sin(t)\left(\zeta\left(\frac{1}{4}-i\,t\right)+\zeta\left(\frac{1}{4}+i\,t\right)\right)}{2\,t}\right)dt = 0.701633$$

$$\int_0^{300}\left(\frac{i\cos(t)\left(-\zeta\left(\frac{1}{2}-i\,t\right)+\zeta\left(\frac{1}{2}+i\,t\right)\right)}{2\,t}+\frac{\sin(t)\left(\zeta\left(\frac{1}{2}-i\,t\right)+\zeta\left(\frac{1}{2}+i\,t\right)\right)}{2\,t}\right)dt = 1.51685$$

$$\int_0^{300}\left(\frac{i\cos(t)\left(-\zeta\left(\frac{3}{4}-i\,t\right)+\zeta\left(\frac{3}{4}+i\,t\right)\right)}{2\,t}+\frac{\sin(t)\left(\zeta\left(\frac{3}{4}-i\,t\right)+\zeta\left(\frac{3}{4}+i\,t\right)\right)}{2\,t}\right)dt = 4.38065$$

Our formula, from the last page is …

$$(\pi/2)\zeta(\alpha) \;+\; \pi e^{-r(1-\alpha)} / (1-\alpha) \qquad (**)$$

With $\alpha = 1/4$, and $r = 1$, (**) computes to ≈ 0.7012, to a few significant digits …
With $\alpha = 1/2$, and $r = 1$, (**) computes to ≈ 1.5170, to a few significant digits …
With $\alpha = 3/4$, and $r = 1$, (**) computes to ≈ 4.3811, to a few significant digits …

Just as in the previous case, with $\alpha > 1$ *outside* the critical strip, the results here *inside* the critical strip are not only encouraging, but stunning as well. The agreement between theory and experiment is close enough, that we feel confident enough to pursue the next level of abstraction, where we will incorporate roots of the zeta function [ζ(s)] into g(s). Not only that, but we will also develop *necessary* and *sufficient* conditions for $\alpha \pm \boldsymbol{i}\varepsilon$ to be a root of ζ(s), using harmonic integrable representations. Finally, we'll look at invariants tied to these representations, and apply Noether's Theorem to gain insight into the symmetries behind these conserved quantities ….

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In this section, we are going to continue the previous discussion, by introducing a new function g(s) and another function h(s). To wit …

$$g(s) = \zeta(\alpha - s + i\varepsilon) \;+\; \zeta(\alpha - s - i\varepsilon)$$
$$h(s) = \zeta(\alpha - s + i\varepsilon) \;-\; \zeta(\alpha - s - i\varepsilon)$$

where $\varepsilon > 0$ is any real number. Again, we are in the critical strip where $0 < \alpha < 1$. We now calculate residues, just as before, noting from above that the *simple* poles are at $s = \alpha - 1 \pm i\varepsilon$ for both g(s) and h(s).

Again, care needs to be taken when working with the *sign* of the residues … but apart from this nuance, things work exactly the same way here as they did in the last section. When the dust settles on the algebra, we have the following, which we'll label (1) and (2), respectively …

$$\int_0^\infty \{(A + C)\sin(yr) \;-\; (B + D)\cos(yr)\}dy / y \;=$$
$$(\pi/2)g(0) \;+\; \pi e^{-r(1-\alpha)}\{2(1-\alpha)\cos(\varepsilon r) - 2\varepsilon\sin(\varepsilon r)\} \,/\, \{(1-\alpha)^2 + \varepsilon^2\}$$

$$\int_0^\infty \{(D - B)\sin(yr) \;+\; (C - A)\cos(yr)\}dy / y \;=$$
$$-(\pi i/2)h(0) \;+\; \pi e^{-r(1-\alpha)}\{2(1-\alpha)\sin(\varepsilon r) + 2\varepsilon\cos(\varepsilon r)\} \,/\, \{(1-\alpha)^2 + \varepsilon^2\}$$

The result is valid *for all* $r > 0$, for all $\varepsilon > 0$, and for all real $0 < \alpha < 1$. Here

$$A \;=\; \{\zeta(\alpha + iy - i\varepsilon) \;+\; \zeta(\alpha - iy + i\varepsilon)\}/\,2$$
$$B \;=\; \{\zeta(\alpha + iy - i\varepsilon) \;-\; \zeta(\alpha - iy + i\varepsilon)\}/\,2i$$
$$C \;=\; \{\zeta(\alpha + iy + i\varepsilon) \;+\; \zeta(\alpha - iy - i\varepsilon)\}/\,2$$
$$D \;=\; \{\zeta(\alpha + iy + i\varepsilon) \;-\; \zeta(\alpha - iy - i\varepsilon)\}/\,2i$$

Notice that as ε ---› 0 … we recover the expression in the last section [p. 263] from (1), since now $A = C$ and $B = D$, and g(0) becomes $2\zeta(\alpha)$. On the other hand, (2) computes to 0 on *both* sides of the equals sign, so more good news.

Now suppose $\varepsilon > 0$ is a root of $\zeta(s)$ for some $0 < \alpha < 1$, so that $\zeta(\alpha \pm i\varepsilon) = 0$. Then both $g(0) = 0$ and $h(0) = 0$ are true, and (1) and (2) reduce to …

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$$\int_0^\infty \{(A + C)\sin(yr) \; - \; (B + D)\cos(yr)\}dy \,/\, y \;=$$

$$\pi e^{-r(1-\alpha)}\{2(1-\alpha)\cos(\varepsilon r) - 2\varepsilon\sin(\varepsilon r)\} \,/\, \{(1-\alpha)^2 + \varepsilon^2\} \qquad \ldots (3)$$

$$\int_0^\infty \{(D - B)\sin(yr) \; + \; (C - A)\cos(yr)\}dy \,/\, y \;=$$

$$\pi e^{-r(1-\alpha)}\{2(1-\alpha)\sin(\varepsilon r) + 2\varepsilon\cos(\varepsilon r)\} \,/\, \{(1-\alpha)^2 + \varepsilon^2\} \qquad \ldots (4)$$

On the other hand, because (1) and (2) are *exact* expressions, the truth of (3) *and* (4) necessarily implies that *both* g(0) and h(0) are identically 0, which means $\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0$. Thus, *together* (3) and (4) are a *necessary* and *sufficient* condition for $\alpha \pm \boldsymbol{i}\varepsilon$ to be a root of $\zeta(s)$. And so we have the following theorem …

$\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0$ *if* and *only if* (3) and (4) are true *for all* $r > 0$ (§§)

Notice too … that (3) and (4) can be thought of as *inverse* Fourier transforms [left side], in the *frequency* domain [y], leading to *signals* [right side], in the *time* domain [r]. The frequency coefficients that are *paired* with the sin(yr) and cos(yr) terms can include the denominator in the integrand, without loss of generality.

In order for (§§) to be a *minimal* energy theorem, we now have to ask ourselves what form the theorem takes if it is to *expend* the *least* amount of energy possible, and what the accompanying *symmetries* might be. From Riemann's functional equation, we know that if $\alpha \pm \boldsymbol{i}\varepsilon$ is a root of $\zeta(s)$, then so is $1 - \alpha \pm \boldsymbol{i}\varepsilon$, in the critical strip. In such a case, (3) and (4) *must* be true for α, *and also* true when we replace α with $1 - \alpha$, leading to *four* equations in (§§), *if* $\alpha \neq ½$.

Only when $\alpha = ½$ does (§§) reduce to *two* equations, and this is the best we can do. The symmetry associated with this *minimal* configuration is reflection across the *x*-axis in the complex plane, along the critical line $\alpha = ½$. And since our theorem is grounded in physical principles from Fourier theory, it is more than likely that (§§) holds, only in this case, and no other.

Now it's time for some verification of our work. We'd like to validate (1) when $\alpha = ½$, $\varepsilon = 14.135$, and $r = 1$. In this case, ε is approximately the first root of $\zeta(s)$ on the critical line. Then we'll validate (2) when $\alpha = ½$, $\varepsilon = 5$, and $r = 1$. Here are the raw snapshots from Wolfram …

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$$\int_0^{300} \frac{1}{t}\left(\frac{1}{2}\left(\zeta\left(\frac{1}{2}+i\,t-14.135\,i\right)+\zeta\left(\frac{1}{2}-i\,t+14.135\,i\right)\right)+\frac{1}{2}\left(\zeta\left(\frac{1}{2}+i\,t+14.135\,i\right)+\zeta\left(\frac{1}{2}-i\,t-14.135\,i\right)\right)\right)\sin(t)\,dt = 0.570246$$

$$\int_0^{300} -\frac{\left(\frac{\zeta\left(\frac{1}{2}+i\,t-14.135\,i\right)-\zeta\left(\frac{1}{2}-i\,t+14.135\,i\right)}{2\,i}+\frac{\zeta\left(\frac{1}{2}+i\,t+14.135\,i\right)-\zeta\left(\frac{1}{2}-i\,t-14.135\,i\right)}{2\,i}\right)\cos(t)}{t}\,dt = -0.838763+0\,i$$

$$\int_0^{300} -\frac{\left(\frac{\zeta\left(\frac{1}{2}+i\,t-5\,i\right)-\zeta\left(\frac{1}{2}-i\,t+5\,i\right)}{2\,i}-\frac{\zeta\left(\frac{1}{2}+i\,t+5\,i\right)-\zeta\left(\frac{1}{2}-i\,t-5\,i\right)}{2\,i}\right)\sin(t)}{t}\,dt = 0.387297$$

$$\int_0^{300} -\frac{\left(\frac{1}{2}\left(\zeta\left(\frac{1}{2}+i\,t-5\,i\right)+\zeta\left(\frac{1}{2}-i\,t+5\,i\right)\right)-\frac{1}{2}\left(\zeta\left(\frac{1}{2}+i\,t+5\,i\right)+\zeta\left(\frac{1}{2}-i\,t-5\,i\right)\right)\right)\cos(t)}{t}\,dt = 0.479759$$

In these snapshots the first two are for the integration in (1), broken into two pieces. The first snapshot is the (A + C) term, and the second is the (B + D) term. Similarly for the last two snapshots. They represent the (D – B) and (C – A) terms, respectively, in (2).

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For (1), the numbers in the first two shots above sum to –0.268517. The first term on the *right*-hand side of (1) is ≈ –0.00011 (technically, it should be 0 for a root, so pretty close here) and the rest of the formula on the *right*-hand side of (1) computes to –0.269252. To *four* significant digits, the integration above is ≈ –0.2685 and the theoretical formula yields ≈ –0.2693. The two agree to 8 parts in 10,000 ! A stunning result, in my opinion, since we are only integrating out to 300 here.

For (2), the numbers in the last two shots above sum to 0.867056. The first term on the *right*-hand side of (2) is ≈ 0.725827, and the rest of the formula on the *right*-hand side of (2) computes to 0.141699. To *four* significant digits, the integration above is ≈ 0.8670 and the theoretical formula yields ≈ 0.8675. The two agree to 5 parts in 10,000 ! Another stunning result, in my opinion, since again, we are only integrating out to 300 here.

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Now we'd like to take the derivative, with respect to r, of the signal [right-hand side] in (3) and (4). They compute to

$$-2\pi e^{-r(1-\alpha)}\cos(\varepsilon r) \quad \ldots\ldots \text{ for (3)}$$

$$-2\pi e^{-r(1-\alpha)}\sin(\varepsilon r) \quad \ldots\ldots \text{ for (4)}$$

Notice that the *ratio* of these derivatives, say the second to the first, is tan(εr), an *invariant*, in so much as it does *not* depend on α. Now from our theorem, recall we have the following duality …...

$$\begin{array}{lll} \zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0 \textit{ if} \text{ and } \textit{only if} \text{ (3) and (4) are true } \textit{for all } r > 0 & & (\S\S) \\ \quad | \qquad\qquad\qquad\qquad\qquad\qquad | & & \\ \quad \boldsymbol{\mathcal{L}} \qquad\qquad\qquad\qquad\qquad\quad \boldsymbol{\mathcal{R}} & & \\ \quad | \qquad\qquad\qquad\qquad\qquad\qquad | & & \\ \zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0 \qquad\qquad\qquad \tan(\varepsilon r) & \text{conserved currents} & \end{array}$$

Thus, if T is the set of α in the interval (0, 1) … for which (§§) holds, then some ε > 0 will *always* exist, such that ζ(α ± ***i*** ε) = 0, *relative* to T. In this sense, the left side [$\boldsymbol{\mathcal{L}}$] of (§§) is *also* an invariant. It is the 'thing which is fixed and equal to 0', for *any* α ϵ T. As to the right side [$\boldsymbol{\mathcal{R}}$], we can treat tan(εr) as the invariant, *relative* to T. It is the 'thing which is *not* changing' for *any* α ϵ T, in so much as it does *not* depend on α.

Now we can invoke Noether, which states that, for the duality under consideration …

> symmetries associated with conserved currents in $\boldsymbol{\mathcal{L}}$ imply
> conservation laws in $\boldsymbol{\mathcal{R}}$, and conversely .. symmetries tied
> to conserved currents in $\boldsymbol{\mathcal{R}}$ .. imply conservation laws in $\boldsymbol{\mathcal{L}}$

The *simplest* symmetry which allows for this duality is again, reflection across the *x*-axis, where we have α = ½. Thus, under Noether, T only has *one* element; that is to say, T = {½}, if we opt for the simplest symmetry …

We can also develop another *invariant* … by noting that the *sum* of the *squares* of the derivatives computes to

$$\mu(r, \alpha) \;=\; 4\pi^2 e^{-2r(1-\alpha)}$$

Thus, if U is the set of $\varepsilon > 0$ for which (§§) holds, then some $\alpha \in (0, 1)$ will *always* exist, such that $\zeta(\alpha \pm i\varepsilon) = 0$, *relative* to U. In this sense, the left side [$\mathcal{L}$] of (§§) is *also* an invariant. It is the ‘thing which is fixed and equal to 0’, for *any* $\varepsilon \in$ U. As to the right side [$\mathcal{R}$], we can treat $\mu(r, \alpha)$ as the invariant, *relative* to U. It is the ‘thing which is *not* changing’ for *any* $\varepsilon \in$ U, in so much as it does *not* depend on ε.

$$\begin{array}{lcll}
\zeta(\alpha \pm i\varepsilon) = 0 & \textit{if} \text{ and } \textit{only if} \text{ (3) and (4) are true } \textit{for all} & r > 0 & \quad (§§) \\
\quad | & & \quad | & \\
\quad \mathcal{L} & & \quad \mathcal{R} & \\
\quad | & & \quad | & \\
\zeta(\alpha \pm i\varepsilon) = 0 & & \mu(r, \alpha) & \quad \text{conserved currents}
\end{array}$$

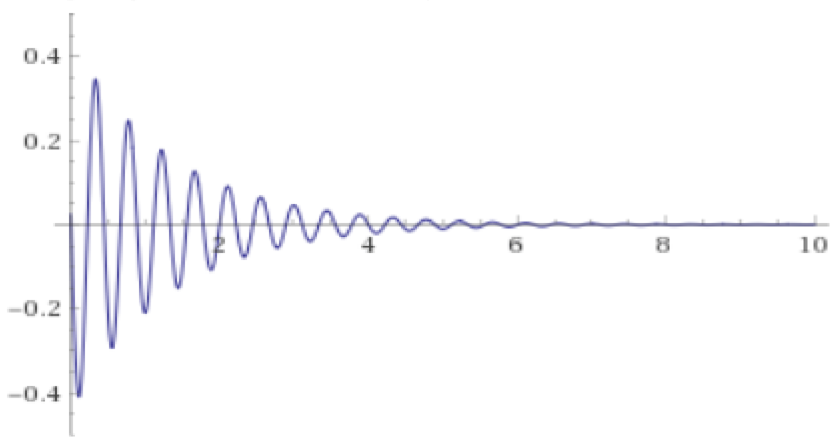
Once again, we can invoke Noether, which states that, for the duality under consideration …

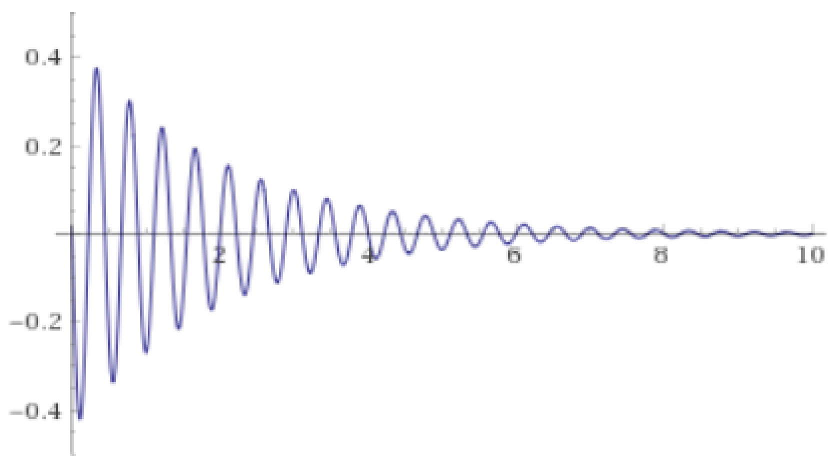
> symmetries associated with conserved currents in $\mathcal{L}$ imply
> conservation laws in $\mathcal{R}$, and conversely .. symmetries tied
> to conserved currents in $\mathcal{R}$ .. imply conservation laws in $\mathcal{L}$

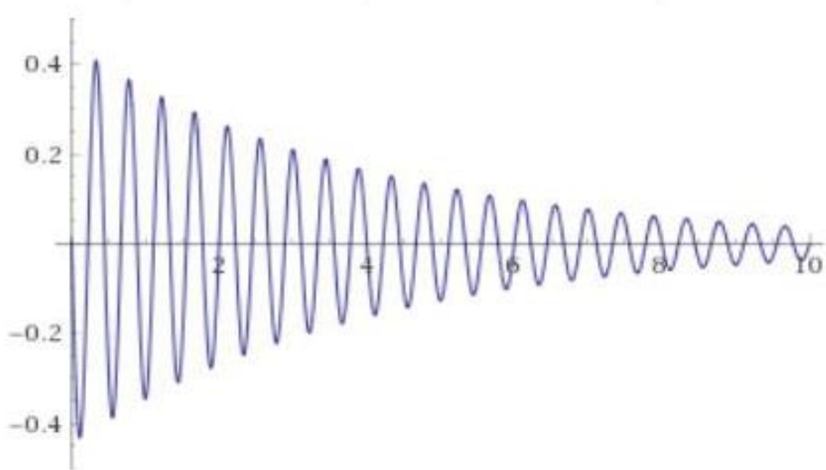
The *simplest* symmetry which allows for this duality is again, reflection across the *x*-axis, where we have $\alpha = ½$. And, since our theorem is grounded in *physical* principles, via Fourier, we have to respect the fact that Nature will always choose the *simplest* path where it can. And here, that means the zeroes of $\zeta(s)$ are, in all likelihood, to be found on the *critical* line ($\alpha = ½$), and nowhere else ….

Some interesting plots follow on the next two pages …

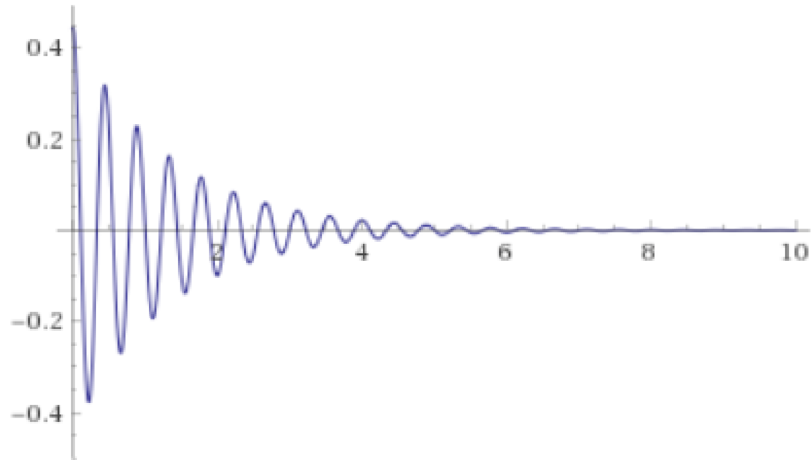
Now for some fun stuff. We'll show a few pictures here of the signal, when ε = 14.135, the approximate first root of ζ(s) on the critical line, when α = ¼, ½, and ¾, respectively. The intent is to give the reader a feel for just how *different* these signals are, and thus how differently behaved ζ(s) is along these vertical lines. Remember, the signal is an *encapsulation* of the *frequency* coefficients in (3) and (4) via Fourier. The plots show α = ¼, ½, and ¾, in that order, for eq'n (3).

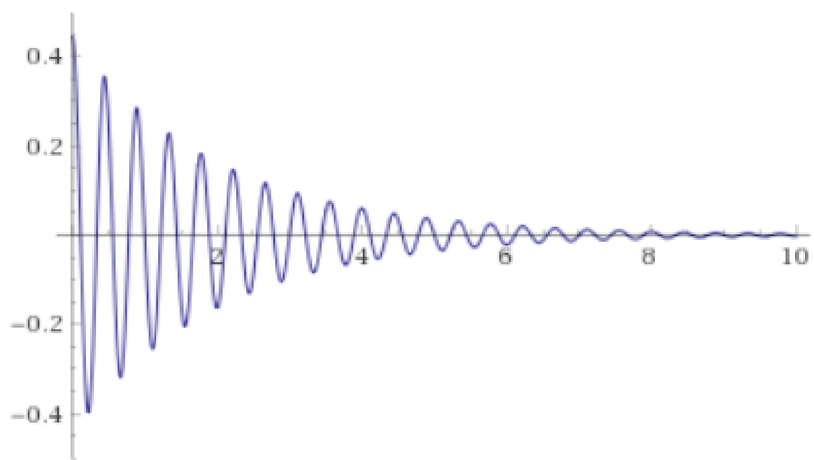


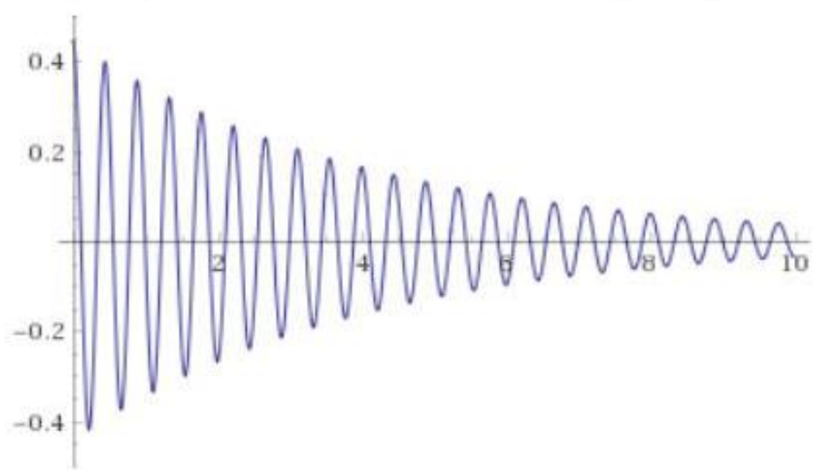


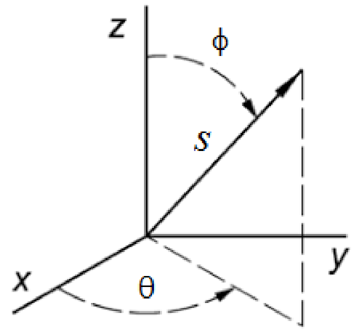


Now for some more stuff. We'll show a few pictures here of the signal, when ε = 14.135, the approximate first root of ζ(s) on the critical line, when α = ¼, ½, and ¾, respectively. The intent is to give the reader a feel for just how *different* these signals are, and thus how differently behaved ζ(s) is along these vertical lines. Remember, the signal is an *encapsulation* of the *frequency* coefficients in (3) and (4) via Fourier. The plots show α = ¼, ½, and ¾, in that order, for eq'n (4).





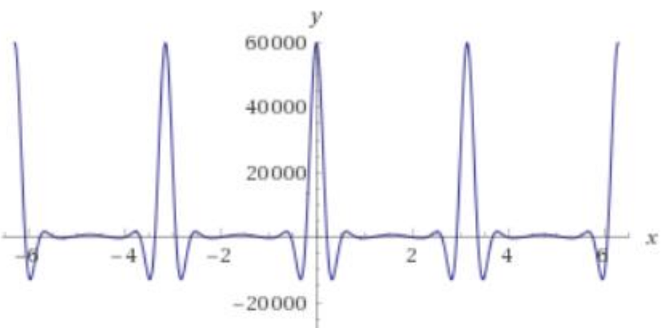




For a 3D star [pp 225-6], using a *simple* inverse density function [$\lambda(s) = \sigma/s$] … with *physical* singularities at the origin [O] and at

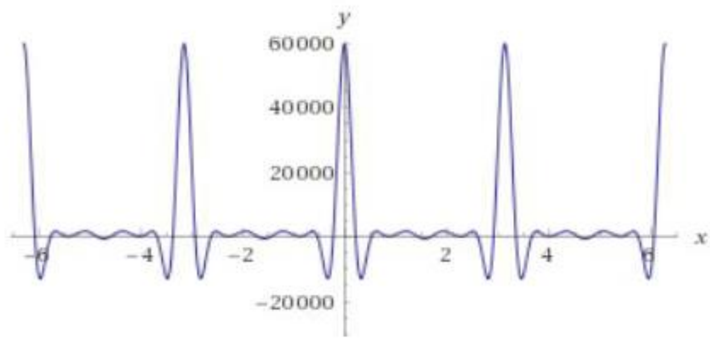
(1, 0, 0) (-1, 0, 0) (0, 1, 0) (0, -1, 0) (0, 0, 1) (0, 0, -1) ,

we can trace the *evolution* of our *quantumlike* dark energy scalar field, by starting with the plane at $\theta = 45^o$ to the *x*-axis, as shown above, and then rotating this plane to 60^o, 72^o, 84^o, and finally 90^o, where we reach the *y*-axis. Because of the location of the singularities above, we expect the scalar field to be *strongest* in the *x-z* and *y-z* planes (and indeed, *equal* in these planes … because of symmetry), and somewhere in between this for our given choices of θ. Here are the plots …



$r = 11$, $\theta = 45^o$, $\phi = x$, 3D star

ripples beginning to form between spikes



$r = 11$, $\theta = 60^o$, $\phi = x$, 3D star

ripples becoming more pronounced

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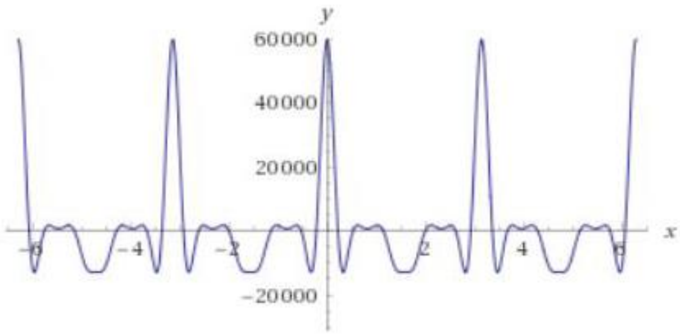
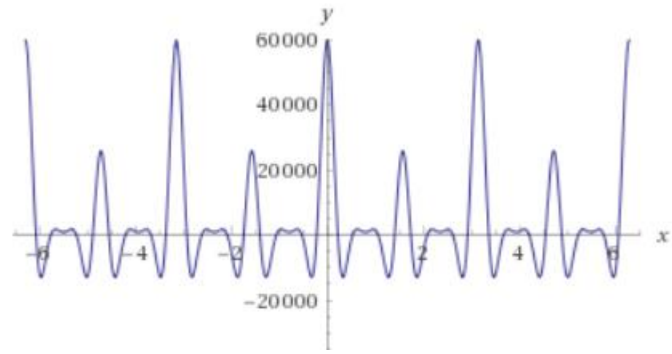


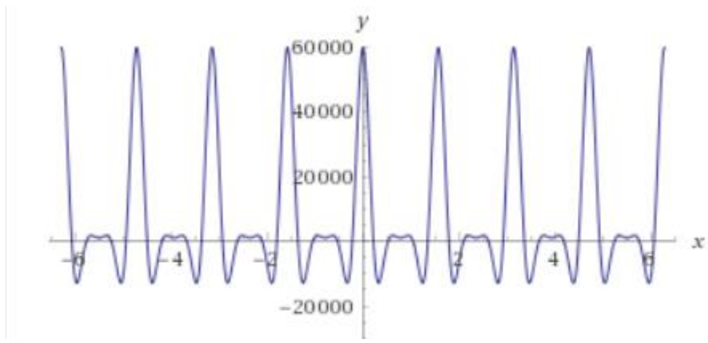
r = 11 , θ = 72^{o} , ϕ = *x* , 3D star

depression now forms between spikes



r = 11 , θ = 84^{o} , ϕ = *x* , 3D star

new spikes emerge from depression



r = 11 , θ = 90^{o} , ϕ = *x* , 3D star

spikes fully formed as we reach y-axis

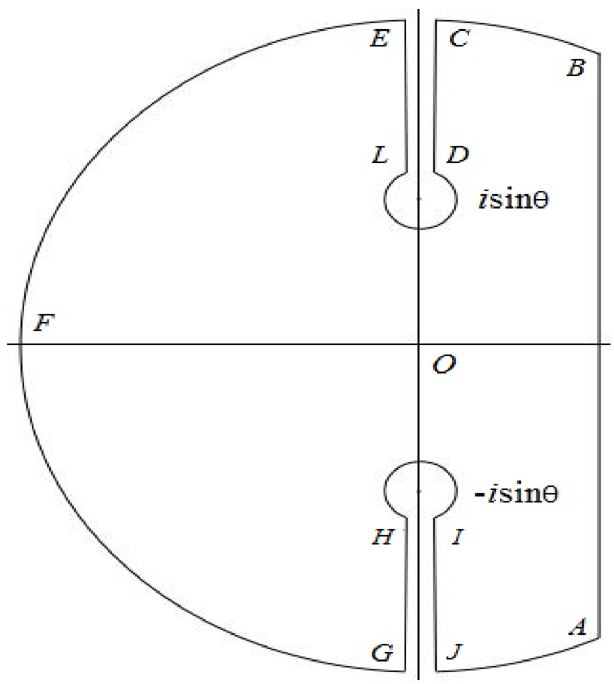
In the essay, we discussed several *density* functions of type $\lambda(s) = \sigma/s^{\mu}$, where $\mu > 0$, that might be suitable candidates for the *quantumlike* dark energy scalar field. And as part of that exercise, we developed various Laplace inversion techniques that allowed us to recover the scalar field from the density function, itself. Here, we're going to give a proof of the following general statement …

$$\Gamma(\nu + ½)u^{-2\nu - 1} \text{ ---> Laplace Inverse ---> } \sqrt{\pi}\,(2\alpha)^{-\nu}r^{\nu}J_{\nu}(\alpha r)\,,\ u = \sqrt{s^2 + \alpha^2}\,,\ \mathrm{Re}(\nu) > -1/2 \quad (*)$$

where $\Gamma()$ is the gamma function, $J_{\nu}()$ is a Bessel function of the first kind, of order ν, and we have $\mu = 2\nu + 1$. This statement was first introduced to us on page 230.

Indeed, we showed (*) to be true when $\nu = 0$ [pp 221-3] … so we'll use that knowledge here to develop the general result above.

Although (*) may seem to be a rather formidable expression, it really isn't that hard to derive, so let's begin with the following *Bessel* contour, which we will use to calculate the Laplace inverse for $\lambda(s)$, which itself, is actually the line integral along AB in the diagram below …



Both ***i***sin(θ) and **-*i***sin(θ) are *branching* points, so we need to set up our *phases* along the arms associated with these points, and here we can choose the phase for CD to be $e^{i\pi}$, and the phase for LE to be $e^{-i\pi}$. Similarly for GH and IJ, where they are $e^{i\pi}$ and $e^{-i\pi}$, respectively.

In doing so, we can calculate the correct value for $\sqrt{-1}$, as we move from one branch to the other, when traversing the contour γ, and indeed, the expression $(-1)^{\nu + ½}$, as well.

Referring back to pp 221-3 … we can now calculate the inverse of $\lambda(s) = 1/(s^2 + \alpha^2)^{\nu + ½}$, where here $\alpha = \sin(\theta)$, and the translation s ---› s – cos(θ) has already been effected for $\xi(s)$.

Thus, we are computing the contour integral

$$\kappa \int_{\gamma} e^{sr}\, \lambda(s)\, ds \qquad (\dagger)$$

where κ is equal to $1/2\pi i$, and $\boldsymbol{i}$ is imaginary and equal to $\sqrt{-1}$. Since the Laplace inverse is actually the line integral along AB, we only need calculate along the arms CD and LE in the upper half-plane, and then their counterparts in the lower half-plane. By letting the large arc BCEFGJA tend to ∞, and the radii of the small circles ---› 0, we will achieve the desired result; namely, Laplace inversion along AB.

Along CD, we integrate (†) from s = $i\infty$ to s = $i\sin(\theta)$. Setting $s = iy$, this means (†) becomes, along this arm, [and remembering our phase (with the constant κ omitted for the time being)] …

$$-e^{-i\pi\nu} \int_{\sin(\theta)}^{\infty} e^{iyr} / (y^2 - \alpha^2)^{\nu + ½}\ dy$$

Then we repeat, along LE, doing exactly the same thing. Remembering our phase here, we have …

$$-e^{i\pi\nu} \int_{\sin(\theta)}^{\infty} e^{iyr} / (y^2 - \alpha^2)^{\nu + ½}\ dy$$

Combining the two gives us, for the *upper* arms …

$$-2\cos(\pi\nu) \int_{\sin(\theta)}^{\infty} e^{iyr} / (y^2 - \alpha^2)^{\nu + ½}\ dy \qquad (1)$$

Now repeat again, but this time on the *lower* arms GH and IJ to obtain …

$$+2\cos(\pi\nu) \int_{\sin(\theta)}^{\infty} e^{-iyr} / (y^2 - \alpha^2)^{\nu + ½}\ dy \qquad (2)$$

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Finally, since we are traversing a contour that has *no* poles, the integration *must* sum to 0, so that …

$$(\dagger) \;=\; \kappa\int_{AB} \;+\; \kappa\int_{\boldsymbol{i}\sin(\theta)\text{ arms}} \;+\; \kappa\int_{\boldsymbol{-i}\sin(\theta)\text{ arms}} \;=\; 0$$

where it is understood, that we are carrying the integrand in (†) in all terms above. Thus, the Laplace inverse along AB is *precisely* the *negative* of (1) and (2) above, *multiplied* by κ. The result is …

$$(2/\pi)\cos(\pi\nu)\int_{\sin(\theta)}^{\infty} \sin(yr) \,/\, (y^2 - \alpha^2)^{\nu + ½}\, dy \qquad (3)$$

Notice that when $\nu = 0$, we recover $J_0(r\sin(\theta))$ [p 227], since again, $\alpha = \sin(\theta)$ has been chosen, rather arbitrarily. The result is valid for $-½ < \nu < ½$, since here we expect the integral in (3) to converge.

On the other hand, $\nu = ½$ is more difficult to determine in (3) … yet as we shall see, this same value of ν has meaning in (*), which we are now going to derive.

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To do this derivation, we need two pieces of information; first the reflection formula for the gamma function …

$$\Gamma(z)\Gamma(1 - z) \;=\; \pi / \sin(\pi z)$$

or, upon mapping z to z + ½,

$$\Gamma(z + ½)\Gamma(½ - z) \;=\; \pi / \cos(\pi z)$$

Next, we need the Mehler-Sonine formula, one of the most elegant expressions in all of Bessel Theory, in my opinion …

$$J_\nu(r) \;=\; [2 \,/\, \{(r/2)^\nu\sqrt{\pi}\; \Gamma(½ - \nu)\}] \int_{1}^{\infty} \sin(yr) \,/\, (y^2 - 1)^{\nu + ½}\, dy \qquad (§)$$

Now let $\theta = \pi/2$, for the time being, so that $\alpha = 1$ and (3) becomes …

$$(2/\pi)\cos(\pi\nu)\int_{1}^{\infty} \sin(yr) \,/\, (y^2 - 1)^{\nu + ½}\, dy \qquad (4)$$

Using the reflection formula for the gamma function, and (§), (4) now becomes …

$$(\sqrt{\pi} / \{\Gamma(½ + \nu)\})\ (r/2)^{\nu}\ J_{\nu}(r) \qquad (5)$$

and this agrees with (*), in this special case. To derive the more general formula, we let r map to rsin(θ) in (4), yielding …

$$(2/\pi)\cos(\pi\nu) \int_{1}^{\infty} \sin(y r\sin(\theta)) / (y^2 - 1)^{\nu + ½}\, dy \qquad (6)$$

Then because (4) and (5) are equivalent, (5) now becomes …

$$(\sqrt{\pi} / \{\Gamma(½ + \nu)\})\ (r\sin(\theta)/2)^{\nu}\ J_{\nu}(r\sin(\theta)) \qquad (7)$$

So at this point, (6) and (7) are equivalent expressions. Now let u = ysin(θ). Then (6) becomes our Laplace inverse,

$$(2/\pi)\cos(\pi\nu) \int_{\sin(\theta)}^{\infty} \sin(ur) / (u^2 - \alpha^2)^{\nu + ½}\, du \qquad (8)$$

where α = sin(θ). And finally, after rearranging terms, (7) becomes …

$$(\sqrt{\pi} / \{\Gamma(½ + \nu)\})\ (r/2\alpha)^{\nu}\ J_{\nu}(r\alpha)\ ,$$

which is what we wanted to show.

Even though values of v, such as $v = ½$, don't return anything meaningful in (3), they actually do in (*), and indeed, the example on pp 233-4 bears this out. And again, that is because at $v = ½$, say, we are *not* dealing with *branching* points. Nonetheless, (*) is general enough, it seems, even in these corner cases, and in this sense, becomes a rather remarkable formula.

Finally, there was nothing particularly special about our choice of α. It could have been anything really, so (*) still holds for more general choices of this variable …

On pp 237-9, we discussed ‘Yukawa-style’ density functions of type

$$\lambda(s) \ = \ \sigma e^{-\mu s}/s \qquad (*)$$

where μ is a *positive* constant, and s is a *measure* of distance. For a *two* dimensional star, with *physical* singularities at the origin [O] of the star, and at (1, 0) and (-1, 0), we saw that the field equations of general relativity could be written as

$$G^{u,v} \ \approx \ \sigma[1 \ + \ \psi(r)]g^{u,v} \qquad (\ddagger)$$

where ψ(r) is one of the two expressions below, with α = sin(θ) and β = cos(θ) …

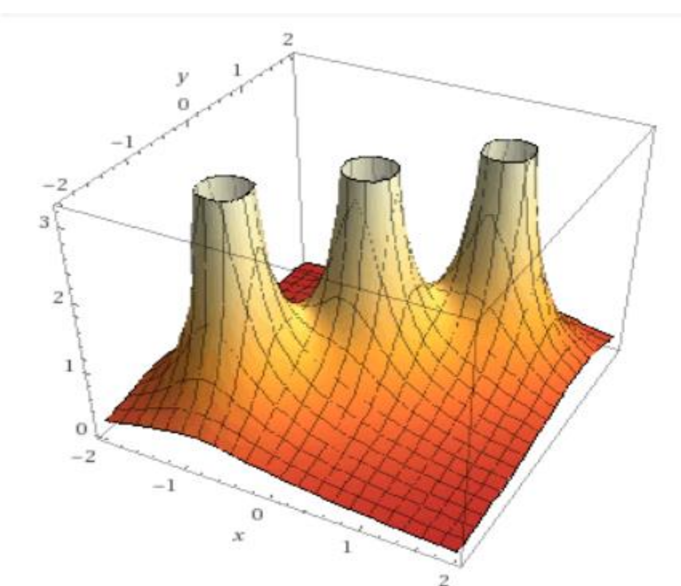
$$2\cosh(r\beta)J_0(\omega\alpha)\ , \ \ r > \mu\ , \ \ \omega \ = \ \sqrt{r^2 - \mu^2} \qquad \ldots \ \ (1)$$

$$2\cosh(r\beta)I_0(\omega\alpha)\ , \ \ r < \mu\ , \ \ \omega \ = \ \sqrt{\mu^2 - r^2} \qquad \ldots \ \ (2)$$

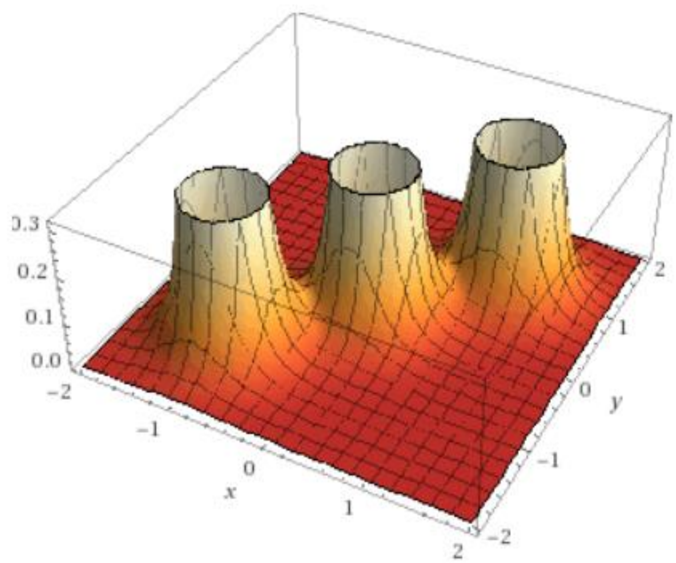
Further to this, we noted that the Laplace inverse of λ(s) is only valid if r > μ … and so we used interpolation methods to extend the result, when r < μ. Here, we’d like to show pictures of the *quantumlike* dark energy scalar field [ξ], which is the bracketed expression in (‡), for *all* r ≥ 0.

In doing so, it will give us a *feel* or *sense* as to how ξ behaves ‘behind the scenes’, when r < μ, and also, a *comparative* view of what ξ is doing, in the more familiar setting, when r > μ.

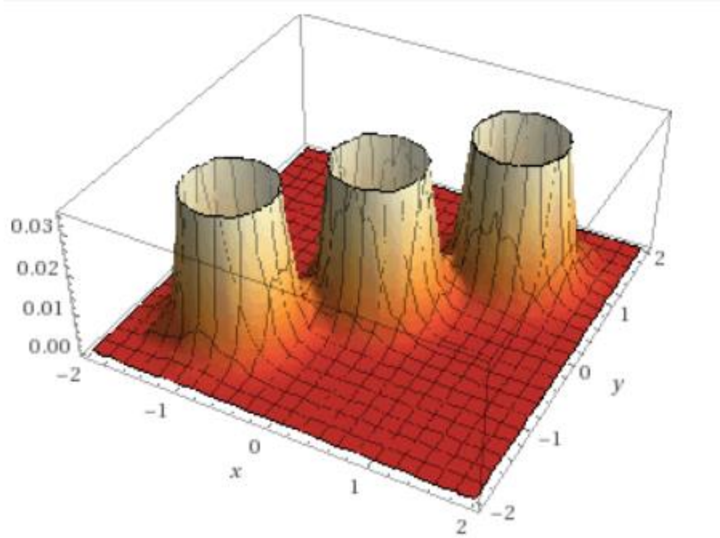
Let’s start, then, with a few 3D pictures of the density function [λ(s)] itself, where σ = 1, and we let μ = 1, 5, 9 for a 2D star, with *physical* singularities at the origin O of the star … *and* also at (1, 0) and (-1, 0). The images, it should be said, have been rotated for better viewing.



μ = 1



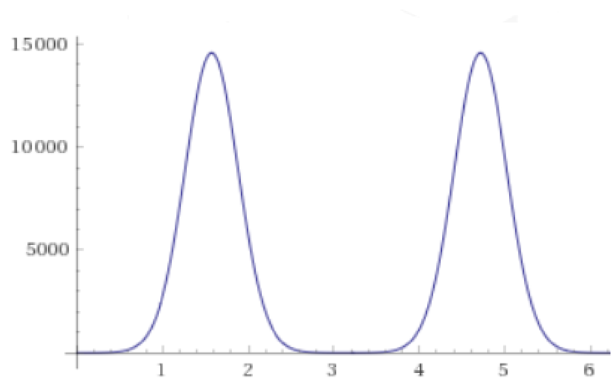
μ = 5



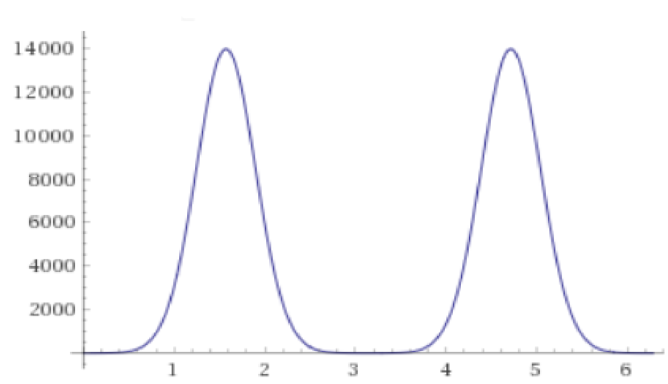
μ = 9

The important thing to notice here, is that as μ *increases*, the density function falls off *much* more rapidly, and this, in turn, has a distinct effect on the *quantumlike* dark energy scalar field [ξ], derived from λ(s), by taking the Laplace inverse.

Now we'd like to show some pictures of how ξ behaves, for r between 0 and 13, in *integer* steps, when μ = 11, and θ [*x*-axis] ranges between 0 and 2π. And again, we are dealing with *three* physical singularities here -- one at the origin O of the star … and the other two at (1, 0) and (-1, 0), respectively. Here, then, are the plots, which we'll discuss afterwards …



r = 0



r = 1

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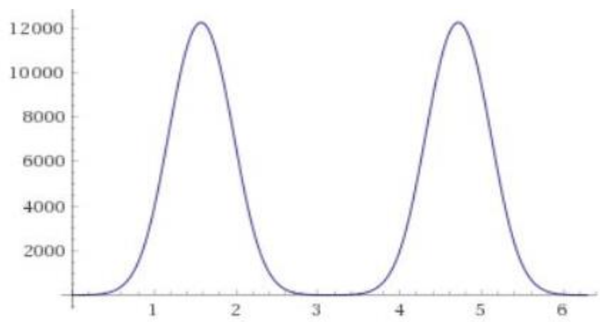
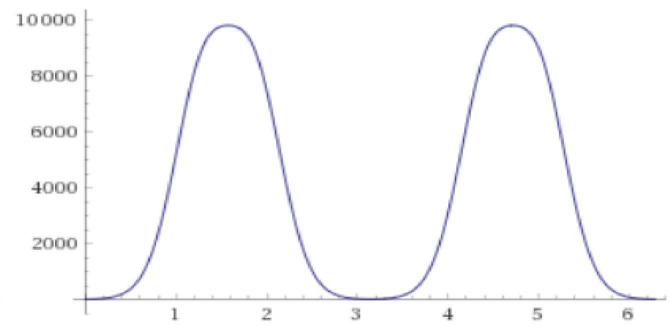
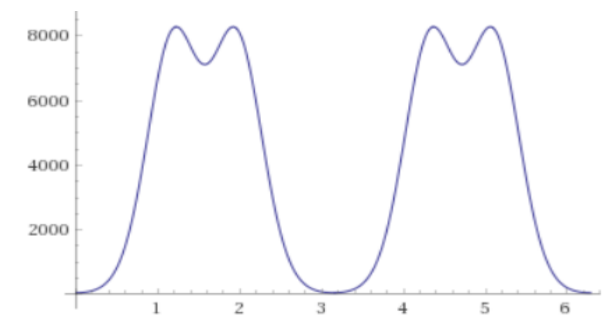


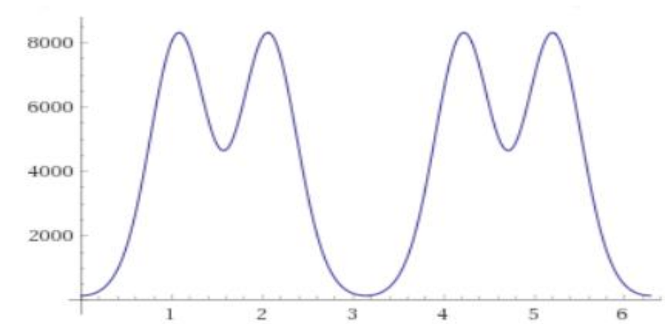
r = 2



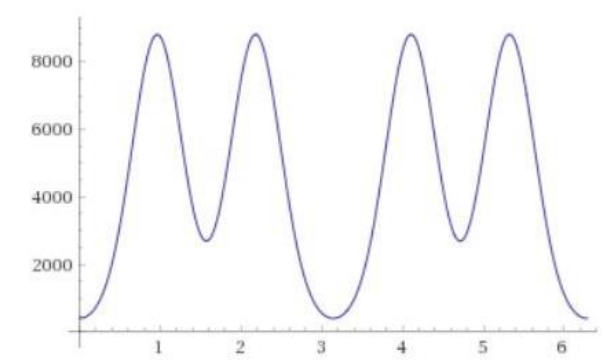
r = 3



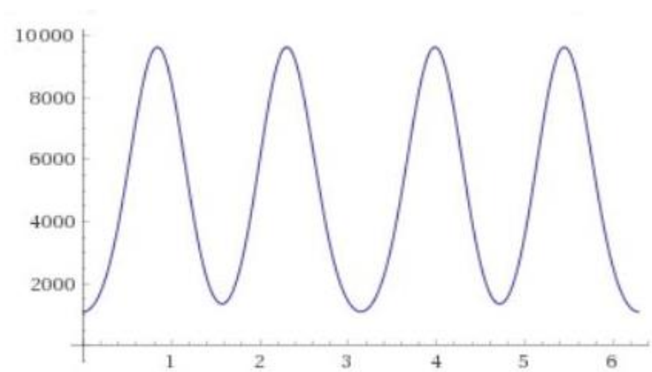
r = 4



r = 5



r = 6



r = 7

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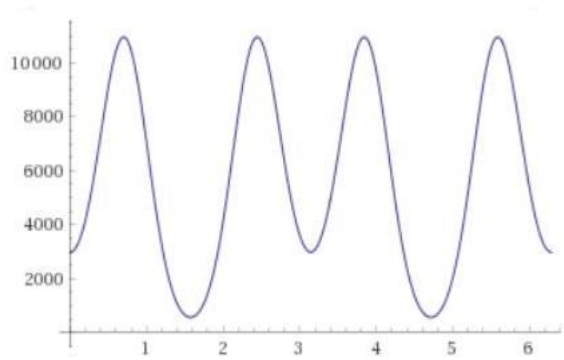
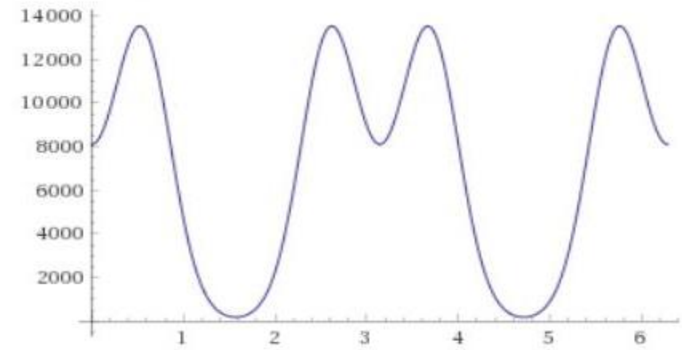
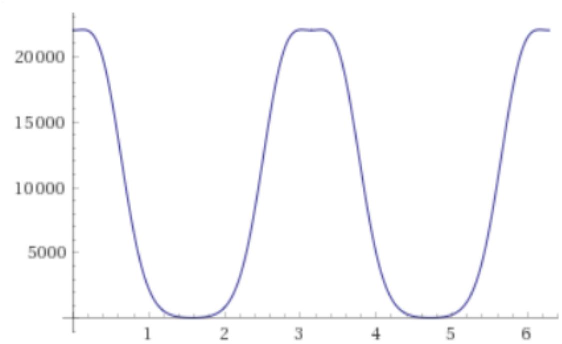
.
.



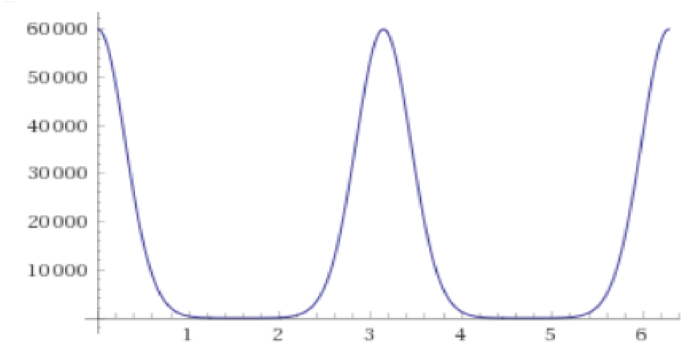
r = 8



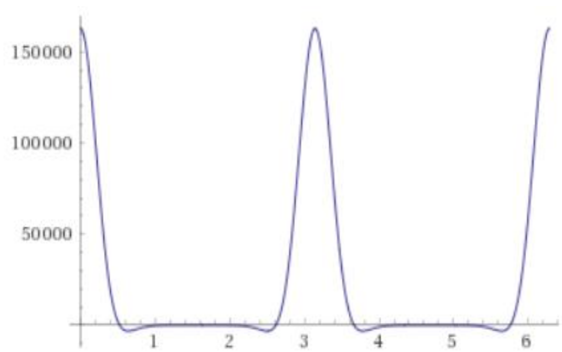
r = 9



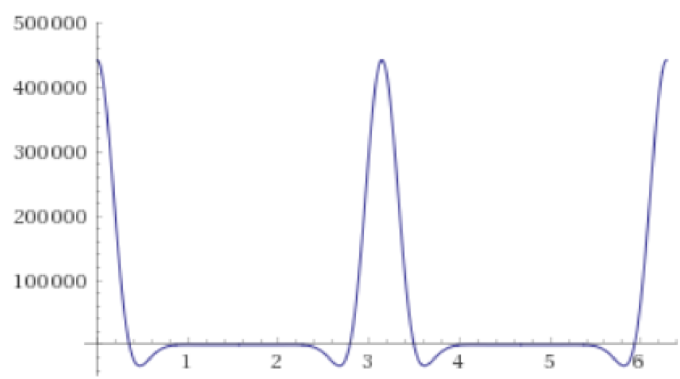
r = 10



r = 11 [barrier]



r = 12



r = 13

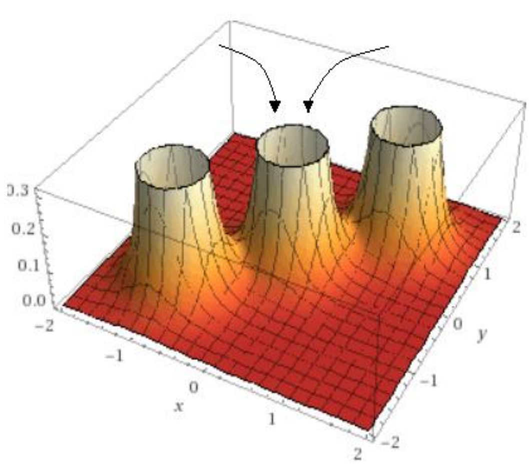
Some discussion of the pictures is in order. First, it is important to understand that what we are seeing here is, in all likelihood, the *quantumlike standing* wave patterns for the dark energy scalar field [ξ], along circles at different radii r, from the origin O of the star, itself. They are fluctuations, in so much as they *flash* on and off, perhaps billions of times per second, or even billions *x* billions of times in a second; seemingly bubbling up out of nothing and then receding into nothing, over and over again. An endless, frenetic dance, you might say, that is driving the expansion of the universe, among other things …

And, if we were to solve the field equations of general relativity [‡], for the gravitational tensor [$g^{u,v}$], in this case [or in any other case we have studied previously], we would probably find that $g^{u,v}$ itself, is *also quantumlike* in nature.

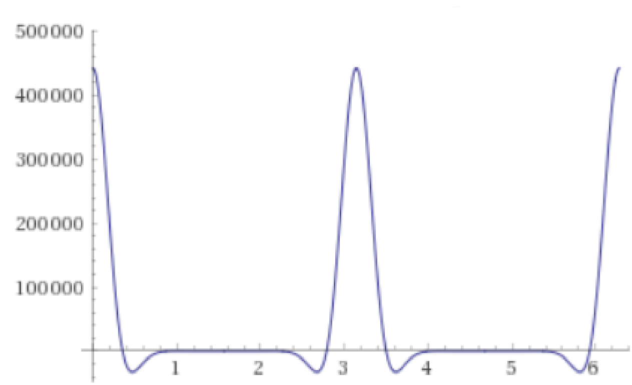
The case of the Yukawa density [*] is most interesting, because of the *barrier* at r = μ, The dark energy needs to find a way to *punch* through this barrier … before going into a kind of *spiky* oscillation, which we see at r = 12, and r = 13, in the pictures. At this point [r > μ] , ξ is both *negative* and *positive*, but *not* before. The positive and negative pieces of ξ should balance out to zero, as r tends to ∞, and we can think of them as the *accelerator* and the *brake*, when considering the expansion of the universe.

In the pictures above, where r runs from 0 to 11, we are … in my opinion, witnessing a kind of *quantum tunnelling* effect – one in which the dark energy, which is always *positive* here, is performing its own sort of dance, in advance of breaking through the barrier at r = μ. And the reason we can see this effect is because we chose μ to be something greater than 0, in this case.

Had we set μ = 0, we would not have seen the effect, because the Yukawa density is now *simple* inverse [$\lambda(s) = \sigma/s$], and only expression (1) applies, when considering ψ(r) in (‡). However, it's *not* that the *tunnelling* effect no longer exists; it probably *does*, but just outside our reality where now r < 0. Thus, dark energy seems to be tunnelling its way into our reality, giving us, perhaps, some evidence for the existence of a *particle* associated with ξ. A *wave-particle* duality, if you would like to characterize it as such …



dark energy tunnelling into our reality
via the singularities associated with λ(s)



our perception of dark energy, obtained
from λ(s), via the Laplace inverse transform

There is one thing that I find rather mystifying in the pictures, and that is when r = 0. Even here, with respect to ξ, the angle of approach *matters* as r tends to 0; something that is *not* true if $\mu = 0$. If I'm standing at the origin O of the star, for example, and I turn through 360 degrees, the dark energy at this point [O], is in fact, *variable*, if $\mu > 0$. Why is that ?

*****

Finally, some thoughts on *covariance*, with respect to (‡), where $G^{u,v} = C^{u,v} - kT^{u,v}$, and $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively …

$$G^{u,v} \approx \sigma[1 + \psi(r)]g^{u,v} \quad (‡)$$

Here, recall that $\psi(r)$ is one of the two expressions below, with $\alpha = \sin(\theta)$ and $\beta = \cos(\theta)$ …

$$2\cosh(r\beta)J_0(\omega\alpha)\,,\ r > \mu\,,\ \omega = \sqrt{r^2 - \mu^2} \quad \ldots \quad (1)$$

$$2\cosh(r\beta)I_0(\omega\alpha)\,,\ r < \mu\,,\ \omega = \sqrt{\mu^2 - r^2} \quad \ldots \quad (2)$$

Note that the left side of (‡) is *fully* covariant, and so is the *first* term in the bracketed expression for (‡), *even* when multiplied by $g^{u,v}$, since the foundational tensor [$g^{u,v}$] is *also* fully covariant. This leaves us with an interpretation of $\psi(r)$, and here we have to recognize that $\psi(r)$ is actually a mathematical *depiction*, in our reality, of the quantum fluctuations that bubble up from the dark energy *streaming* into our universe, via the *singularities* associated with the underlying *dark energy* density function [$\lambda(s)$]. These fluctuations *emerge* and then *recede*, almost continuously, and are derived by taking the Laplace inverse of $\lambda(s)$ [see pp 168-9 for more on this subject].

Thus, if we are to assign a *measure* of covariance to $\psi(r)$, it would have to be done at the quantum level, and indeed, this makes sense, since our perception of *dark energy* is the *bracketed* expression in (‡). It is *both* classical and *quantumlike*. and indeed, joins together the classical piece of general relativity with its quantumlike counterpart ...

*****

Perhaps it might be fitting to close with a comment from the French mathematician … Jacques Hadamard, who once said …

> 'The shortest path between two truths in the real domain passes through the complex domain'

$$G^{u,v} \approx \kappa \int_\gamma e^{sr}\,\lambda(s)g^{u,v}ds \quad (**)$$

Could it also be true, when looking for a solution to the quantum gravity problem, according to (**) above ?

This follow-on research note [**II**] is also dedicated to a deeper understanding of the Riemann zeta function [ζ(s)], using some of the material in the original essay. Specifically, as in the first note [pp 258-71, which we'll label **I**], we will develop *harmonic* integrable representations for variants of ζ(s), both in the critical strip S, where 0 < Re(z) < 1 and z is complex, *and* to the right of S as well, using a *Yukawa* density. The technical details will be kept to a minimum, and along the way, we shall also verify our work, so that theory and experiment line up, as they should. The hope is that this note, too, will lead to a better understanding of the Riemann Hypothesis [p 181], within the context of harmonic representations.

The Yukawa density is defined as [pp 237-9], for $\mu \geq 0$,

$$\lambda(s) = \sigma e^{-\mu s}/s$$

and its Laplace inverse is, for $r > \mu$, where J_0 is a Bessel function of order 0 …

$$e^{-\mu\nu}/\nu \text{ ---› Laplace Inverse ---› } J_0(\omega\alpha)\,,\ \nu = \sqrt{s^2+\alpha^2}\,,\ \omega = \sqrt{r^2-\mu^2} \qquad (\dagger)$$

Now by traversing the contour γ shown on page 258, and noting that the Laplace inverse is actually the line integral along AB, we can derive (†) just as we did for a certain *class* of *density functions* [pp 274-77] in this supplementary material, according to the following Yukawa integral …

$$\kappa\int_\gamma e^{sr}\, exp(-\mu\sqrt{(s-isin(\theta))(s+isin(\theta))}\,)\,/\sqrt{(s-isin(\theta))(s+isin(\theta))}\ ds \qquad (*)$$

The result is,

$$(2/\pi)\int_{\sin(\theta)}^{\infty} \sin(yr)\cos(\mu\sqrt{y^2-\sin^2\theta})\,/\sqrt{y^2-\sin^2\theta}\ dy$$

and this is actually $J_0(\omega\alpha)$, where $\alpha = \sin(\theta)$ [see page 86, #33, Bateman Manuscript Project, Tables of Integral Transforms, Volume I, for example].

Now we'll *couple* the function $g(s) = \zeta(\alpha - s)$ to the Yukawa density in (*), where α is any *real* number > 1. The constant κ remains and is equal to $1/2\pi i$, where $\boldsymbol{i}$ is imaginary and equal to $\sqrt{-1}$, and so we have, for our contour integral …

$$\kappa\int_\gamma e^{sr} g(s)\, exp(-\mu\sqrt{(s-isin(\theta))(s+isin(\theta))}\,)\,/\sqrt{(s-isin(\theta))(s+isin(\theta))}\ ds \qquad (\S)$$

As θ ---› 0, the integration along the arms *extends* from 0 to ∞ along the *imaginary* axis, in both the upper and lower-half planes … according to the following description below, in the case of Yukawa [refer back to pp 258-61 for the general methodology in **I**]. And the residue in (§)

computes to ζ(α), as θ ---› 0, *because* we are dealing with a *simple* pole here, at the origin [recall, in this case, that 0 < Re(AB) < α – 1] …

$$-2\int_0^\infty e^{iyr} g(iy)\cos(\mu\sqrt{(y)(y)}\,)\,/\sqrt{(y)(y)}\,dy \qquad 2\int_0^\infty e^{-iyr} g(-iy)\cos(\mu\sqrt{(y)(y)}\,)\,/\sqrt{(y)(y)}\,dy$$

isin(θ) arms ***-i***sin(θ) arms

And so, with this, we now have all the information we need to write down our first harmonic expression, when α is any *real* number > 1, in the case of a Yukawa coupling …

$$\int_0^\infty \{A\sin(yr) - B\cos(yr)\}\cos(\mu y)dy\,/\,y = (\pi/2)\zeta(\alpha) \qquad (\ddagger)$$

The result is true *for all* μ ≥ 0, for all r > μ, and for all real α > 1. Here

$$A = \{\zeta(\alpha + iy) + \zeta(\alpha - iy)\}/\,2$$

$$B = \{\zeta(\alpha + iy) - \zeta(\alpha - iy)\}/\,2i$$

and it should be said that when μ = 0, we recover the expression on page 260. Notice too, that the right-hand side of (‡) does *not* depend on μ, nor does it depend on r.

Now it’s time for some verification of our work. We’ll choose r = 1, α = 3/2, and μ = 1/2 and 3/2. We should see agreement in (‡) when μ = 1/2, but *not* when μ = 3/2. Here are the snapshots from Wolfram (the range of integration will vary, depending on how much execution time we have) …

$$\int_0^{400} \cos\left(\frac{t}{2}\right)\left(\frac{i\cos(t)\left(-\zeta\left(\frac{3}{2}-i\,t\right)+\zeta\left(\frac{3}{2}+i\,t\right)\right)}{2t}+\frac{\sin(t)\left(\zeta\left(\frac{3}{2}-i\,t\right)+\zeta\left(\frac{3}{2}+i\,t\right)\right)}{2t}\right)dt =$$
$$4.1029$$

$$\int_0^{350} \cos\left(\frac{3t}{2}\right)\left(\frac{i\cos(t)\left(-\zeta\left(\frac{3}{2}-i\,t\right)+\zeta\left(\frac{3}{2}+i\,t\right)\right)}{2t}+\frac{\sin(t)\left(\zeta\left(\frac{3}{2}-i\,t\right)+\zeta\left(\frac{3}{2}+i\,t\right)\right)}{2t}\right)$$
$$dt = 2.53464$$

To a few significant digits,

$$(\pi/2)\zeta(3/2) \approx 4.1035$$

. . .

and we see that, given our integration range, theory and experiment agree to 6 parts in 10,000 ! However, just as expected, when r = 1 and μ = 3/2, there is no agreement at all …

*****

Now we'd like to move into the *critical* strip where $0 < \alpha < 1$. Our function $g(s) = \zeta(\alpha - s)$ remains the same, but since $\zeta(\alpha - s)$ has a *simple* pole at $s = \alpha - 1 < 0$, we can no longer exclude it when traversing the *Bessel* contour γ, as shown on page 258. The vertical line AB in this contour *must* be to the *right* of the imaginary axis so that we trap the *branching* points $\boldsymbol{i}\sin(\theta)$ and $\boldsymbol{-i}\sin(\theta)$, and thus, by default, the pole associated with $\zeta(\alpha - s)$ is automatically included. Here, $0 < \mathrm{Re(AB)} < \alpha$.

Now since the Yukawa integral, reproduced below,

$$\kappa\int_\gamma e^{sr}g(s)\,exp(-\mu\sqrt{(s-isin(\theta))(s+isin(\theta))})\,/\sqrt{(s-isin(\theta))(s+isin(\theta))}\;ds \qquad (\S)$$

has *square roots* in both the *numerator* and the *denominator* of the integrand, *both* must be taken into account when considering the *sign* of the residues, just as we did in **I** [pp 262-64]. In so doing, we have the following, where $\beta = \alpha - 1 < 0$ …

$$\zeta(\alpha) - e^{(r+\mu)\beta}/\beta - e^{(r-\mu)\beta}/\beta \quad = -\kappa\int_{\boldsymbol{i}\sin(\theta)\text{ arms}} + \quad -\kappa\int_{\boldsymbol{-i}\sin(\theta)\text{ arms}}$$

Now we can finish off the exercise, just as we did when we were to the *right* of the critical strip with $\alpha > 1$, and we find that

$$\int_0^\infty \{A\sin(yr) - B\cos(yr)\}\cos(\mu y)\,dy\,/\,y = (\pi/2)\zeta(\alpha) + \pi e^{-r(1-\alpha)}\cosh(\mu(1-\alpha))\,/\,(1-\alpha)$$

This result is true *for all* $\mu \geq 0$, for all $r > \mu$, and for all real $0 < \alpha < 1$. Here

$$A = \{\zeta(\alpha + iy) + \zeta(\alpha - iy)\}/\,2$$

$$B = \{\zeta(\alpha + iy) - \zeta(\alpha - iy)\}/\,2i$$

Notice, too, that when μ = 0 in the expression above, we recover the original formula on page 263 in section **I**.

Now it's time for some verification of our work, so we'll choose r = 1, α = 1/2, and μ = 1/2. Here is the snapshot from Wolfram …

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$$\int_0^{260} \cos\left(\frac{t}{2}\right)\left(\frac{i\cos(t)\left(-\zeta\left(\frac{1}{2}-i\,t\right)+\zeta\left(\frac{1}{2}+i\,t\right)\right)}{2t}+\frac{\sin(t)\left(\zeta\left(\frac{1}{2}-i\,t\right)+\zeta\left(\frac{1}{2}+i\,t\right)\right)}{2t}\right)dt =$$

1.63754

Our formula from the last page, for r = 1, α = 1/2, and μ = 1/2, evaluates approximately to …

$$(\pi/2)\zeta(\alpha) \;+\; \pi e^{-r(1-\alpha)}\cosh(\mu(1-\alpha))\,/\,(1-\alpha) \;\approx\; 1.63674$$

and we see here that *theory* and *experiment* agree, for our range of integration, to 8 parts in 10,000 !

Just as in the previous case, with α > 1 *outside* the critical strip, the results here *inside* the critical strip are not only encouraging, but stunning as well. The agreement between theory and experiment is close enough, that we feel confident enough to pursue the next level of abstraction, which we will do in the next section, for Yukawa densities …

*****

In this section, we are going to continue the previous discussion, by introducing a new function g(s) and another function h(s) [**I** pp 265-271]. To wit …

$$g(s) = \zeta(\alpha - s + i\varepsilon) \;+\; \zeta(\alpha - s - i\varepsilon)$$
$$h(s) = \zeta(\alpha - s + i\varepsilon) \;-\; \zeta(\alpha - s - i\varepsilon)$$

where ε > 0 is any real number. Again, we are in the critical strip where 0 < α < 1. We now calculate residues, just as before, noting from above that the *simple* poles are at $s = \alpha - 1 \pm i\varepsilon$ for both g(s) and h(s), but because we are dealing with a Yukawa density, we again have to respect the *sign* of the residues, just as we did in the last section.

Now define the functions …

$$\Psi(x) \;=\; e^{-x(1-\alpha)}\{2(1-\alpha)\cos(\varepsilon x) - 2\varepsilon\sin(\varepsilon x)\} \;/\; \{(1-\alpha)^2 + \varepsilon^2\}$$

$$\Omega(x) \;=\; e^{-x(1-\alpha)}\{2(1-\alpha)\sin(\varepsilon x) + 2\varepsilon\cos(\varepsilon x)\} \;/\; \{(1-\alpha)^2 + \varepsilon^2\}$$

Then it is not too hard to show that

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.
.
.

$$\int_0^\infty \{(A + C)\sin(yr) \;-\; (B + D)\cos(yr)\}\cos(\mu y)\,dy\,/\,y \;=$$

$$(\pi/2)g(0) \;+\; (\pi/2)\{\,\Psi(r + \mu) \;+\; \Psi(r - \mu)\,\} \qquad \ldots (1)$$

$$\int_0^\infty \{(D - B)\sin(yr) \;+\; (C - A)\cos(yr)\}\cos(\mu y)\,dy\,/\,y \;=$$

$$-\,(\pi\boldsymbol{i}/2)h(0) \;+\; (\pi/2)\{\,\Omega(r + \mu) \;+\; \Omega(r - \mu)\,\} \qquad \ldots (2)$$

The result is valid *for all* $\mu \geq 0$, for all $r > \mu$, for all $\varepsilon > 0$, and for all real $0 < \alpha < 1$. Here

$$\begin{aligned} A &= \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) \;+\; \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/\,2 \\ B &= \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) \;-\; \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/\,2\boldsymbol{i} \\ C &= \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) \;+\; \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/\,2 \\ D &= \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) \;-\; \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/\,2\boldsymbol{i} \end{aligned}$$

Notice that at $\mu = 0$, both (1) and (2) assume their original forms in **I** [p 265], and that as ε ---› 0, we recover the expression in the last section [p. 286] from (1), since now A = C and B = D, and g(0) becomes $2\zeta(\alpha)$. On the other hand, (2) computes to 0 on *both* sides of the equals sign, so more good news.

Setting g(0) and h(0) to *zero* gives us …

$$\int_0^\infty \{(A + C)\sin(yr) \;-\; (B + D)\cos(yr)\}\cos(\mu y)\,dy\,/\,y \;=$$

$$(\pi/2)\{\,\Psi(r + \mu) \;+\; \Psi(r - \mu)\,\} \qquad \ldots (3)$$

$$\int_0^\infty \{(D - B)\sin(yr) \;+\; (C - A)\cos(yr)\}\cos(\mu y)\,dy\,/\,y \;=$$

$$(\pi/2)\{\,\Omega(r + \mu) \;+\; \Omega(r - \mu)\,\} \qquad \ldots (4)$$

And lastly, the original theorem [**I** p. 266] in its *new* form still holds, where $\alpha \pm \boldsymbol{i}\varepsilon$ is a root of $\zeta(s)$ in the *critical* strip …

$\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0$ *if* and *only if* (3) and (4) are true *for all* $\mu \geq 0$, for all $r > \mu$ (§§)

…

Now it's time for some verification of our work. We'd like to validate (1) when $\alpha = ½$, $\varepsilon = 14.135$, $r = 1$, and $\mu = ½$. In this case, ε is approximately the first root of $\zeta(s)$ on the critical line. Then we'll validate (2) when $\alpha = ½$, $\varepsilon = 5$, $r = 1$, and $\mu = ½$. Here are the raw snapshots from Wolfram …

Definite integral:

$$\int_0^{300} \frac{1}{t}\left(\frac{1}{2}\left(\zeta\left(\frac{1}{2} + i\,t - 14.135\,i\right) + \zeta\left(\frac{1}{2} - i\,t + 14.135\,i\right)\right) + \frac{1}{2}\left(\zeta\left(\frac{1}{2} + i\,t + 14.135\,i\right) + \zeta\left(\frac{1}{2} - i\,t - 14.135\,i\right)\right)\right) \sin(t)\cos\left(\frac{t}{2}\right)dt = 1.43166$$

Definite integral:

$$\int_0^{300} -\frac{1}{t}\left(\frac{\zeta(\frac{1}{2} + i\,t - 14.135\,i) - \zeta(\frac{1}{2} - i\,t + 14.135\,i)}{2\,i} + \frac{\zeta(\frac{1}{2} + i\,t + 14.135\,i) - \zeta(\frac{1}{2} - i\,t - 14.135\,i)}{2\,i}\right)\left(\cos(t)\cos\left(\frac{t}{2}\right)\right)dt = -1.62877 + 0\,i$$

$$\frac{\pi}{2}\exp(-0.75) \times \frac{\cos(14.135 \times 1.5) - 28.27\sin(14.135 \times 1.5)}{0.25 + 14.135^2}$$

Result:

$-0.0769986\ldots$

$$\frac{\pi}{2}\exp(-0.25) \times \frac{\cos(14.135 \times 0.5) - 28.27\sin(14.135 \times 0.5)}{0.25 + 14.135^2}$$

Result:

$-0.117781\ldots$

In these pictures, we are dealing with the first case (1), where $\alpha = ½$, $\varepsilon = 14.135$, $r = 1$, and $\mu = ½$. The first two pictures show the integral for (1), broken into two pieces, reflecting the (A + C) and (B + D) terms, respectively. The last two snapshots show the last two terms on the *right*-hand side of (1) and the *first* term on the *right*-hand side of (1) is ≈ –0.0001076 … nearly zero since we are

dealing with an approximate root of ζ(s). And it should be said that *all* calculations for sin() and cos() are to be done in *radians*.

Adding together the integral contributions yields ≈ –0.19711. Adding together *all* three terms on the *right*-hand side of (1) yields ≈ –0.19489. The two agree to about 2 parts in 1000, and this could be improved *significantly* if the integration range was *increased* … something that requires more execution time on the Wolfram site ! Now on to the second validation …

$$\int_0^{300} -\frac{\left(\frac{\zeta\left(\frac{1}{2}+i\,t-5\,i\right)-\zeta\left(\frac{1}{2}-i\,t+5\,i\right)}{2\,i}-\frac{\zeta\left(\frac{1}{2}+i\,t+5\,i\right)-\zeta\left(\frac{1}{2}-i\,t-5\,i\right)}{2\,i}\right)\left(\sin(t)\cos\left(\frac{t}{2}\right)\right)}{t}\,dt = 0.578853$$

$$\int_0^{300} -\frac{1}{t}\left(\frac{1}{2}\left(\zeta\left(\frac{1}{2}+i\,t-5\,i\right)+\zeta\left(\frac{1}{2}-i\,t+5\,i\right)\right)-\frac{1}{2}\left(\zeta\left(\frac{1}{2}+i\,t+5\,i\right)+\zeta\left(\frac{1}{2}-i\,t-5\,i\right)\right)\right)\left(\cos(t)\cos\left(\frac{t}{2}\right)\right)dt = -0.0830889+0\,i$$

Exact result:

$$-\frac{1}{2}\,i\,\pi\left(\zeta\left(\frac{1}{2}+5\,i\right)-\zeta\left(\frac{1}{2}-5\,i\right)\right)$$

Decimal approximation:

0.72582730986250184542300663!

$$\frac{\pi}{2}\exp(-0.75)\times\frac{\sin(5\times1.5)+10\cos(5\times1.5)}{0.5^2+25}$$

Result:

0.129425…

$$\frac{\pi}{2}\exp(-0.25)\times\frac{\sin(5\times0.5)+10\cos(5\times0.5)}{0.5^2+25}$$

Result:

−0.359151…

In these five pictures just above, we are dealing with the second case (2), where $\alpha = ½$, $\varepsilon = 5$, $r = 1$, and $\mu = ½$. The first two pictures show the integral for (2), broken into two pieces, reflecting the (D – B) and (C – A) terms, respectively. The last three snapshots show the three terms on the *right*-hand side of (2), where again sin() and cos() are calculated in *radians*.

Adding together the integral contributions yields ≈ 0.49576. Adding together *all* three terms on the *right*-hand side of (2) yields ≈ 0.49610. The agreement is stunning; they agree to about 3 parts in 10,000 !

CONNECTIONS TO NOETHER'S THEOREM

Just as in **I** [pp 268-9], we can develop *invariants* here that satisfy Noether's Theorem, relative to (§§) on page 288. From our definitions of $\Psi(x)$ and $\Omega(x)$ [p 287], we see that the following *derivatives*, with respect to r, hold true for the functions in the *rightmost* column …

$$-2\pi e^{-(r+\mu)(1-\alpha)}\cos(\varepsilon(r+\mu)) \quad \ldots.. \text{ for } \pi\Psi(r+\mu)$$

$$-2\pi e^{-(r-\mu)(1-\alpha)}\cos(\varepsilon(r-\mu)) \quad \ldots.. \text{ for } \pi\Psi(r-\mu)$$

$$-2\pi e^{-(r+\mu)(1-\alpha)}\sin(\varepsilon(r+\mu)) \quad \ldots.. \text{ for } \pi\Omega(r+\mu)$$

$$-2\pi e^{-(r-\mu)(1-\alpha)}\sin(\varepsilon(r-\mu)) \quad \ldots.. \text{ for } \pi\Omega(r-\mu)$$

Taking the *ratio* of the third to the first term, and then the *ratio* of the fourth to the second term, for example, and *adding* them together, yields the following *invariant*, which does *not* depend on α …

$$\tan(\varepsilon(r+\mu)) + \tan(\varepsilon(r-\mu))$$

And, if $\mu = 0$, the expression above now reduces to our *original* invariant, $\tan(\varepsilon r)$ [**I** p 268], after dividing by 2.

We can also develop *another* invariant, for example, by taking a *sum* of *squares* over all four derivatives [combining the first and third terms, and then the second and fourth terms]. This yields the following, which does *not* depend on ε …

$$8\pi^2 e^{-2r(1-\alpha)}\cosh(2\mu(1-\alpha))$$

And, if $\mu = 0$, the expression above reduces to our *original* invariant, $4\pi^2 e^{-2r(1-\alpha)}$ [**I** p 269], after dividing by 2.

Thus, we can apply Noether's Theorem, just as we did in **I** [pp 268-9], and conclude here, just as we did there, that the zeroes of $\zeta(s)$ are, in all likelihood, to be found on the *critical* line ($\alpha = ½$), and nowhere else …

ADDITIONAL EXPERIMENTAL TESTING

For the first case (1), where α = ½, ε = 14.135, r = 1, and μ = ½, the range of integration was, at last, successfully *increased* to 399. Here are the snapshots from Wolfram …

$$\int_0^{399} \frac{1}{t}\left(\frac{1}{2}\left(\zeta\left(\frac{1}{2}+i\,t-14.135\,i\right)+\zeta\left(\frac{1}{2}-i\,t+14.135\,i\right)\right)+\frac{1}{2}\left(\zeta\left(\frac{1}{2}+i\,t+14.135\,i\right)+\zeta\left(\frac{1}{2}-i\,t-14.135\,i\right)\right)\right)\sin(t)\cos\left(\frac{t}{2}\right)dt = 1.4216$$

$$\int_0^{399} -\frac{1}{t}\left(\frac{\zeta\left(\frac{1}{2}+i\,t-14.135\,i\right)-\zeta\left(\frac{1}{2}-i\,t+14.135\,i\right)}{2\,i}+\frac{\zeta\left(\frac{1}{2}+i\,t+14.135\,i\right)-\zeta\left(\frac{1}{2}-i\,t-14.135\,i\right)}{2\,i}\right)\left(\cos(t)\cos\left(\frac{t}{2}\right)\right)dt = -1.61661+0\,i$$

These pictures show the integral for (1), broken into two pieces, reflecting the (A + C) and (B + D) terms, respectively. Adding them together yields –0.19501. And from page 290, we know the three terms on the *right*-hand side of (1) sum to –0.19489. The agreement is stunning; the two *agree* to about 1 part in 10,000 …

On pages 274-77, we discussed Laplace *inversion* techniques for a certain class of density functions, and saw that for $\alpha = \sin(\theta)$, the expression

$$(2/\pi)\cos(\pi\nu)\int_{\sin(\theta)}^{\infty} \sin(yr) \,/\, (y^2 - \alpha^2)^{\nu + ½}\, dy \qquad (*)$$

formed our Laplace inverse for densities of type

$$\lambda(s) = 1/(s^2 + \alpha^2)^{\nu + ½} ,$$

provided $-½ < \nu < ½$. And at $\nu = ½$, we also saw that a Laplace inverse could *still* be inferred from (*), according to the expression below, as per the example on pages 233-34 …

$$\Gamma(\nu + ½)u^{-2\nu - 1} \;\text{---›}\; \text{Laplace Inverse} \;\text{---›}\; \sqrt{\pi}\,(2\alpha)^{-\nu}r^{\nu}J_{\nu}(\alpha r)\;,\; u = \sqrt{s^2 + \alpha^2}\;,\; \mathrm{Re}(\nu) > -1/2 \quad (§)$$

Thus, by *analytic continuation*, which is what we see here in (§), it is possible to deduce what the dark energy [ξ] in creation must look like, when the underlying dark energy *density* function [λ(s)] is inverse *square* [recall that ξ is derived from λ(s) by taking its Laplace inverse].

Such an operation raises an interesting *philosophical* question, which can be posed as follows ... …

‘ does analytic continuation [$\mathcal{A}$] drive the physical
laws of creation [$\mathcal{L}$] or is it the reverse ’

Clearly if there are *no* physical laws, there is nothing, and so the *existence* of $\mathcal{A}$ necessarily implies the existence of $\mathcal{L}$. But what about the reverse ? Does $\mathcal{L}$ imply $\mathcal{A}$? It almost has to, for dark energy [ξ] is, itself, an *inherent* part of the physical laws [$\mathcal{L}$] , derived via (§), which, in turn, is a representation of $\mathcal{A}$. Thus, the laws necessarily depend on *analytic continuation*, which means that $\mathcal{L}$ implies $\mathcal{A}$. And so we have the interesting result that $\mathcal{L}$ and $\mathcal{A}$ are *equivalent* notions.

*****

In the essay, all *foundational* laws are treated as *dualities*, grounded in *Noetherian* principles, so that symmetries imply the laws, and conversely, by way of some *action*. Since we now know that $\mathcal{L}$ and $\mathcal{A}$ are *equivalent* notions, it stands to reason that *certain* dualities arrived at, via some application of $\mathcal{A}$, could *also* be grounded in Noetherian principles.

Such an example is our work on *harmonic integrals* for the *analytic continuation* of the Riemann zeta function [ζ(s)], in this supplementary material, and most notably … the *physical* theorem (§§) concerning the zeroes of ζ(s) in the *critical* strip [see pp 266 and 288].

If the theorem (§§) is *fundamentally* Noetherian here (and, in fact, we have made it so via the *invariants*), then most surely the zeroes of ζ(s) in the critical strip will *only* be found on the *critical* line $\alpha = ½$, and nowhere else. Indeed, the equivalency of $\mathcal{L}$ and $\mathcal{A}$ allows us to put forward a

physical argument, as opposed to a *mathematical* one, and so we can, with confidence, argue in favor of the *simplest* [and hence, *strongest*] symmetry, when enquiring about these zeroes. Perhaps a quote from Einstein might be fitting at this point, when he was talking about the physical laws of creation. Simply put, he said

‘ … God does not like weak symmetries … ’

*****

And finally, since ‘the beast’ [ζ(s)] is bigger than the thing that built it [the human mind], we may have *no* choice but to take a *physical* approach when tackling the Riemann Hypothesis. Indeed, a mathematical approach may never give us an answer, one way or the other …

*****

We would now like to turn our attention to the Riemann Hypothesis, which asserts that the non-trivial zeroes of the Riemann zeta function [ζ(s)] lie *only* on the *critical* line ($\alpha = ½$) in the *critical* strip, and nowhere else. We'll demonstrate here, through our work below, that it is *not* possible to tell which α-line a zero is coming from, if it lies *off* the critical line, in the critical strip.

Now recall, from page 266, our theorem regarding the zeroes of ζ(s) in the critical strip, where we have $0 < \alpha < 1$, and $\varepsilon > 0$ …

$\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0$ *if* and *only if* (3) and (4) are true *for all* $r > 0$ (§§)

$$\int_0^\infty \{(A + C)\sin(yr) - (B + D)\cos(yr)\}dy / y = \pi e^{-r(1-\alpha)}\{2(1-\alpha)\cos(\varepsilon r) - 2\varepsilon\sin(\varepsilon r)\} / \{(1-\alpha)^2 + \varepsilon^2\} \quad \ldots (3)$$

$$\int_0^\infty \{(D - B)\sin(yr) + (C - A)\cos(yr)\}dy / y = \pi e^{-r(1-\alpha)}\{2(1-\alpha)\sin(\varepsilon r) + 2\varepsilon\cos(\varepsilon r)\} / \{(1-\alpha)^2 + \varepsilon^2\} \quad \ldots (4)$$

Here, the following coefficients *implicitly* reference the α-line,

$$\begin{aligned} A &= \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) + \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/ 2 \\ B &= \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) - \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/ 2\boldsymbol{i} \\ C &= \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) + \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/ 2 \\ D &= \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) - \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/ 2\boldsymbol{i}, \end{aligned}$$

and it is to be noted that an integration, carried out (implicitly) on the α-line, in (3) and (4), yields a *signal* on the *right*-hand side containing terms in $(1 - \alpha)$.

Now from the Bateman Manuscript Project, Tables of Integral Transforms [Volume 1, page 9], we see that the Fourier *cosine* transform of

$$f(y) \;=\; (\varepsilon + y) / (\beta^2 + (\varepsilon + y)^2) \;+\; (\varepsilon - y) / (\beta^2 + (\varepsilon - y)^2)$$

is equal to

$$\pi e^{-r\beta} \sin(\varepsilon r) \,,$$

where $\beta = 1 - \alpha$, and the Fourier *cosine* transform, itself, is

$$\int_0^\infty f(y)\cos(yr)dy \,.$$

Similarly, on page 65 of Volume 1, we see that the Fourier *sine* transform of

$$g(y) \;=\; (\varepsilon + y) / (\beta^2 + (\varepsilon + y)^2) \;-\; (\varepsilon - y) / (\beta^2 + (\varepsilon - y)^2)$$

is equal to

$$\pi e^{-r\beta} \cos(\varepsilon r) \,.$$

Thus, we can *completely* rebuild the *exact* signal [right-hand side] in (3), for example, by factoring in suitable constants for *both* the Fourier cosine and sine transforms of f(y) and g(y), respectively, and then adding them together. These constants are …

$-2\varepsilon \,/\{(1 - \alpha)^2 + \varepsilon^2\}$ … for the Fourier *cosine* transform of f(y)

$+2(1 - \alpha) \,/\{(1 - \alpha)^2 + \varepsilon^2\}$ … for the Fourier *sine* transform of g(y)

Notice now, that the *denominators* in f(y) and g(y), can be written as

$$(\beta^2 + (\varepsilon + y)^2) \;=\; (\beta + (\varepsilon + y)\boldsymbol{i})(\beta - (\varepsilon + y)\boldsymbol{i})$$

$$(\beta^2 + (\varepsilon - y)^2) \;=\; (\beta + (\varepsilon - y)\boldsymbol{i})(\beta - (\varepsilon - y)\boldsymbol{i}) \,,$$

where $\boldsymbol{i}$ is imaginary and equal to $\sqrt{-1}$, so that our Fourier *cosine* and *sine* integrations occur (implicitly) along the vertical line $\beta = 1 - \alpha$ in the complex plane, yielding the *same* signal as we see in (3).

Thus, for the case where $\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0$ in the *critical* strip, we may now ask *which line* the signal came from, since we have two *distinct* integrations to choose from. If we say (3), then the signal was generated from the α-line; but if we choose the Fourier *cosine* and *sine* approach, then this *same* signal was generated from the β-line, where $\beta = 1 - \alpha$, and it is *not* possible to distinguish between the two paths.

In other words, it is *not* possible to distinguish the root on the α-line from the root on the β-line, using this methodology, unless, of course, $\alpha = ½$ [Riemann's functional equation tells us *both* roots exist for ζ(s), *if* one of them does, but says nothing about the *undecidable* nature of these roots].

And finally, a similar argument can be applied to (4) using the Bateman Manuscript Tables, but we omit it here …

UNDECIDABILITY AND THE RIEMANN HYPOTHESIS

Suppose now, $\omega = \alpha \pm \boldsymbol{i}\varepsilon$ is a root of ζ(s) in the *critical* strip, where $0 < \alpha < 1$. By what we have just said, there is no *mathematical* method $\mathcal{M}$ that can prove, *without* ambiguity, that $\alpha < ½$. Similarly $\mathcal{M}$ cannot prove that $\alpha > ½$, *without* ambiguity, since there is no way to decide if the root came from the α-line or the β-line, according to our methodology above.

About all we can say is that *if* ω exists, it came from the α-line or the β-line, but which … we'll never know. It simply cannot be decided, which suggests to me anyway, that the Riemann Hypothesis itself, is undecidable, *if* $\alpha \neq ½$. Indeed, there is *no* way to verify the existence of ω.

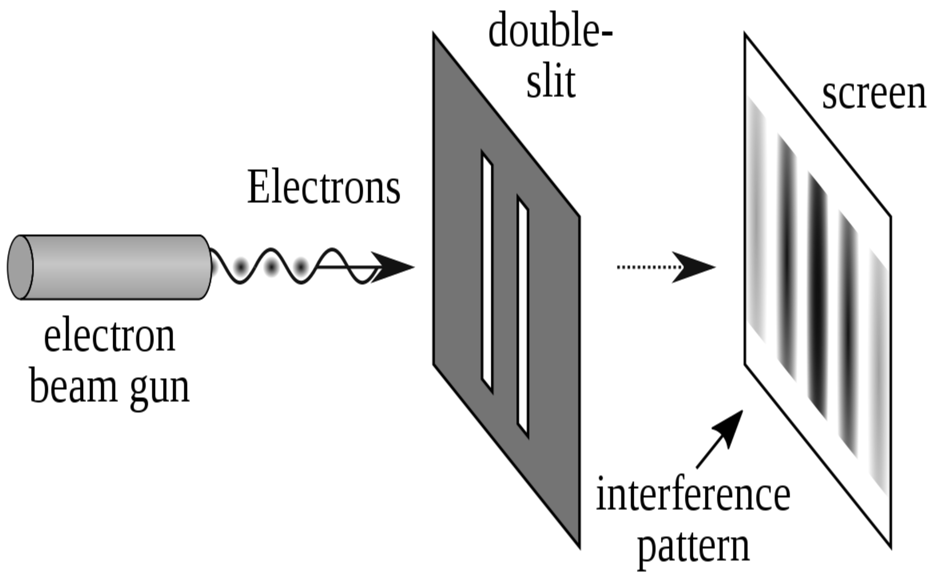
To further amplify on this conundrum, suppose we had two drawers of unpaired socks of the same shape and size, where the number of socks in each drawer was equal. And suppose further that one, and *only* one, of the drawers contained a red sock. If you were asked to say which drawer contained the red sock, but were *colorblind*, you couldn't decide. And this is what we're facing here with the root ω of ζ(s) in the *critical* strip. We can't tell if it came from the α-line or the β-line, *if* $\alpha \neq ½$.

Now our methodology above states that we cannot tell if the signal [***S***] was generated from α or β, so that *relative* to ***S***, it is not possible to distinguish between the α-line and the β-line. But ***S*** exists *if* and *only if* ω is a root of ζ(s) in the *critical* strip, according to our theorem (§§) on page 294. Thus, when we say 'relative to ***S***' it is equivalent to saying 'relative to ω'. And so we may conclude that relative to $\omega = \alpha \pm \boldsymbol{i}\varepsilon$, in the *critical* strip, it is *also not* possible to distinguish between α and β.

To see this a bit more clearly, suppose ω is a root. Then we have already shown that ***S*** exists, *independent* of the path [α or β]. Suppose now, we *assume* ***S*** exists, *independent* of the path. Then ***S*** can be generated from α or β. If we say α, then (§§) applies on page 294, and ω is a root. If we say β, then ***S*** exists as a *proper* signal, and since, by assumption, ***S*** exists *independent* of the path, it must *also* exist as a *proper* signal, according to (§§), implying again that ω is a root. Thus, ω and ***S***

are *equivalent* to one another in this case, so that 'relative to $\boldsymbol{S}$' means 'relative to ω', as we said above.

In some ways, it reminds me of the famous double-slit experiment in physics. Electrons travelling toward these slits can take either path, but because of the strangeness of quantum mechanics, there is no way to formulate a coherent picture in our minds … of how this actually happens. Perhaps, in the end, we'll find out that the Riemann zeta function may be quantumlike in nature as well, exhibiting both *wavelike* and *particle-like* properties at the same time …



Electrons 'passing through' the slits produce a *wave* pattern on the screen, not unlike the diffraction pattern produced by a plane wave approaching these same slits (photo courtesy of Johannes Kalliauer, from the Wikipedia pages, and NekoJaNekoJa~commonswiki). Thus the electrons have both *wavelike* and *particle-like* properties, simultaneously.

OTHER CONSIDERATIONS

Suppose now, for $\alpha \neq \frac{1}{2}$, a mathematical method $\mathcal{M}$ exists that can show $|\,\zeta(\alpha \pm \boldsymbol{i}\varepsilon)\,| > 0$, for some choice of $0 < \alpha < 1$, and for *all* $\varepsilon > 0$, where $|\,.\,|$ means 'take the modulus of'. Then we are led to believe ζ(s) has no roots on this α-line, *nor* does it have any roots on the β-line, where $\beta = 1 - \alpha$. Is this a 'rock solid' conclusion ? In other words, can we trust $\mathcal{M}$?

Well, suppose for the sake of argument, a root ω of ζ(s) really *does* exist on this α-line, so that we may write $\zeta(\omega) = 0$. By assumption, $\mathcal{M}$ will report that $|\,\zeta(\omega)\,| > 0$, but this must *also* be so from our work above, since we already know there is *no* mathematical method that can *verify* the existence of ω, if $\alpha \neq \frac{1}{2}$. In particular, $\mathcal{M}$ is *not* allowed to conclude that $|\,\zeta(\omega)\,| = 0$, *even* if ω is a root (cont'd …)

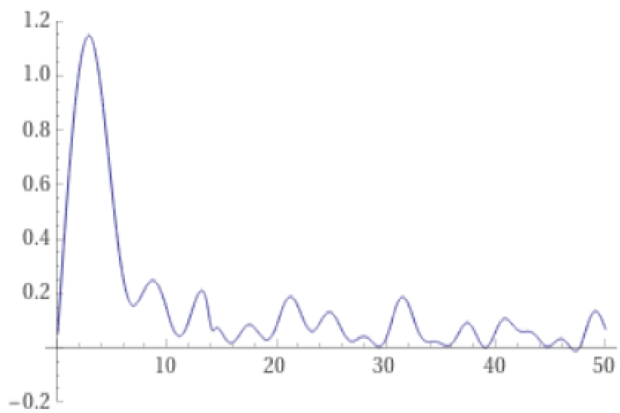
Thus, $\mathcal{M}$ *cannot* tell us if ω really *is* or *isn't* a root on the α-line, under this scenario, for either way, it's going to say | ζ(ω) | > 0. In turn, this means one of two things: no such $\mathcal{M}$ exists, or if it does exist, $\mathcal{M}$ cannot distinguish between roots and *non*-roots of ζ(s) on *any* α-line, if α ≠ ½. To $\mathcal{M}$, everything seems like a *non*-root, even though this may not actually be the case.

The more reasonable conclusion, in my opinion, is that no such $\mathcal{M}$ exists [at least not in the case, for *all* ε > 0]. Indeed, the *existence* or *inexistence* of a root ω on the α-line really isn't the issue anyway; but rather, what $\mathcal{M}$ does with ω. And perhaps it might be worth emphasizing, too, that our theorem (§§) on page 294 is really an *existence* theorem that doesn't tell us anything about *verification*. These ideas, it goes without saying, are two entirely different notions; for it is one thing to say a root ω exists, but quite another to verify this statement, where 'verify' means 'prove without ambiguity'.

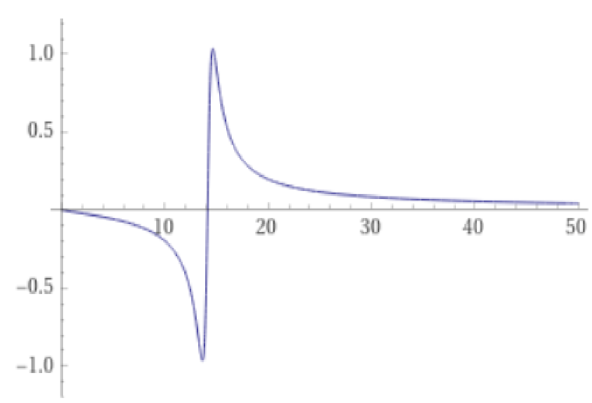
To me, at least, this makes the Riemann Hypothesis *undecidable* in the *critical strip*, if α ≠ ½. Said another way, *any* root of ζ(s) that can be *verified* in the critical strip, necessarily lies on the *critical* line (α = ½), and nowhere else. Beyond that, we are really in no man's land ...

SOME INTERESTING PICTURES

Below are some interesting plots of the coefficient function (A + C) / y, referenced in equation (3) on page 294, and the coefficient function g(*y*), referenced on page 295. Both are used, in part, to create *trigonometric* Fourier transforms, that ultimately lead us back to the *same* signal [right-hand side] in (3). It is quite amazing, really, that two such disparate coefficients [the first being a variant of the Riemann zeta function, and the second essentially algebraic in nature], could actually do this. And it is curiously odd, too, that whilst A and C *implicitly* reference the α-line, the denominator in g(*y*) *implicitly* references the β-line, where β = 1 – α. Exactly where the roots of ζ(s) lie, according to Riemann's functional equation …



(A + C)/y, α = 1/2, ε = 14.135



g(*y*), β = 1/2, ε = 14.135

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.
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SOME ADDITIONAL REMARKS

One might be tempted to see our theorem (§§), on page 294, as something *more* than a statement concerning the *existence* of zeroes of the Riemann zeta function [ζ(s)], in the *critical* strip. For example, we could conclude that (§§) is true for α, and *also* true when we replace α with β = 1 – α, thereby convincing ourselves that the theorem is able to *validate* a root of ζ(s) on the α-line, and *also* separately, on the β-line.

But while it is true that (§§) holds for both α and β, it does so only *because* of Riemann's functional equation [RFE] and *not* in spite of it. And since RFE is commentary *only* on the *existence* of roots of ζ(s), but says nothing about the decidable or undecidable nature of these roots, we have to see (§§) in the same light. Our theorem, therefore, is an *existence* theorem, valid in the critical strip, just as we said earlier. It doesn't tell us anything about the *verification* of non-trivial zeroes of ζ(s), but offers some hope in the case where α = ½; for here, there is no *ambiguity* concerning the α-line versus the β-line.

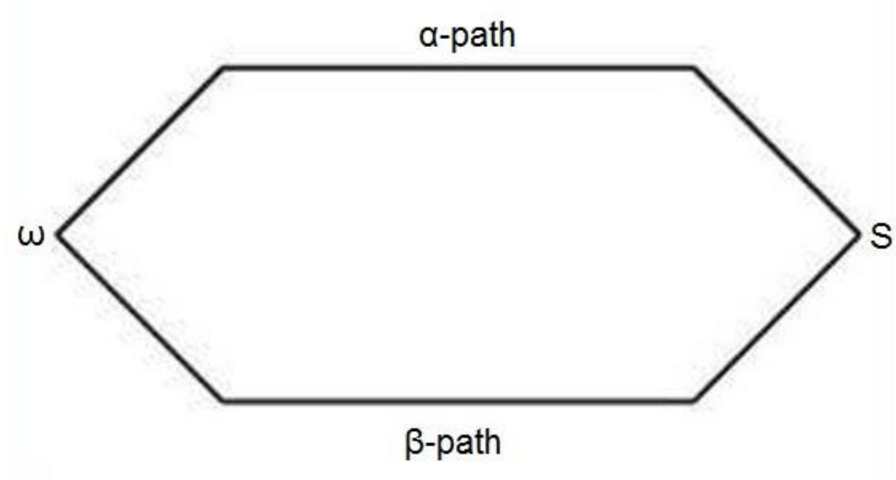
Thus, our preceding logic is correct, in my opinion, where we concluded that any root of ζ(s) that lies in the critical strip, but *off* the critical line [α = ½], can never be verified, where 'verify' means 'prove without ambiguity'. If this is so, then what does it say about the Riemann Hypothesis, in general ?

If we think back to the 'sock and drawer' analogy on page 296, the *colorblind* observer will never be able to decide which drawer the red sock is in. So in what sense, then, does the red sock even exist, as far as this observer [$\mathcal{O}$] is concerned ? To $\mathcal{O}$, it exists because that is what he or she was told, but in reality, $\mathcal{O}$ has no way of verifying this assertion. The existence *or* inexistence of the red sock, therefore, is of no consequence to $\mathcal{O}$, since either way, $\mathcal{O}$ cannot decide which drawer it is in, all else being equal [perhaps, for example, the red sock was really blue, unbeknownst to $\mathcal{O}$].

And so it is with the roots of ζ(s) in the critical strip, that lie *off* the critical line [α = ½]. Maybe they are there, and maybe they are not; but regardless, there is *no* mathematical method $\mathcal{M}$ that can verify these roots on any α-line, if α ≠ ½. Similarly, $\mathcal{M}$ cannot 'walk the entire α-line' and conclude, without ambiguity, that there are *no* roots here either [α ≠ ½], for what is it going to say if it encounters one ?

> walking the α-line and saying there are no roots, is equivalent to walking the α-line and saying there are roots that cannot be verified. For logically, there is no difference between the two statements concerning ζ(s), in the critical strip, if α ≠ ½ …

Finally, are we any better off traversing the *entire* critical strip *horizontally*, as opposed to *vertically*, so that α is free to roam, where 0 < α < 1, and ε > 0. I don't believe we are, for again, there is no mathematical method $\mathcal{M}$ that can verify *any* roots of ζ(s) that lie *off* the critical line, *no* matter the direction of traversal. Thus, just as before, $\mathcal{M}$ cannot 'walk the entire ε-line horizontally' [meaning for *all* α, such that 0 < α < 1, α ≠ ½], and conclude, *without* ambiguity, that there are *no* roots here either; for yet again, what is $\mathcal{M}$ going to say if it encounters one … for some ε > 0, however large ?



SUMMING IT UP

In the diagram above, ω is a root of ζ(s) in the *critical* strip, and ***S*** is the signal, which can be generated via the α-line *or* the β-line, as per our discussions on pages 294 and 295. If we accept this methodology, then we have *no* choice, but to conclude, that roots of ζ(s) which lie in the critical strip, but *off* the critical line, can never be verified …

Finally, if ***ℛ*** is the set of *all* zeroes of ζ(s) in the critical strip, does it make sense to speak about the cardinality of ***ℛ***, knowing that roots *off* the critical line can never be verified ? Is it **N** (the integers), or perhaps **c** (the reals), or can it even be decided any longer ? And to what extent should we *trust* traditional methods in mathematics, that are normally used to identify the zeroes of a function, when in the case of ζ(s), they can't even be verified *away* from the critical line ? Time will tell, I suppose, if these questions really have any answers …

THE SPECIAL CASE WHEN $\varepsilon = 0$

In the special case when $\varepsilon = 0$, in the *critical* strip, where $0 < \alpha < 1$, we can still make a comment on roots of $\zeta(\alpha)$, *if* they exist. Recalling our theorem on page 263, which is valid *for all* $r > 0$ and for all *real* $0 < \alpha < 1$,

$$\int_0^\infty \{A\sin(yr) - B\cos(yr)\}dy / y = (\pi/2)\zeta(\alpha) + \pi e^{-r(1-\alpha)} / (1-\alpha)$$

we see that *if* $\zeta(\alpha) = 0$, this reduces to

$$\int_0^\infty \{A\sin(yr) - B\cos(yr)\}dy / y = \pi e^{-r(1-\alpha)} / (1-\alpha) \qquad (\dagger)$$

Here,

$$A = \{\zeta(\alpha + iy) + \zeta(\alpha - iy)\}/\,2$$

$$B = \{\zeta(\alpha + iy) - \zeta(\alpha - iy)\}/\,2i$$

Now let ε ---› 0 in g(*y*) on page 295, so that the Fourier *sine* transform of g(*y*) becomes $\pi e^{-r\beta}$, where β = 1 – α. Factoring in the constant 1/ β reproduces (†) above, and now the same argument put forth on pages 295-6 applies, if α ≠ ½. Thus, any roots of ζ(α) that lie in the *critical* interval (0,1), but *off* the critical line [α = ½], can never be verified …

UNDECIDABILITY AND THE RIEMANN HYPOTHESIS WHEN μ ≥ 0

In this case, when μ ≥ 0, the general theorem on page 288 applies, which is reproduced below, for the *critical* strip …

$$\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0 \textit{ if} \text{ and } \textit{only if} \text{ (3) and (4) are true } \textit{for all } \mu \geq 0, \text{ for all } r > \mu \qquad (\S\S)$$

$$\int_0^\infty \{(A + C)\sin(yr) - (B + D)\cos(yr)\}\cos(\mu y)\,dy\,/\,y =$$
$$(\pi/2)\{\,\Psi(r + \mu) + \Psi(r - \mu)\,\} \qquad \ldots (3)$$

$$\int_0^\infty \{(D - B)\sin(yr) + (C - A)\cos(yr)\}\cos(\mu y)\,dy\,/\,y =$$
$$(\pi/2)\{\,\Omega(r + \mu) + \Omega(r - \mu)\,\} \qquad \ldots (4)$$

Here,

$$A = \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) + \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/\,2$$
$$B = \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) - \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/\,2\boldsymbol{i}$$
$$C = \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) + \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/\,2$$
$$D = \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) - \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/\,2\boldsymbol{i}$$

and it is to be noted that the expressions under the integral signs in (3) and (4), *modulated* by the cos(μy) term, produce a pair of *translates* on the right-hand side, where Ψ(*x*) and Ω(*x*) are defined on page 287. This is a minor miracle, in and of itself, deduced from residue theory and the Laplace transform, but something that is not readily obvious in going from μ = 0 to μ > 0, just using Fourier theory.

Now define the functions

$$\xi(x) = e^{-x(1-\alpha)} \sin(\varepsilon x) \text{ and } \eta(x) = e^{-x(1-\alpha)} \cos(\varepsilon x),$$

so that the Fourier *cosine* transform of f(y), defined on page 295, is just $\pi\xi(r)$. Similarly, the Fourier *sine* transform of g(y), also defined on page 295, is $\pi\eta(r)$.

Using the Bateman Manuscript Project, Tables of Integral Transforms [Volume 1, pages 7 and 63], we see that the Fourier *cosine* transform of f(y)cos(μy) now becomes

$$(\pi/2)(\xi(r + \mu) + \xi(r - \mu)),$$

and similarly, the Fourier *sine* transform of g(y)cos(μy) becomes

$$(\pi/2)(\eta(r + \mu) + \eta(r - \mu)).$$

Factoring in the constants $-2\varepsilon/\{(1-\alpha)^2 + \varepsilon^2\}$ for $\xi(x)$, and $+2(1-\alpha)/\{(1-\alpha)^2 + \varepsilon^2\}$ for $\eta(x)$, and *adding* the two expressions just above together, reproduces the *right*-hand side of (3) on page 301.

Thus, the arguments put forth on pages 294-6 apply, which means that for our theorem (§§) on page 301, *any* root $\omega = \alpha \pm i\varepsilon$ in the *critical* strip determined by (§§) , but *off* the *critical* line (where $\alpha = ½$), can never be verified. This statement is true, no matter our choice of $\mu \geq 0$ …

Finally, letting ε ---› 0 takes care of the case where we are in the *critical* interval (0,1), so that our result is now true for *all* $\varepsilon \geq 0$ …

UNDECIDABILITY AND THE RIEMANN HYPOTHESIS WHEN ζ(s) IS NON-ZERO

In this particular case, we'd like to discuss *undecidable* states when ζ(s) evaluates to a *non*-zero value, but before doing so, a few comments are in order concerning the undecidable nature of the root ω in the *critical* strip, where $\omega = \alpha \pm i\varepsilon$. For $\alpha \neq ½$, we said on page 296, that relative to the signal ***S***, it was *not* possible to distinguish between the α-line and the β-line, where $\beta = 1 - \alpha$, and $\zeta(\alpha \pm i\varepsilon) = 0$. But notice now that if ω is a root, then necessarily (and obviously) the α-line exists. Conversely, if the α-line exists (and it does for all $0 < \alpha < 1$ in the critical strip), then necessarily it *must* contain the ω-root, in order to *reach* a state of undecidability, provided $\alpha \neq ½$.

Thus, the α-line and ω imply each other, which means they are *equivalent* notions. And so, to say that `relative to ***S***, it is not possible to distinguish between the α-line and the β-line`, is equivalent to saying that `relative to ***S***, it is not possible to distinguish the α-root from the β-root`. This is the gist of our remarks at the top of page 296.

Now we'd like to consider, briefly, the case when we can reach a state of undecidability, where ζ(s) is *not* zero. Let $\zeta(\alpha + i\varepsilon) = a + ib$, where *not* both a and b are 0, so that by conjugation, we may write $\zeta(\alpha - i\varepsilon) = a - ib$. *Assume* now, $\zeta(\beta + i\varepsilon) = a + ib$, so that again $\zeta(\beta - i\varepsilon) = a - ib$.

Then in short form, $\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = \zeta(\beta \pm \boldsymbol{i}\varepsilon)$, so that we may enquire as to whether we can distinguish the points $\alpha \pm \boldsymbol{i}\varepsilon$ from the points $\beta \pm \boldsymbol{i}\varepsilon$, where $\beta = 1 - \alpha$, and $\alpha \neq ½$. Referring back to our general theorem on page 265 (see below), we note that the signals here [second term, *right*-hand side], in (1) and (2), are augmented by the offsets $(\pi/2)g(0)$, and $-(\pi\boldsymbol{i}/2)h(0)$, respectively; and they can be moved over to the *left*-hand side in (1) and (2), functioning as offsets to the integrations now. These offsets, it should be said, compute to πa and πb, as seen on the *right*-hand side of (1) and (2), whether we calculate them from the α-line *or* the β-line.

Thus, we can generate the signal, via the offset *and* the integration, from the α-line in (1) and (2), *and* we can also generate this *same* signal, via the integration from the β-line, using the methods outlined on pages 294-6. Since $\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = \zeta(\beta \pm \boldsymbol{i}\varepsilon)$, we may now ask 'which line the signal came from'. And the answer is, it is *not* possible to decide. And so, by our opening remarks in this section, it is *also* not possible to distinguish between the points $\alpha \pm \boldsymbol{i}\varepsilon$ on the α-line, and the *corresponding* points $\beta \pm \boldsymbol{i}\varepsilon$ on the β-line.

Theorem From Page 265 Reproduced Here

$$\int_0^\infty \{(A + C)\sin(yr) - (B + D)\cos(yr)\}dy / y =$$

$$(\pi/2)g(0) + \pi e^{-r(1-\alpha)}\{2(1-\alpha)\cos(\varepsilon r) - 2\varepsilon\sin(\varepsilon r)\} / \{(1-\alpha)^2 + \varepsilon^2\} \qquad (1)$$

$$\int_0^\infty \{(D - B)\sin(yr) + (C - A)\cos(yr)\}dy / y =$$

$$-(\pi\boldsymbol{i}/2)h(0) + \pi e^{-r(1-\alpha)}\{2(1-\alpha)\sin(\varepsilon r) + 2\varepsilon\cos(\varepsilon r)\} / \{(1-\alpha)^2 + \varepsilon^2\} \qquad (2)$$

The result is valid *for all* $r > 0$, for all $\varepsilon > 0$, and for all real $0 < \alpha < 1$. Here

$$A = \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) + \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/ 2$$
$$B = \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) - \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/ 2\boldsymbol{i}$$
$$C = \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) + \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/ 2$$
$$D = \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) - \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/ 2\boldsymbol{i}$$

$$g(s) = \zeta(\alpha - s + \boldsymbol{i}\varepsilon) + \zeta(\alpha - s - \boldsymbol{i}\varepsilon)$$
$$h(s) = \zeta(\alpha - s + \boldsymbol{i}\varepsilon) - \zeta(\alpha - s - \boldsymbol{i}\varepsilon)$$

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COMPLETING THE CIRCUIT

Suppose $\omega = \alpha \pm \boldsymbol{i}\varepsilon$ is a root of $\zeta(s)$ in the *critical* strip, where $0 < \alpha < 1$, and $\alpha \neq ½$. Then we have already shown that relative to the signal [$\boldsymbol{S}$], we can't distinguish between the α-line and the β-line [p. 296], where $\beta = 1 - \alpha$. Since ω and $\boldsymbol{S}$ are *equivalent* to one another, it means that relative to ω, it is *also not* possible to distinguish between the α-line and the β-line [p. 296].

But this means that relative to ω, it is not possible to distinguish between the α-root and the β-root. And thus, because ω and $\boldsymbol{S}$ are *equivalent* to one another, it must be the case that relative to $\boldsymbol{S}$, it is *also not* possible to distinguish between the α-root and the β-root. And this is what we said in the last section on page 302, only here, we are strictly using the $\omega \equiv \boldsymbol{S}$ equivalency to make our case.

And, in fact, these remarks apply when ω is a *pseudo*-root, where for any real a and b, it is the case that

$$\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = \zeta(\beta \pm \boldsymbol{i}\varepsilon) = a \pm ib$$

Finally, the logic also works in reverse order, so that ultimately we may conclude 'relative to $\boldsymbol{S}$, it is not possible to distinguish between the α-line and the β-line' … is *equivalent* to saying 'relative to $\boldsymbol{S}$, it is not possible to distinguish between the α-root and the β-root'. There is no difference between the two statements here, even though $\boldsymbol{S}$, itself, is an algebraic construct.

And indeed, it is the signal's frame of reference (or lens) through which we really *perceive* [or glean information about] the undecidable nature of the Riemann Hypothesis, in our reality. We shall refer to this as the *external* frame of reference, going forward; and when talking about the signal $\boldsymbol{S}$, shall tacitly assume that we are talking about the *pair* of signals in (1) and (2) on page 303, simultaneously, just as we have in the past.

Miscellaneous Topics

On The Self-Consistent Nature of $\zeta(s)$ Within The Context of Our Theorem (§§)

We have already put forward the notion that our duality (§§) on page 266 is an *existence* theorem, in the *critical* strip, and one could ask whether $\zeta(s)$ 'evaluates itself' properly within this context. After all, we believe from previous research that roots of $\zeta(s)$ that lie in the critical strip, but *off* the critical line, can never be verified, so what happens, then, if $\zeta(s)$ encounters such a root in the integrals below ?

Supposing $\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0$, we see from (§§) that (3) and (4) hold true, and in particular, as y ---› 0, the coefficients A, B, C, and D all ---› 0, inside the integrals, based on our supposition. That is to say, $\zeta(\alpha \pm \boldsymbol{i}\varepsilon)$ is evaluating itself correctly under the integral signs, when y ---› 0. And since our choice of $\varepsilon > 0$ is arbitrary, it must be the case that $\zeta(s)$ *always* evaluates itself correctly, inside the integrals, whenever it encounters a root, for *any* choice of $y \geq 0$, for which this is so.

Conversely, if (3) and (4) hold true, then our general theorem on page 265 [and reproduced on page 303] implies both g(0) and h(0) are identically *zero*, which means $\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0$. Thus, the coefficients A, B, C, and D are *forced* to 0 as y ---› 0 inside the integrals below; and again, the arbitrary nature of $\varepsilon > 0$ means $\zeta(s)$ *always* evaluates itself correctly, under the integral signs, whenever it encounters a root, for *any* choice of $y \geq 0$, for which this is so.

$$\int_0^\infty \{(A + C)\sin(yr) - (B + D)\cos(yr)\}dy / y =$$
$$\pi e^{-r(1-\alpha)}\{2(1-\alpha)\cos(\varepsilon r) - 2\varepsilon\sin(\varepsilon r)\} / \{(1-\alpha)^2 + \varepsilon^2\} \quad \ldots (3)$$

$$\int_0^\infty \{(D - B)\sin(yr) + (C - A)\cos(yr)\}dy / y =$$
$$\pi e^{-r(1-\alpha)}\{2(1-\alpha)\sin(\varepsilon r) + 2\varepsilon\cos(\varepsilon r)\} / \{(1-\alpha)^2 + \varepsilon^2\} \quad \ldots (4)$$

$\zeta(\alpha \pm \boldsymbol{i}\varepsilon) = 0$ *if* and *only if* (3) and (4) are true *for all* $r > 0$ (§§)

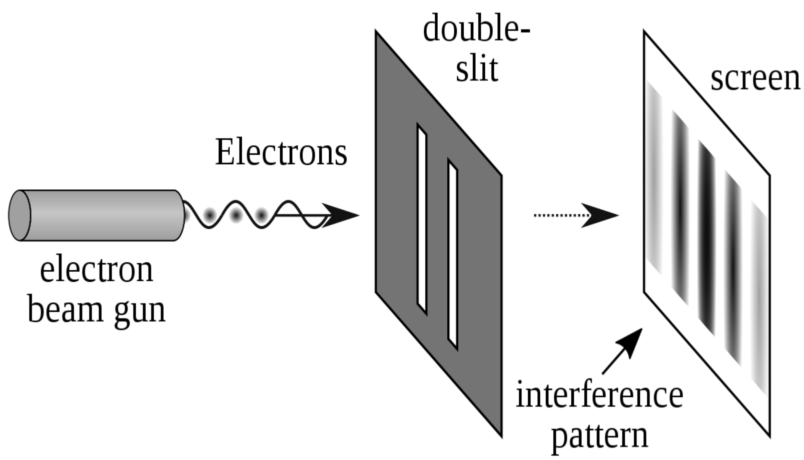
Here,

$$A = \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) + \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/ 2$$
$$B = \{\zeta(\alpha + iy - \boldsymbol{i}\varepsilon) - \zeta(\alpha - iy + \boldsymbol{i}\varepsilon)\}/ 2\boldsymbol{i}$$
$$C = \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) + \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/ 2$$
$$D = \{\zeta(\alpha + iy + \boldsymbol{i}\varepsilon) - \zeta(\alpha - iy - \boldsymbol{i}\varepsilon)\}/ 2\boldsymbol{i}\ .$$

Thus, not only is (§§) an *existence* theorem, but it is *also* a theorem in which ζ(s) always evaluates itself correctly inside the integrals, whenever it encounters a root. Knowing that this is so, now allows us to put forward an *undecidability* argument for the Riemann Hypothesis, grounded in reasonable mathematics, which the reader will find in the last research note [pp 293-304].

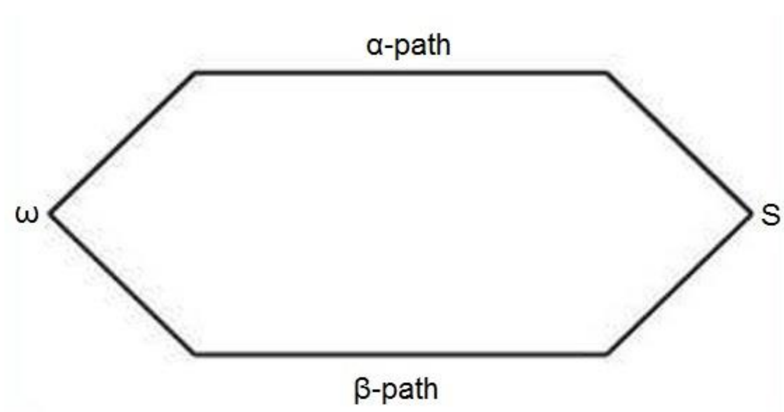
On The Parallels Between The Riemann Hypothesis and The Double Slit Experiment

The similarities between our approach to the Riemann Hypothesis and the Double Slit Experiment in physics are actually quite striking, and indeed, we alluded to this on page 297. In the Double Slit Experiment, electrons are fired at a pair of slits, one after the other, and those that 'make it through' create an interference pattern on the screen, as shown below …



One would think the pattern would be an image of the two slits themselves, but instead, what you get is an image that is *akin* to the diffraction pattern produced by a *plane* wave passing through these same slits. Thus, the electron has both 'wavelike' and 'particle-like' properties, at the same time.

In the diagram below, ω is a root of ζ(s) in the *critical* strip, and ***S*** is the signal, which can be generated via the α-line *or* the β-line, as per our discussions on pages 294-6. In fact, each line contains a copy of the root, according to Riemann's Functional Equation, so that we can think of ω as the electron *e* 'passing through either slit', and generating an interference pattern that parallels ***S***.



In the same way, then, that we glean information about the electron, by studying the interference pattern [$\mathscr{P}$] on the screen, we do, in a similar way, glean information about the undecidable nature of the root ω, by studying the signal $\mathcal{S}$. In both cases we are in the *external* frame of reference, and learning something about an *object* that carries its own *internal* frame of reference; namely *e* or ω.

And while it may be true that *relative* to ω, it is not possible to distinguish between the α-line and the β-line (and indeed, we have established this via the $\omega \equiv \mathcal{S}$ equivalency); and while it may *also* be true that relative to *e*, it is not possible to distinguish between slit A and slit B, we must always keep in mind that these are *internal* frames of reference. In our reality, however, we learn about the *ambiguous* nature of the electron, or the *undecidable* nature of ω through the *external* frame.

Having said this, it is hard to miss the striking parallels between the two as outlined above. Indeed, the parallels are so close that it makes one wonder if a quantumlike 'wave function' of sorts might well exist for the roots of ζ(s). It is an interesting question, to say the least, and one which we shall address in future notes, in this section.

On The Equivalency of Hidden Variables and Analytic Continuation

On page 293 we established an equivalency between the physical laws of creation [$\mathcal{L}$], and analytic continuation [$\mathcal{A}$]. In physics today, there seems to be a fairly strong interest in the notion of 'hidden variables' and what they might mean. Is there a connection here between the laws and these variables ?

In this note, we are not going to try and interpret the meaning of hidden variables [$\mathcal{H}$] per se, but rather, demonstrate how they might be incorporated into the field equations of general relativity, assuming they exist.

Let us begin, then, by supposing that an equivalency between $\mathcal{L}$ and $\mathcal{H}$ does, in fact, exist, so that the laws necessarily imply the existence of hidden variables, and conversely. Since we know from previous research that $\mathcal{L}$ and $\mathcal{A}$ are equivalent to one another [p 293], it now stands to reason that $\mathcal{H}$ and $\mathcal{A}$ are *also* equivalent to each other.

Thus, when searching for a candidate for $\mathcal{H}$, within the context of general relativity [GR], we are probably searching for some mathematical object that exists in the complex plane. We have such a candidate already … the Riemann zeta function [ζ(s)].

Now define the function

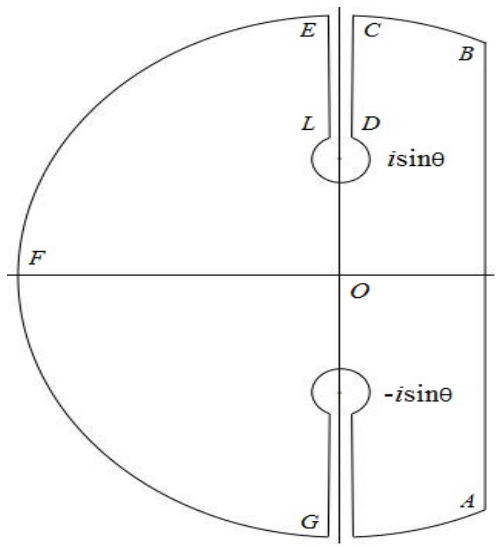
$$\mu(\alpha,s) \;=\; \lambda(s)\zeta(\alpha - s),$$

where λ(s) is the underlying density function associated with dark energy. Then from previous research in this essay, the field equations of general relativity now become …

$$G^{u,v} \;\approx\; \kappa\int_{\gamma} e^{sr}\,\mu(\alpha,s)g^{u,v}ds \qquad (^{**})$$

where γ denotes the Laplace inverse, according to some contour, and $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively. And here, we will only deal with the case where the gravitational tensor $g^{u,v}$ is *coupled* to $\mu(\alpha,s)$.

Now let us assume only one dark energy singularity exists at the center of the star, whose density function is $\lambda(s) \approx \sigma / s$, where σ is some constant. We'll take our contour to be the one shown on page 258, and reproduced here, *but* the branching points and their associated arms are to be omitted in the calculations. They no longer exist, as it were. Thus, we are traversing the large contour ABFA, to obtain the Laplace inverse along AB, itself.



For $\alpha > 1$, since $\zeta(s)$ has a *simple* pole at $s = 1$, $\zeta(\alpha - s)$ has a *simple* pole at $s = \alpha - 1 > 0$, and so we *exclude* it by choosing $0 < \text{Re}(AB) < \alpha - 1$. That leaves us with a *simple* pole at the origin O, associated with $\lambda(s)$. The result is …

$$G^{u,v} \approx \sigma g^{u,v}(0) \cdot \zeta(\alpha) \qquad (\dagger)$$

For $0 < \alpha < 1$, and $0 < \text{Re}(AB) < \alpha$, there are two *simple poles* … one at O and one at $\beta = \alpha - 1 < 0$. Calculating the residues here leads to …

$$G^{u,v} \approx \sigma\{g^{u,v}(0)\zeta(\alpha) + e^{r\beta} g^{u,v}(\beta) / \beta\}$$

Now since $\beta < 0$ is relatively small, we can replace $g^{u,v}(\beta)$ with $g^{u,v}(0)$, and in the *far-field* view, with $r > 0$ large, the *second* term in the bracketed expression will essentially vanish.

$$G^{u,v} \approx \sigma\{g^{u,v}(0)\zeta(\alpha) + e^{r\beta} g^{u,v}(0) / \beta\}$$

$$= \sigma g^{u,v}(0)\{\zeta(\alpha) + e^{r\beta} / \beta\}$$

This leaves us with the *same* expression as we see in (†) … the only difference being the *sign* of ζ(α).

For α > 1, ζ(α) is strictly *positive*, and for 0 < α < 1, ζ(α) is strictly *negative*. I would submit to the reader that ζ(α) is acting as a *hidden* variable here … behaving one way in the *critical* strip [the *substrate* level *below* our reality, where 0 < α < 1], and in another way in *our* reality, when α > 1.

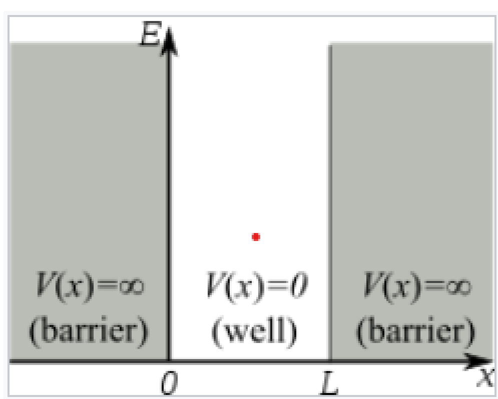
Thus, from (†) above, we might infer that the field equations of general relativity operate in *two* spaces simultaneously, where one is *positive* and the other *negative*, and that somehow these spaces are intimately connected to the Riemann zeta function [ζ(s)], itself.

A Quantumlike Wave Function For The Roots of ζ(s)

Previously, we talked about the parallels between the Riemann Hypothesis and the Double Slit Experiment, and surmised that there just might be a *quantumlike* wave function associated with the roots of ζ(s). After all, we learned in a previous research note [pp 293-304] that roots of ζ(s) that lie *off* the critical line, in the critical strip $\mathcal{C}$, can never be verified. Thus, there is no mathematical method $\mathcal{M}$ that can verify these roots, where verify means 'prove without ambiguity'.

So what, then, will $\mathcal{M}$ do if it encounters such a root, *away* from the critical line in $\mathcal{C}$? To answer this question, we have to better understand what the word 'encounter' really means within this context, and more specifically, from the perspective of a quantumlike wave function ψ, that might be associated with the roots of ζ(s) in $\mathcal{C}$.

In quantum mechanics, 'particle in a box theories' imagine a scenario where there is an infinite (repulsive) potential, running the length of two vertically parallel lines, separated by a distance of L units, say.

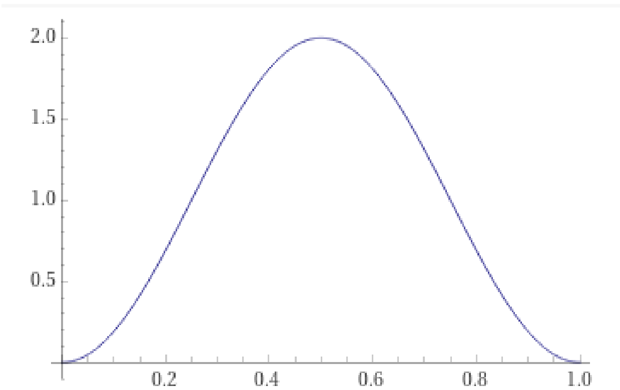


A particle, such as an electron *e*, bounces back and forth between these lines, and when solving Schrodinger's equation for this particular setup, a wave function Ψ emerges, that tells us something about the particle's whereabouts.

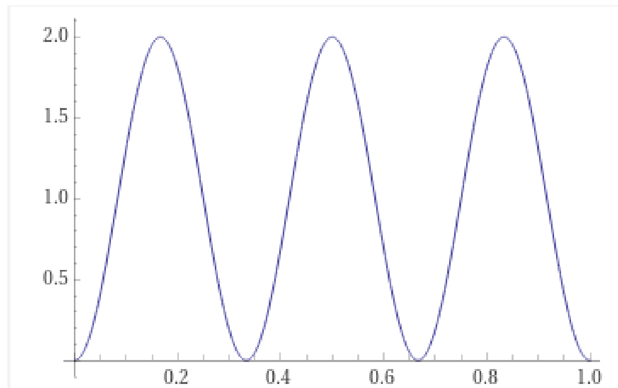
Specifically, a probability distribution can be calculated from Ψ, for infinitely many 'position states' that *e* might be in, at any given time. These probability states (n = 1, 2, 3, …) are tied directly to the *energy* of the electron, itself, and if L is equal to 1 in the diagram above, compute to

$$2\sin^2(n\pi x)\ ,$$

where $0 < x < 1$. The energy of *e*, it should be said, is proportional to n^2.



$n = 1$



$n = 3$

For our root ω that lies in the critical strip, but *off* the critical line, we now have a mechanism by which we can can interpret an 'encounter' by the mathematical method $\mathcal{M}$ with our root ω. A quantumlike wave function ψ, associated with the roots of $\zeta(s)$ in the critical strip, but *away* from the critical line, tells us that $\mathcal{M}$ can *never* speak *precisely* about the location of such a root in $\mathcal{C}$; but rather, just the *probability* of finding a root at a certain location.

And, roots with 'more energy' – perhaps meaning, further away from the *x*-axis in $\mathcal{C}$, will of course, be associated with higher values of the integer *n*.

The Spinning Spheres

Then I saw two perfectly shaped spheres, suspended in mid-air, as it were. Perhaps they were made from some type of glass, or maybe even light itself. But they were side-by-side, motionless and maybe six feet apart, as we might measure distance.

Inside each sphere, and suspended in mid-air, as it were, I could see small metal bars, or maybe they were made from light as well. I don't know. But they too were completely motionless.

Then the spheres began to spin, and as this happened, the metal bars in the first sphere began to talk or communicate with the bars in the second. I was astonished by what I was seeing, and could not understand it, at first.

Then the spheres stopped spinning, and when they did, the communication between the bars also ceased.

I was in the autoscopic view at the time, and the person explaining this effect to me, did so with perfect clarity. But alas, when I woke up, I could only remember what I saw, and nothing more.

Except for one small detail which I shall leave as a mystery for others to ponder … it is intimately connected to the behavior of the Riemann zeta function …

Can A Mathematical Method Show Roots of $\zeta(s)$ Lie Only On The Critical Line ?

In my opinion, the short answer to this question is no. For suppose such a method $\mathcal{M}$ exists, so that $\mathcal{M}$ can state, *unequivocally*, that *any* and *all* roots of $\zeta(s)$ lie *on* the critical line, in the critical strip $\mathcal{C}$, and nowhere else.

Then by *implication*, $\mathcal{M}$ is stating, *without* ambiguity, that *no* roots of $\zeta(s)$ lie in $\mathcal{C}^*$, where $\mathcal{C}^*$ is the critical strip *minus* the critical line. However, we already know there is *no* mathematical method $\mathcal{M}'$ that can verify roots in $\mathcal{C}^*$ [pp 293-304], so that $\mathcal{M}'$ cannot say, *without* ambiguity, that no roots of $\zeta(s)$ exist in $\mathcal{C}^*$ [see, for example, pp 297-99].

Thus, if we were to believe that $\mathcal{M}$ existed, we would, by implication, arrive at an impasse, in so much as $\mathcal{M}$ is now contradicting $\mathcal{M}'$.

But now, if $\mathcal{M}$ doesn't exist and $\mathcal{M}'$ is *inherently* ambiguous, it leaves us with no choice, but to conclude, that the Riemann Hypothesis has no answer … it too, is *inherently* ambiguous, and therefore undecidable …

```
 |                      |
 |  (no roots)          | ω
 |                      |
 |                      |
 |                      |
 |                      |
 A                      B
α-line      α ≠ ½     α-line
```

In scenario A, $\mathcal{M}'$ is a mathematical method that walks the *entire* α-line in $\mathcal{C}^*$, and concludes there are no roots. In scenario B, a root ω of $\zeta(s)$ exists somewhere on this α-line, but this time $\mathcal{M}'$ will *also* conclude there are no roots, since here they can't be verified. To be sure, $\mathcal{M}'$ will encounter the root ω, but it will *not* report it as such, according to our methodology.

So which of the two scenarios above should we believe, relative to $\mathcal{M}'$? The answer is … it is *not* possible to decide. Both A and B could be true, but we will never know which …

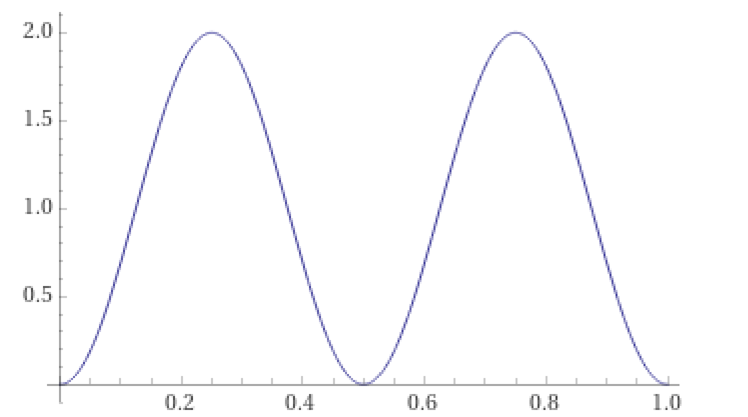
Close Encounters of The ω Kind – A Case Study in Classical Versus Quantumlike

Let $\mathcal{M}'$ be a mathematical method that walks the *entire* α-line in $\mathcal{C}^*$, according to scenario B above. Then in the *classical* sense, $\mathcal{M}'$ will encounter the root $\omega = \alpha \pm i\varepsilon$, for some $\varepsilon > 0$, however large, and report it as a *non*-root. Most likely, as a discontinuity of sorts, as I think about it now. For example, the function $f(x) = |\, x \,|$ has no meaningful derivative at the origin O, even though it is continuous over the real line. And this is, perhaps, the best way to think of what happens when $\mathcal{M}'$ encounters ω directly, as seen from a *classical* perspective.

From a *quantumlike* perspective, the roots of $\zeta(s)$ in $\mathcal{C}^*$ are endowed with a *wave* function ψ [see pages 309-10], from which probability states ($n = 1, 2, 3, \ldots$) can be calculated, that tell us how the roots on any given ε-line might be distributed.

In this case study, we will assume the 'energy level' associated with ω is $n = 2$, and so, without loss of generality, using the probability distribution mentioned on page 310, this is, for $0 < \alpha < 1$ …

$$P_2(\alpha) \;=\; 2\sin^2(2\pi\alpha)$$



$P_2(\alpha)$ for $0 < \alpha < 1$, when $n = 2$,
that is associated with the ε-line

Thus, when $\mathcal{M}'$ walks the *entire* α-line in $\mathcal{C}^*$, it will eventually encounter the ε-line, and now, even though ω may very well *exist* on this α-line, $\mathcal{M}'$ can only 'make contact' with ω via the probability distribution $P_2(\alpha)$. And here there are two probability zones, as shown in the diagram above, where the *expected* values are at $\alpha = ¼$ and $\alpha = ¾$. Notice too, that at $\alpha = ½$, $P_2(\alpha) = 0$.

Thus, from the perspective of $\mathcal{M}'$, ω lies somewhere on this ε-line, according to $P_2(\alpha)$, and that is the best we can do. There is no such thing as $\mathcal{M}'$ encountering ω *directly* anymore, as there was in the *classical* sense; rather, just probability distributions that $\mathcal{M}'$ must operate through. This, then, is the *quantumlike* perspective, but to be sure, whether we choose *classical* or *quantumlike*, roots in $\mathcal{C}^*$ can never be verified, where verify means 'prove without ambiguity'.

*****

It is important, I think, to summarize our findings at this point. In the *classical* sense, $\mathcal{M}'$ walks the *entire* α-line in $\mathcal{C}^*$, according to scenario B, and reports the root $\omega = \alpha \pm \boldsymbol{i}\varepsilon$ as a *non*-root. Here, ω exists by supposition.

In the *quantumlike* sense, the experiment is repeated, but this time … $\mathcal{M}'$ defers to the probability distribution $P_2(\alpha)$, and concludes that the root exists *somewhere* on the ε-line, but can say no more. Thus, $\mathcal{M}'$ cannot *verify* the existence of ω on the α-line, in particular, in $\mathcal{C}^*$.

But in this particular case [quantumlike], we have assumed the root $\omega = \alpha \pm i\varepsilon$ exists as a *static* object on the α-line, and might ask the question … could ω be *dynamic* ? That is to say, could ω be bouncing back and forth on the ε-line, between $\alpha = 0$ and $\alpha = 1$, just like the electron *e* does in the 'particle in a box' scenario, described on pages 309-10 ?

The answer is, it doesn't matter. For if ω is *static* and is assumed to exist at $\alpha \pm i\varepsilon$, $\mathcal{M}'$ will defer to $P_2(\alpha)$, and if ω is *dynamic*, $\mathcal{M}'$ will still defer to $P_2(\alpha)$, and conclude *either* way, that it cannot verify the existence of ω on the α-line … *without* ambiguity. The most $\mathcal{M}'$ can say is that ω exists somewhere on the ε-line, according to $P_2(\alpha)$, but nothing more.

But this is very good news for us, because it means we can adopt the 'particle in a box' model for the electron *e*, just as it is, and use it to describe the probabilistic layout of the roots of ζ(s) in $\mathcal{C}^*$, which, in turn, is inherited from the *wave* function ψ, itself. To be a little more specific, these probability states are derived from $|\psi|^2$.

In my opinion, the *quantumlike* approach has several advantages over the *classical* approach, when considering the roots of ζ(s) in $\mathcal{C}^*$, assuming they exist. First, we don't have to wrestle with the idea that $\mathcal{M}'$ will report a root as a *non*-root, should it encounter one. Second, the parallels between our approach to the Riemann Hypothesis and the Double Slit Experiment [pp 306-7] are so close, that we can't ignore the likely prospect of a wave function ψ for roots of ζ(s) in $\mathcal{C}^*$. Third, a resolution to the Riemann Hypothesis has never been found, and perhaps the *quantumlike* approach described here best explains why. And finally, what about those spinning spheres [p 311], which almost surely, are depicting some kind of quantumlike behavior in their own right … be it entanglement, action at a distance, consciousness, and so forth …

But to be sure, as we said before, whether we choose *classical* or *quantumlike*, roots of ζ(s) in $\mathcal{C}^*$ can never be verified, according to our methodology [pp 293-304], where verify means 'prove without ambiguity'. And this is really why the Riemann Hypothesis has no answer, in my opinion …

On The Nature of Static and Dynamic Roots For ζ(s) in $\mathcal{C}^*$

In the *static* case, $\omega = \alpha \pm i\varepsilon$ is a root of ζ(s) in $\mathcal{C}^*$, the critical strip *minus* the critical line, and in the *classical* sense, a mathematical method $\mathcal{M}'$ makes *direct* contact with ω, as it walks the *entire* α-line. Thus, we are to think of ω as *fixed* in location here, and indeed, this is how roots of ζ(s) are perceived to this day – even in $\mathcal{C}^*$, supposing, as we may, that they exist in this region.

Now we endow the roots of ζ(s) in $\mathcal{C}^*$ with a *wave* function ψ, so that we are transitioning away from *classical* mode, and into *quantumlike* mode, where the roots are now governed by probability distributions [$P_n(\alpha)$ n =1, 2, 3, …], inherited from ψ. How then, should we perceive ω, in this case ?

If we still want to think of ω as a *static* object on the α-line, so that $\omega = \alpha \pm i\varepsilon$, we can do so, but it's a bit like mixing parts of the *classical* and *quantum* worlds together. On the one hand, ω is *fixed* in location [a classical notion], but on the other … $\mathcal{M}'$ perceives its location through $P_n(\alpha)$ [a quantumlike notion].

As such, it is better to make a clean break from the *classical* world altogether, by perceiving ω as a *dynamic* object now, that bounces back and forth between $\alpha = 0$ and $\alpha = 1$, on any given ε-line. This, in turn, brings us into complete alignment with the 'particle in a box' scenario [$\mathcal{B}$] for the electron *e*, and here we know how ψ is to be calculated.

And indeed, there is no harm or foul in perceiving ω as *dynamic*, since $\mathcal{M}'$ always defers to $P_n(\alpha)$ anyway, in the quantum world. The most $\mathcal{M}'$ can say is that ω exists somewhere on the ε-line, according to $P_n(\alpha)$, but nothing more. And so, whether or not ω is bouncing back and forth is somewhat immaterial to $\mathcal{M}'$, but its *dynamic* nature *is* material if we are going to maintain consistency with $\mathcal{B}$.

What, perhaps, is more difficult here, is for us to *change* the way we think about the Riemann Hypothesis, and the nature of the roots of ζ(s) in $\mathcal{C}^*$. To leave behind the *classical* world and embrace the *quantum* world, as we've described it here, is not an easy thing to do. But if we can make the leap, we might find there is much to be gained by doing so; and indeed, we will have succeeded in confirming what we knew already; namely, that roots of ζ(s) in $\mathcal{C}^*$ can never be verified, where verify means 'prove without ambiguity'. Alas, the quantum world does this for us, automatically …

A Physical Interpretation For The Roots and Pseudo-Roots of ζ(s) in $\mathcal{C}^*$

Pseudo-roots were discussed on pages 302-4, but essentially, anything we've said so far in the Miscellaneous Topics section applies to *pseudo*-roots, as well. Recall that a pseudo-root is one for which

$$\zeta(\alpha \pm i\varepsilon) = \zeta(\beta \pm i\varepsilon) = a \pm ib \qquad (*)$$

where *a* and *b* are real, and $\beta = 1 - \alpha$, in the *critical* strip. Now suppose, in the diagram below, that P_A and P_B are two *exotic* particles, associated, say, with consciousness or even dark energy, that bounce back and forth between $\alpha = 0$ and $\alpha = 1$, as per the 'particle in a box' scenario [$\mathcal{B}$], described on pages 309-10.

```
|                      |
|                      |
| P_A --->     <--- P_B |  (ε-line)
|                      |
|                      |
|                      |

α = 0                 α = 1
```

P_A and P_B are identical in every respect, and possess attributes that are *conserved*, like, for example, the notion of *spin*. P_A begins to travel toward $\alpha = 1$ at the same time that P_B begins to travel toward $\alpha = 0$. They'll meet in the middle at $\alpha = ½$, *but* pass through each other in 'ghost-like' fashion, and continue on their merry way. The particles remain *entangled* in this 'bouncing back and forth' scenario, in so much as any attributes associated with them are always conserved. And the ε-line,

it should be said, corresponds to the energy of these particles … in the quantum sense, as we mentioned in earlier research notes here.

From a mathematical perspective, (*) is telling us that attributes exist and are *invariant*, but that they may be complex-valued (in this case the pseudo-root $\alpha \pm \boldsymbol{i}\varepsilon$ is to be associated with P_A and the pseudo-root $\beta \pm \boldsymbol{i}\varepsilon$ with P_B). In *our* reality, by looking at the *real* components *a* and *b*, or even something like $a^2 + b^2$, we infer at the *quantum* level [a physical notion], what these attributes might be. For example, an expression like $a^2 + b^2$ might denote *spin*, up to sign.

Thus, when the mathematician enquires about roots or even pseudo-roots in $\mathcal{C}^*$, the critical strip *minus* the critical line, it might be better if the physicist enquires about attributes associated with these exotic particles, in a lab setting, similar to $\mathcal{B}$. For it seems to me, as our understanding of $\zeta(s)$ evolves, that we are really dealing with a mathematical function here, that depicts [or characterizes] a reality at the *quantum* level, that we know very little about.

The research note on ‘Hidden Variables’ [pages 307-9] is also another example of how $\zeta(s)$, ‘acting through dark energy’, may have an effect or subtle influence on the equations of General Relativity. I’m sure there is no end to what’s possible, if you think about it …

A Poem

I once knew a man named Infinity,
Who stood tall in the fields with you and me.
I said, go away, you’re not welcome today,
But he laughed, and he said, I’m the Remedy …

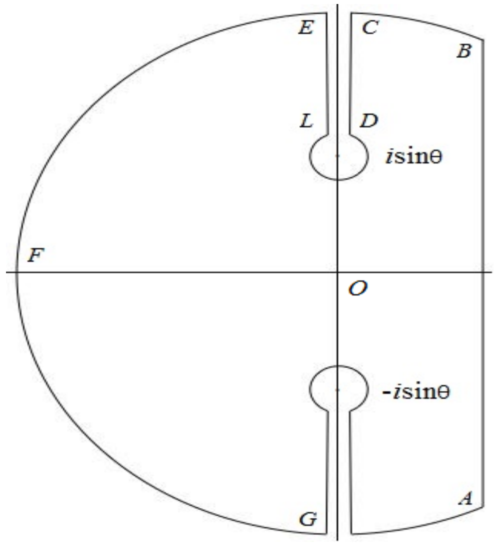
A Harmonic Representation For The Gravitational Tensor $g^{u,v}$ When $\lambda(s) \approx \sigma / s$

In the same way that we developed harmonic representations for the Riemann zeta function [$\zeta(s)$], we can do something similar for the gravitational tensor $g^{u,v}$, using a *simple* inverse density function [$\lambda(s)$] for dark energy [ξ]. In doing so, we can learn more about the constraints placed upon $g^{u,v}$ in the complex plane $\mathbb{C}$, and indeed, that is something we are going to do now, in this research note.

Recall from previous research [pp 197-243], that the dark energy *contour* integral takes the *exact* form

$$\kappa\sigma e^{r\cos(\theta)} \int_\gamma e^{sr} g^{u,v}(s + \cos(\theta)) / \sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\, ds \qquad (*)$$

before any averaging or smoothing on $g^{u,v}$ [see, for example, pages 235-6]. Here, the *physical* singularities for $\lambda(s)$ are at $(\pm1, 0)$, and in particular, $(+1, 0)$ is associated with (*), whilst $(-1, 0)$ is associated with its counterpart, where $s + \cos(\theta)$ maps to $s - \cos(\theta)$, and $r\cos(\theta)$ to $-r\cos(\theta)$, in the integral above. The contour γ is the one shown below, and we shall focus on (*), for the time being. And here, κ is equal to $1/2\pi i$, where $\boldsymbol{i}$ is imaginary and equal to $\sqrt{-1}$.



Since we are interested in a harmonic expression for $g^{u,v}$, we can delete the $\sigma e^{r\cos(\theta)}$ term, leaving us with

$$\kappa\int_\gamma e^{sr} g^{u,v}(s + \cos(\theta)) / \sqrt{(s - isin(\theta))(s + isin(\theta))}\, ds \qquad (\dagger)$$

All we need to do now, is repeat the procedure outlined on pages 258-60, realizing too, that in the case where $g^{u,v}$ is coupled *directly* to λ(s), it is *well-behaved* throughout the complex plane ℂ [the series of notes titled 'Schwarzschild, Perfect Stars and The Dark Energy Contour Integral' (pages 197-243), discusses this issue in several places, but in particular, see the summary on page 232].

Thus, when considering the integral (†) above, we don't have to worry about encountering any *poles* associated with $g^{u,v}$, like we did with ζ(s), when developing harmonic expressions; and so, the methods outlined on pages 258-60 will work just fine … for *any* physical singularity (α, 0) associated with λ(s), where α > 0. Indeed, we expect the same to be true, even as α ---› 0, but for the time being, we'll deal with the case where α = 1.

Following our methodology [pp 258-60], we see that the residue in (†) computes to $g^{u,v}(\alpha)$, where here α = 1, and θ ---› 0. At the same time, the arms associated with the *branching* points ***i***sin(θ) and **-*i***sin(θ) compute to …

$$-2\int_0^\infty e^{iyr} g^{u,v}(\alpha + iy) / \sqrt{(y)(y)}\, dy \qquad\qquad 2\int_0^\infty e^{-iyr} g^{u,v}(\alpha - iy) / \sqrt{(y)(y)}\, dy$$

isin(θ) arms　　　　　　　　　**-*i***sin(θ) arms

And, since it is the case that

$$g^{u,v}(\alpha) \;=\; -\kappa\int_{\boldsymbol{i}\sin(\theta)\text{ arms}} \;+\; -\kappa\int_{-\boldsymbol{i}\sin(\theta)\text{ arms}}$$

we see, in turn, that

$$g^{u,v}(\alpha) \;=\; 2\kappa\int_0^\infty \{e^{iyr}g^{u,v}(\alpha + iy) \;-\; e^{-iyr}g^{u,v}(\alpha - iy)\}dy \,/\, y$$

Collecting terms in the expression above leads to the following, which we'll label (§) …

$$g^{u,v}(\alpha) \;=\; 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(\alpha + iy) \;-\; g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) \;+\; g^{u,v}(\alpha - iy)]\}\; dy \,/\, y$$

And since $g^{u,v}(\alpha)$ is going to be *real-valued*, so must the integral above. One way to ensure this happens (since κ is imaginary), is by insisting that

$g^{u,v}(\alpha + iy) \;-\; g^{u,v}(\alpha - iy)$ be purely *imaginary* for all $y > 0$
$g^{u,v}(\alpha + iy) \;+\; g^{u,v}(\alpha - iy)$ be purely *real* for all $y > 0$

Such a constraint is true by default, for a function like $\zeta(s)$, because $\zeta(s^*) \;=\; \zeta(s)^*$, where * means 'take the conjugate of'. However, for the gravitational tensor $g^{u,v}$, we might not be so lucky. And finally, there was nothing particularly special about $\alpha = 1$. It could have been any *real* number > 0, matching the *physical* singularity for $\lambda(s)$ at $(\alpha, 0)$. Similarly, we could have chosen any $\alpha < 0$, by selecting the counterpart to (*), described on page 316. And lastly, letting α ---› 0 shouldn't cause any problems either, since we believe $g^{u,v}$ is *well-behaved* in the complex plane $\mathbb{C}$ … in the case where it is coupled directly to $\lambda(s)$.

As a quick confirmation, suppose (hypothetically) that one of the gravitational components $[g^{u,v}]$ is a *constant*, so that $g^{u,v} = c$. Then the bracketed term associated with cos(yr) in (§) above is 0, and the bracketed term associated with $\boldsymbol{i}$sin(yr) is $2c$. The integral in (§) now computes to (for *any* $r > 0$)

$$4\kappa\boldsymbol{i}c\int_0^\infty \sin(yr)\,dy \,/\, y \;=\; 4\kappa\boldsymbol{i}c \cdot \pi/2 \;=\; c$$

since κ is equal to $1/2\pi i$.

While there is no guarantee that the integral in (§) will always converge, it is still, in my opinion, an interesting expression that is worth pondering. Indeed, the fact that we can actually produce a harmonic expression for $g^{u,v}$ is progress in its own right, and could be of some value down the road, as we learn more about how to solve the field equations of general relativity, when $g^{u,v}$ is coupled *directly* to the underlying dark energy density function λ(s). To wit,

$$G^{u,v} \approx \kappa \int_{\gamma} e^{sr} \lambda(s) g^{u,v} ds$$

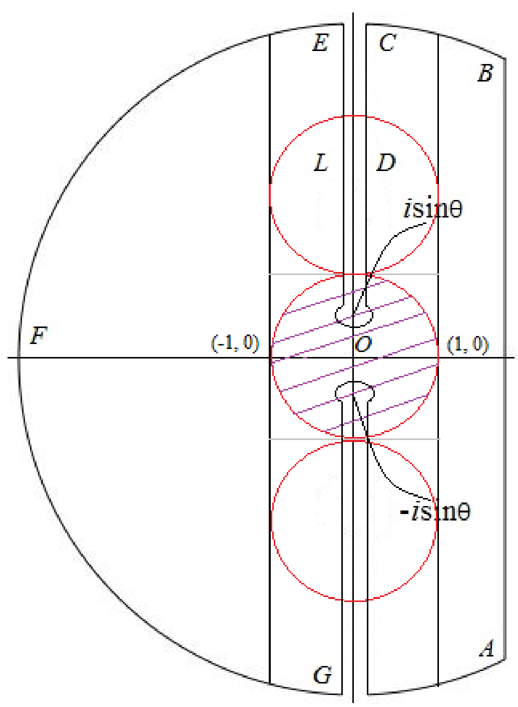
where here, $G^{u,v} = C^{u,v} - kT^{u,v}$, and $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively …

Revisiting Our Smoothing Methodology For $g^{u,v}$ When $\lambda(s) \approx \sigma / s$

On pages 235-6, we developed an *averaging* methodology for the gravitational tensor $g^{u,v}$ in the expression below, where the argument to $g^{u,v}$, which is really s + cos(θ), has been replaced by *s*.

$$\kappa\sigma e^{r\cos(\theta)} \int_{\gamma} e^{sr} g^{u,v}(s) / \sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\, ds \qquad (*)$$

Recalling the methodology, using the diagram below, a vertical line in the complex plane C runs from cos(θ) + *i*∞ to cos(θ) + *i*sin(θ), where θ is between 0 and π. The line will pass through *all* disks in the strip *above* the *x-axis*, except the one [$\mathfrak{D}$] centered at O, where the line will halt on $\partial\mathfrak{D}$ [the boundary of $\mathfrak{D}$], at some point on or *above* the *x-axis*. For example, the line associated with θ = 0 would halt at (1, 0) … whereas the line associated with θ = π/2 would halt at (0, *i*).



The same is true for all disks *below* the *x-axis*, where in this case, the lines from $\cos(\theta) - i\infty$ to $\cos(\theta) - i\sin(\theta)$ halt on $\partial\mathcal{D}$, on or *below* the *x-axis*. And, according to our methodology on pages 235-6, the value of $g^{u,v}$ at these points on $\partial\mathcal{D}$, whether above or below the *x-axis*, would all be mapped to $g^{u,v}(0)$; which, in turn, just happens to be the *average* value of $g^{u,v}$ over the whole of $\partial\mathcal{D}$.

Thus, when considering the integral (*) as $\theta \dashrightarrow 0$ [where γ is now $\partial\mathcal{D}$], the question of what to choose for an 'averaged or smoothed out' version of $g^{u,v}$ becomes clear. We select it in exactly the same way as we did for any other disk in the diagram above [namely $g^{u,v}(\xi)$, where ξ is the center of that disk]; but here, we infer what that must be by looking only at $\partial\mathcal{D}$.

After averaging $g^{u,v}$, where $\theta \dashrightarrow 0$ and γ is now $\partial\mathcal{D}$, this allows us to write (*) as

$$\kappa\sigma e^r g^{u,v}(0)\int_\gamma (e^{sr} / s)\, ds \;=\; \sigma e^r g^{u,v}(0)\,,$$

which is *exactly* what we would have gotten, had we calculated (*) just as it is, when $\theta \dashrightarrow 0$! The rest of the details can be found on pages 228-9, where the field equations of general relativity are developed for small θ, in the case of a *coupling* between $g^{u,v}$ and $\lambda(s)$.

A Metaphysical Interpretation For The Gravitational Tensor $g^{u,v}$ When $\lambda(s) \approx \sigma / s$

Let us recall our harmonic representation for $g^{u,v}$ on page 318, where α is any *real* number …

$$g^{u,v}(\alpha) \;=\; 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(\alpha + iy) \;-\; g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) \;+\; g^{u,v}(\alpha - iy)]\}\, dy / y$$

What is interesting to me about this equation … is that $g^{u,v}(\alpha)$, a *real-valued* expression, can be manufactured from a Fourier transform, ostensibly along the α-line in the complex plane $\mathbb{C}$. But the gravitational components $[g^{u,v}(\alpha)]$, as we perceive them, seem very *real*, and indeed, are the bedrock of General Relativity, itself. In fact, our whole understanding of *space* and *time* is intimately connected to these components.

Yet the *realness* of $g^{u,v}(\alpha)$ is actually built from its *imaginary* counterparts, by way of a Fourier transform in $\mathbb{C}$. How then, does something quite *real* to us $[g^{u,v}(\alpha)]$, come from something wholly *imaginary*, meaning of the *mind* or of the *collective consciousness*, in creation ? Could it be that $g^{u,v}(\alpha)$ is actually an *illusion*, built from artifacts by some external intelligence or agency, that we know nothing about ? Certainly the mathematical expression above has to make you wonder …

*****

The literature seems to suggest that for any *holomorphic* function g() in the complex plane $\mathbb{C}$, which is *real-valued* along the *x-axis*, it is always the case that $g(s^*) = g(s)^*$, where * means 'take the

conjugate of '. If this is so, then the same property would also apply to $g^{u,v}$, when it is coupled *directly* to the dark energy density function $\lambda(s)$ … since here we believe $g^{u,v}$ is *well-behaved* throughout C.

*****

A Proposed Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$

In the case where we are dealing with a *simple* inverse density function [$\lambda(s)$], associated with dark energy [ξ], with *physical* singularities at the origin O of the star, and at $(\pm 1, 0)$, we are going to provide a formulation for the field equations of General Relativity, in the case of a *coupling* between $g^{u,v}$ and $\lambda(s)$. This formulation will cover off the *entire* two-dimensional plane, so that we are no longer restricted to small θ. And here, we'll deal with θ between 0 and π, due to symmetry.

Let us recall in the *uncoupled* case [pp 221-7], that the field equations take the form

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))]g^{u,v} \qquad (*)$$

where $G^{u,v} = C^{u,v} - kT^{u,v}$, and $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively. And, it should be said, both $G^{u,v}$ and $g^{u,v}$ are functions of (r, θ) in (*) above.

In the *coupled* case, the Laplace inverse takes the *exact* form $\sigma g^{u,v}(0)$ [pp 228-9] for the singularity associated with $\lambda(s)$ at O, and at $(\pm 1, 0)$, the Laplace inverse takes the *exact* form [page 316] …

$$\sigma e^{r} g^{u,v}(+1) \text{ at } (+1, 0) \ldots \text{ as } \theta \dashrightarrow 0 \quad \text{and} \quad \sigma e^{-r} g^{u,v}(-1) \text{ at } (-1, 0) \ldots \text{ as } \theta \dashrightarrow 0$$

However, because of symmetry, we want the *same* to be true as $\theta \dashrightarrow \pi$, and this can be arranged by rewriting the above as

$$\sigma e^{r\cos(\theta)} g^{u,v}(\cos(\theta)) \ldots \text{ as } \theta \dashrightarrow 0 \text{ or } \pi \quad \text{and} \quad \sigma e^{-r\cos(\theta)} g^{u,v}(-\cos(\theta)) \ldots \text{ as } \theta \dashrightarrow 0 \text{ or } \pi$$

Combining all three, and factoring in the $J_0(r\sin(\theta))$ term to achieve *symbolic* agreement with (*), we then have, for the coupled case, with θ between 0 and π …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + \{e^{r\cos(\theta)} g^{u,v}(\cos(\theta)) + e^{-r\cos(\theta)} g^{u,v}(-\cos(\theta))\} J_0(r\sin(\theta))] \qquad (\dagger)$$

Now let's compare (†) with (*) in three separate cases. First, let $\theta \dashrightarrow 0$. Then (*) becomes

$$G^{u,v} \approx \sigma[1 + 2\cosh(r)]g^{u,v} \qquad \text{(uncoupled)}$$

whilst (†) becomes

$$G^{u,v} \approx \sigma[g^{u,v}(0) + \{e^{r} g^{u,v}(+1) + e^{-r} g^{u,v}(-1)\}] \qquad \text{(coupled)}$$

Notice that the *coupled* form agrees *symbolically* with the *uncoupled* form *if* we *approximate* both $g^{u,v}(+1)$ and $g^{u,v}(-1)$ with $g^{u,v}(0)$. This is the smoothing effect we spoke about in earlier notes.

Now let $\theta \dashrightarrow \pi$. Then (*) becomes

$$G^{u,v} \approx \sigma[1 + 2\cosh(r)]g^{u,v}$$

whilst (†) becomes

$$G^{u,v} \approx \sigma[g^{u,v}(0) + \{e^{r}g^{u,v}(+1) + e^{-r}g^{u,v}(-1)\}]$$

and the remarks above apply again, *identically* so. That is to say, there is *no* difference between the cases $\theta \dashrightarrow 0$ and $\theta \dashrightarrow \pi$, as it should be, where symmetry is present in these equations.

Finally, let $\theta \dashrightarrow \pi/2$. Then (*) becomes

$$G^{u,v} \approx \sigma[1 + 2J_0(r)]g^{u,v}$$

whilst (†) becomes

$$G^{u,v} \approx \sigma[1 + 2J_0(r)]g^{u,v}(0)$$

The two agree *symbolically*, but … the interesting difference here is that in the case of a coupling, the term $g^{u,v}(0)$ shows up, which makes sense. The further away we are from the *physical* singularities $(\pm 1, 0)$ associated with $\lambda(s)$, the *less* of an influence they should have on the field equations of general relativity, when there is a coupling. These singularities $[(\pm 1, 0)]$ would have their greatest impact when θ is close to 0 or π.

There is always a tradeoff when dealing with equations like (†) above. For example, when θ is 0 or π, we could attempt to solve (†) as is, but it would probably require numerical analysis and a computer. Even so, if it could be done, it would tell us not only how $g^{u,v}$ is behaving at O, but *also* at $(\pm 1, 0)$. Whereas if we smooth (†), by replacing $g^{u,v}(\pm 1)$ with $g^{u,v}(0)$, the equations are much simpler to solve, but the behavior of $g^{u,v}$ at $(\pm 1, 0)$ would not be as accurate as it would be at O.

Finally, it should be said that $g^{u,v}(+1)$ means 'look at the value of $g^{u,v}$ one unit to the *east* of O', whilst $g^{u,v}(-1)$ means 'look at the value of $g^{u,v}$ one unit to the *west* of O'. And, if we wanted to scale up to three dimensions from two, it wouldn't be a problem. The research notes on pages 197-243 give several examples of how to do this, and this note, in particular, tells us what sort of formulation we might be searching for, in the case of a *coupling*, when looking at (†).

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We can simplify (†) by noting that $g^{u,v}(\cos(\theta))$ and $g^{u,v}(-\cos(\theta))$ are going to be the same value, because of symmetry. Thus, (†) becomes

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta))] \qquad (\S)$$

where θ is between 0 and π, say. This is probably a fairly decent estimate for the field equations of General Relativity, where there are *physical* singularities at O and at (± 1, 0), associated with $\lambda(s)$, in the case of a *coupling*. The solution to (§) will be $g^{u,v}(r, \theta)$, but unlike the *uncoupled* case, we expect the gravitational tensor $g^{u,v}$ to be *well-behaved* everywhere here; that is, for *all* $r \geq 0$, and for *all* θ.

As to the dark energy [ξ], in this case, it is made up of two parts within (§). First, the term σ that is associated with $g^{u,v}(0)$, and second, the term $2\sigma\cosh(r\cos(\theta))J_0(r\sin(\theta))$ associated with $g^{u,v}(\cos(\theta))$. In total, this computes to

$$\xi = \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))] ,$$

which *exactly* matches what we see in (*), in the *uncoupled* case.

Scaling Up When $\lambda(s) \approx \sigma / s$

If we decide to scale up to *five* physical singularities, by adding the north and south points (0, 1) and (0, -1), the field equations of General Relativity become [pp 225-6], in the *uncoupled* case …

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta)) + 2\cosh(r\sin(\theta))J_0(r\cos(\theta))]g^{u,v}$$

In the *coupled* scenario, we would write, where θ is between 0 and π, say …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta)) + 2\cosh(r\sin(\theta))J_0(r\cos(\theta))g^{u,v}(\sin(\theta))]$$

Thus, the argument to $g^{u,v}$ is always anchored to the argument in the cosh( ) term … save for the radius r, and so in *three* dimensions [pp 225-6], the *coupled* equations would look like …

$$\begin{aligned} G^{u,v} \approx \sigma[g^{u,v}(0) &+ 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) + \\ &2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) + \\ &2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] . \end{aligned}$$

And here, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, and the *physical* singularities associated with $\lambda(s)$ are at the origin O, and at

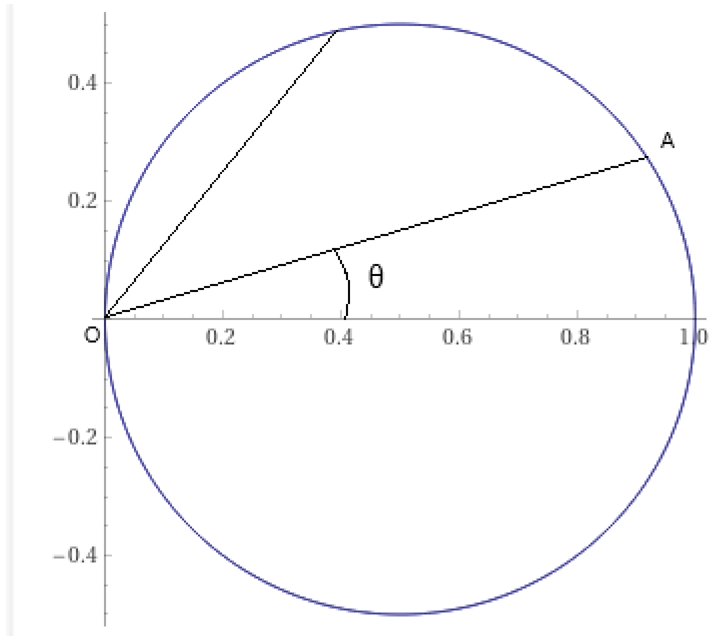
$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) .$$

A Physical Interpretation For $g^{u,v}(\cos(\theta))$ In The Coupled Case When $\lambda(s) \approx \sigma / s$

In this note, we want to better understand the meaning of $g^{u,v}(\cos(\theta))$, within the context of our form for the field equations of General Relativity, when there is a *coupling*. Recall that our form is

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta))] , \quad (§)$$

where the *physical* singularities associated with $\lambda(s)$ are at the origin O and at $(\pm1, 0)$. Now the diagram below is a *polar* plot of $\cos(\theta)$, as θ ranges between 0 and $\pi/2$. Thus, when $\theta = 0$, $\cos(\theta)$ is



the length of the segment running from O to $(1, 0)$. Similarly, at an angle of θ, $\cos(\theta)$ is the length of the segment OA. These segments become shorter and shorter, until finally at $\theta = \pi/2$, the length of the segment is *zero*, which of course, is $\cos(\pi/2)$.

When $\theta = 0$, the term $g^{u,v}(1)$ shows up in our equation (§), because the form is *exact* here, and $\cos(0) = 1$. And this makes sense, because the singularities at $(\pm1, 0)$ have their *greatest* impact on the field equations when θ is close to 0 or π [see pp 272-73, where the evolution of the quantumlike dark energy scalar field is shown in different frames, as θ moves from 45^{o} to 90^{o} in a 3D setting. Note that the second set of spikes *only* emerge when we get *very* close to 90^{o}].

Now let θ start to move northward in our diagram above. Intuitively, one would guess that the field equations should have the same form, but that the argument to $g^{u,v}$ should *weaken*, because the *physical* singularities associated with $\lambda(s)$ at $(\pm1, 0)$ … do not have the same impact on the gravitational tensor, itself. And indeed, $g^{u,v}(\cos(\theta))$ is a *measure* of how much the gravitational tensor is influenced by the singularities at $(\pm1, 0)$, along the radial line $\boldsymbol{\ell}_{\theta}$.

Finally, at $\theta = \pi/2$, the physical singularities at $(\pm1, 0)$ have all but lost their influence, and only the term $g^{u,v}(0)$ shows up in (§), which is what we might expect.

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Although we have decided to parametrize $g^{u,v}$ via $\cos(\theta)$, there are other options. However, they would most certainly be more complex, making (§) even more difficult to solve than it is already. I'm not sure we should go down that path, for it seems to me that what we want here, is the simplest algebraic and geometric formulation possible, without giving up too much in the process.

For example, if instead of (§) we were to choose

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))]g^{u,v}(0)$$

as our form, things get much simpler, but we most certainly will lose information about how $g^{u,v}$ behaves near the *physical* singularities (±1, 0), when solving the equation above. As such, I do believe (§) is our best option, and one which, someday, might yield a solution of interest …

Finally, for small r ---› 0, (§) becomes …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2g^{u,v}(\cos(\theta))] ,$$

so that the angle of approach still matters. Approaching the origin along $\theta = 0$ produces a different result than, for example, approaching the origin along $\theta = \pi/2$.

A Heuristic Derivation For The Field Equations Near $\pi/2$ When $\lambda(s) \approx \sigma / s$

Recall, from page 316, our expression for the dark energy *contour* integral … in the case of a coupling, for the *physical* singularity at (+1, 0) associated with $\lambda(s)$ …

$$\kappa\sigma e^{r\cos(\theta)}\int_\gamma e^{sr}g^{u,v}(s + \cos(\theta)) \,/\sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\, ds \qquad (*)$$

Now let θ ---› π/2, so that (*) becomes

$$\kappa\sigma\int_\gamma e^{sr}g^{u,v}(s) \,/\sqrt{(s - i)(s + i)}\, ds \qquad (\dagger)$$

If we compute (†) along the arms associated with the *branching* points (0, ±i) … and follow the methods outlined on pages 316-19, we arrive at the following expression for our Laplace inverse …

$$2\kappa\sigma\int_1^\infty \{\cos(yr)[g^{u,v}(iy) - g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) + g^{u,v}(-iy)]\}\, dy \,/\sqrt{y^2 - 1} \qquad \ldots (1)$$

Here, we can't equate this to anything, because the branching points don't 'merge into a *pole*' like they did when θ ---› 0. But the expression above *is* 'similar to' our general harmonic expression on page 318, and reproduced here, when $\alpha = 0$ …

$$g^{u,v}(0) \;=\; 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(iy) \;-\; g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) \;+\; g^{u,v}(-iy)]\}\,dy\,/\,y \qquad \ldots\ (2)$$

Now since (1) involves a *square root* term that resembles 'Bessel-like' behavior, when integrating from 1 to ∞, we might surmise, when looking at (2), that (1) might be 'similar to' $\sigma g^{u,v}(0)J_0(r)$.

We can confirm this in a very simple case, where the gravitational component is a *constant* … say $g^{u,v} = c$. Then (1) computes to,

$$4\kappa\boldsymbol{i}\sigma c\int_1^\infty \sin(yr)\,dy\,/\sqrt{y^2-1} \;=\; \sigma c\cdot 2/\pi\int_1^\infty \sin(yr)\,dy\,/\sqrt{y^2-1} \;=\; \sigma g^{u,v}(0)J_0(r)$$

after 'aligning symbolically' with (2), since κ is equal to $1/2\pi i$.

Repeating the exercise, using the counterpart to (*) [p 316] for the *physical* singularity at (–1, 0), would lead to the same result as above, so that in total we would have $2\sigma g^{u,v}(0)J_0(r)$. Combined with the physical singularity at the origin O, would lead to

$$G^{u,v} \;\approx\; \sigma[1 + 2J_0(r)]g^{u,v}(0)$$

for the field equations, when $\theta \dashrightarrow \pi/2$, in this case.

Thus, from a *heuristic* point of view, we get a result which is consistent with what we found on pages 321-3. But while there can be no assurance that it is *exact*, except in very special cases, the result is encouraging, and when we combine it with previous research notes, only bolsters our view that

$$G^{u,v} \;\approx\; \sigma[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta))]$$

is probably a fairly decent estimate for the field equations of General Relativity, where there are *physical* singularities at O and at (±1, 0), associated with $\lambda(s)$, in the case of a *coupling*.

A *Less* Heuristic Derivation For The Field Equations Near $\pi/2$ When $\lambda(s) \approx \sigma / s$

Let us begin by introducing the following expression, from the previous research note, but *modified* now to suit our purposes …

$$2\kappa\int_\varepsilon^\infty \{\cos(yr)[g^{u,v}(iy) \;-\; g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) \;+\; g^{u,v}(-iy)]\}\,dy\,/\sqrt{y^2-\varepsilon^2} \qquad \ldots\ (1)$$

Here ε is any *real* number between 0 and 1. Notice that when $\varepsilon \dashrightarrow 0$, we recover the harmonic expression on page 318 when $\alpha = 0$ …

$$g^{u,v}(0) \;=\; 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(iy) \;-\; g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) \;+\; g^{u,v}(-iy)]\}\,dy\,/\,y \qquad \ldots (2)$$

Now in (1) there are two pieces to consider; first, the cos(yr) term paired with the denominator $\sqrt{y^2-\varepsilon^2}$, and then, the sin(yr) term, also paired with $\sqrt{y^2-\varepsilon^2}$. From Bessel theory, it is known that

$$2/\pi \int_1^\infty \sin(yr)\,dy\,/\sqrt{y^2-1} \;=\; J_0(r)\,,$$

so that upon mapping r to rε [page 227], this becomes …

$$2/\pi \int_\varepsilon^\infty \sin(yr)\,dy\,/\sqrt{y^2-\varepsilon^2} \;=\; J_0(r\varepsilon)\,.$$

Similarly,

$$-\,2/\pi \int_\varepsilon^\infty \cos(yr)\,dy\,/\sqrt{y^2-\varepsilon^2} \;=\; Y_0(r\varepsilon)\,.$$

Here J_0 and Y_0 are Bessel functions of the *first* and *second* kind, with one singular difference. As ε tends to 0 (ε ---› 0), J_0 converges to a meaningful value, but Y_0 does *not* (in fact, $Y_0(0) = -\infty$).

Thus, when considering (1), as ε ---› 0, we're going to recover (2), but now, since $g^{u,v}$ is *well-behaved* because it is *coupled* to the underlying dark energy density function λ(s), we have to rule out the presence of Y_0 when evaluating (1) for *small* ε. In other words, $g^{u,v}(0)$ is a *finite* value, which forces this conclusion.

That leaves us with a combination of $g^{u,v}(.)$ and J_0 here, as the dominant factors, and we'll *presume* this can be written as $g^{u,v}(.)\,J_0(r\varepsilon)$, where the argument to $g^{u,v}$ either *does* or *doesn't* depend on *small* ε.

If it *does*, then the same should be true as ε ---› 1. The argument to $g^{u,v}$ should *still* depend on ε, yet as we have seen in the last research note, ε approaching 1 is equivalent to evaluating the field equations as θ tends to π/2. And here the Laplace inverse computes to

$$2\kappa\int_1^\infty \{\cos(yr)[g^{u,v}(iy) \;-\; g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) \;+\; g^{u,v}(-iy)]\}\,dy\,/\sqrt{y^2-1}$$

where σ has been normalized to 1. Setting the gravitational component to a *constant* now, so that $g^{u,v} = c$, leads to the following, for the expression just above [see page 326] …

$$4\kappa \boldsymbol{i} c \int_1^\infty \sin(yr)\,dy \,/\sqrt{y^2-1} \;=\; c\cdot 2/\pi \int_1^\infty \sin(yr)\,dy\,/\sqrt{y^2-1} \;=\; g^{u,v}(0)J_0(r)\,,$$

and we see here, by *arbitrarily* choosing $g^{u,v}(0) = c$, that the argument to $g^{u,v}$ *needn't* change at all. It is still *zero*, just as it was when ε ---› 0.

Thus, for *small* ε, there is no need to presume the argument to $g^{u,v}$ should depend on ε, in which case we are left with the formulation $g^{u,v}(0)\,J_0(r\varepsilon)$ … for *any* choice of ε between 0 and 1, when evaluating (1) above. In particular, when ε = 1, this is $g^{u,v}(0)J_0(r)$, which is what we concluded in the previous research note.

OTHER CONSIDERATIONS

In considering (1) again,

$$2\kappa\int_\varepsilon^\infty \{\cos(yr)[g^{u,v}(iy) \;-\; g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) \;+\; g^{u,v}(-iy)]\}\,dy\,/\sqrt{y^2-\varepsilon^2} \qquad \ldots (1)$$

we could suppose the integral does, in fact, contain terms in $g^{u,v}(0)$, $J_0(r\varepsilon)$, *and* $r\varepsilon\cdot Y_0(r\varepsilon)$, after it is evaluated. For example, something like

$$g^{u,v}(0)[J_0(r\varepsilon) \;+\; r\varepsilon\cdot Y_0(r\varepsilon)]$$

might be feasible, since the expression just above *does* converge, as r or ε ---› 0. In fact, $J_0(r\varepsilon)$ would tend to 1, whilst $r\varepsilon\cdot Y_0(r\varepsilon)$ would tend to 0.

And so, as ε ---› 1, we would have, for the *coupled* case,

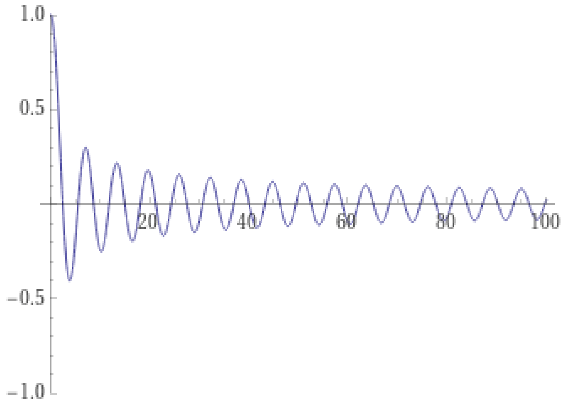
$$G^{u,v} \;\approx\; \sigma[1 + 2J_0(r) + 2rY_0(r)]g^{u,v}(0)$$

as our form for the field equations, as θ tends to π/2, presuming of course, that the argument to $g^{u,v}$ does *not* depend on ε.

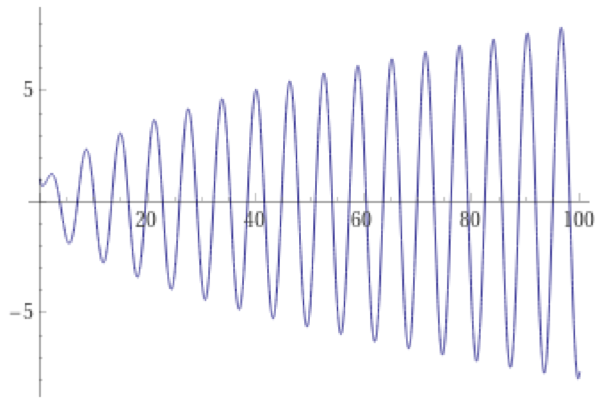
However, the form above would *not* be consistent with the *uncoupled* case below, as θ ---› π/2 …

$$G^{u,v} \;\approx\; \sigma[1 + 2J_0(r)]g^{u,v}\,,$$

so that our *perception* of dark energy [the bracketed expression, which we'll label ξ], would differ between the two. This, to me, is an *untenable* outcome. Our perception of ξ should *always* agree, when comparing the uncoupled and coupled forms, no matter the choice of θ.



Plot of $J_0(r)$ at $\theta = \pi/2$



Plot of $J_0(r) + rY_0(r)$ at $\theta = \pi/2$

Thus, ruling out Y_0 seems to be the only plausible path for us, when evaluating (1), and this takes us back to the top of page 328, where we concluded that $g^{u,v}(0)J_0(r\varepsilon)$ is the most likely form for (1), when ε is between 0 and 1.

In turn, this means the expression

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta))]$$

is probably the most *tenable* form, in the *coupled* case, where the *physical* singularities associated with $\lambda(s)$ are at the origin O and at $(\pm 1, 0)$. For here, we have complete consistency with this research note, and the previous one, when $\theta \dashrightarrow \pi/2$, *and* we have complete consistency with its *uncoupled* counterpart [page 323]

$$G^{u,v} \approx \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))]g^{u,v} ,$$

when looking at dark energy.

A *Less* Heuristic Derivation For The Field Equations For Any θ When $\lambda(s) \approx \sigma / s$

Let us bring back our original harmonic expression, but modified again to suit our purposes, which we'll label (1) [and throughout, we'll assume θ is between 0 and π] …

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\alpha + iy) - g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) + g^{u,v}(\alpha - iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

Notice that when $\varepsilon \dashrightarrow 0$, we recover $g^{u,v}(\alpha)$, where α is any real number [page 318], and that when $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, (1) is actually the Laplace inverse of our dark energy *contour* integral (*), in the case of a coupling, for the *physical* singularity at (+1, 0) associated with $\lambda(s)$. Here we are normalizing σ to 1, for the time being, and omitting the $e^{r\cos(\theta)}$ term, found in (*) below.

$$\kappa\sigma e^{r\cos(\theta)} \int_\gamma e^{sr} g^{u,v}(s + \cos(\theta)) \,/ \sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\; ds \qquad (*)$$

Thus from our two previous research notes, we may conclude that the likely form for (1) is going to be $g^{u,v}(\alpha) \cdot J_0(r\varepsilon)$, so that after factoring in the other *physical* singularity at (–1, 0), and also at O, the field equations become, in the case of a *coupling* …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] \qquad (\dagger)$$

Here, we've brought back both the σ term and the cosh() term, which are associated with (*) and its counterpart on page 316 [for the counterpart, $s + \cos(\theta)$ maps to $s - \cos(\theta)$, and $r\cos(\theta)$ to $-r\cos(\theta)$, in the integral above]. Notice too, that $g^{u,v}(\alpha) = g^{u,v}(-\alpha)$, because of symmetry, so that (†) can be written in the way that it is.

But, in fact, (†) is actually the form

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta))]\,, \qquad (\S)$$

which we proposed earlier, before doing any heuristic analysis ! See pages 321-23 here, for the details.

Thus, by taking a heuristical approach and doing some comparative functional analysis, we see that the form we originally proposed … is probably correct. To me, at least, this is a very encouraging milestone, in our journey to better understand how gravity behaves in the presence of dark energy singularities, where $g^{u,v}$ is *coupled* directly to the underlying dark energy density function λ(s).

Some More Validation of Our Harmonic Representation For The Gravitational Tensor $g^{u,v}$

Let us recall the harmonic representation on page 318 for $g^{u,v}$, where $\alpha = 0$ and $\kappa = 1/2\pi i$ … …

$$g^{u,v}(0) = 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(iy) - g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) + g^{u,v}(-iy)]\}\,dy\,/\,y \qquad \ldots (*)$$

Here, we are going to let $g^{u,v}(s) = \zeta(\beta - s)$, where ζ(.) is the Riemann zeta function, and β is now any *real* number > 1. In this region, ζ(s) is well-behaved, and has no *poles*. Now since it is the case that $\zeta(s^*) = \zeta(s)^*$, where * means 'take the conjugate of', (*) can be rewritten as

$$2/\pi \int_0^\infty \{A\sin(yr) - B\cos(yr)\}dy\,/\,y \qquad (\dagger)$$

where here …

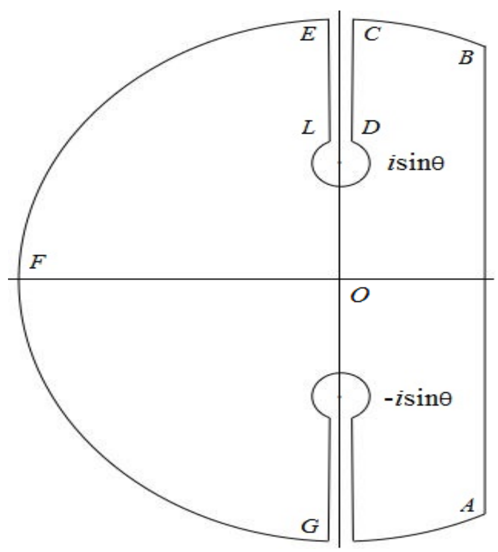
$$A = \{\zeta(\beta + iy) + \zeta(\beta - iy)\}/\,2$$

$$B = \{\zeta(\beta + iy) - \zeta(\beta - iy)\}/\,2i$$

But from page 260, (†) is actually $\zeta(\beta)$, which just happens to be $g^{u,v}(0)$, in this case. In general, the statement

$$g^{u,v}(\alpha) = 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(\alpha + iy) - g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) + g^{u,v}(\alpha - iy)]\}\,dy\,/\,y$$

will *always* be true if $g^{u,v}(.)$ is *well-behaved* in the complex plane $\mathbb{C}$ … or at least in the region enveloped by the contour γ, as shown below [see pages 316-19, where we tacitly assume the dark energy contour integral actually exists]. However, for this case involving $\zeta(\beta - s)$, we must have that $0 < \mathrm{Re}(AB) < \beta - 1$, since $\zeta(s)$ has a *simple* pole at $s = 1$. In doing so, we *exclude* the pole when traversing γ, as we shall see below …



Now let's take a closer look 'under the hood', as to why things really work here. To begin with, then, suppose the *physical* singularities associated with $\lambda(s)$ were located at $(\pm\delta, 0)$, where δ is any real number > 0.

Then our dark energy *contour* integral becomes, for the physical singularity at $(\delta, 0)$ …

$$\kappa\sigma e^{r\delta\cos(\theta)}\int_\gamma e^{sr} g^{u,v}(s + \delta\cos(\theta))\,/\sqrt{(s - i\delta sin(\theta))(s + i\delta sin(\theta))}\,ds \qquad (\ddagger)$$

Now for the purposes of generating a harmonic expression, we don't need the term $\sigma e^{r\delta\cos(\theta)}$, so after dropping it, (‡) becomes

$$\kappa\int_{\gamma} e^{sr} g^{u,v}(s + \delta\cos(\theta)) \,/\sqrt{(s - i\delta sin(\theta))(s + i\delta sin(\theta))}\; ds \qquad (\sim)$$

Thus, for *very small* δ, the numerator term $g^{u,v}(s + \delta\cos(\theta))$ will become $g^{u,v}(s)$, and the residue in (~) will compute to $g^{u,v}(0)$, as both δ and θ ---› 0. Additionally, the integration along the arms associated with the branching points $\pm i\delta\sin(\theta)$ will compute to

$$-2\int_{0}^{\infty} e^{iyr} g^{u,v}(iy) \,/\sqrt{(y)(y)}\; dy \qquad\qquad 2\int_{0}^{\infty} e^{-iyr} g^{u,v}(-iy) \,/\sqrt{(y)(y)}\; dy$$

$\boldsymbol{i}\delta\sin(\theta)$ arms $\qquad\qquad$ $\boldsymbol{-i}\delta\sin(\theta)$ arms

so that finally, the following expression emerges, after equating the residue with the arms [see pages 316-19 for more on this methodology] …

$$g^{u,v}(0) \;=\; 2\kappa\int_{0}^{\infty} \{\cos(yr)[g^{u,v}(iy) \;-\; g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) \;+\; g^{u,v}(-iy)]\}\; dy \,/\, y \qquad \ldots (*)$$

What's important to see here, is that the Laplace inverse in (~) above makes sense as δ ---› 0, because we have defined $g^{u,v}(s)$ to be $\zeta(\alpha - s)$, where α is *any* real number > 1. As such, we are assured of convergence to 0 along the large arc in the contour γ above, as its radius tends to ∞ [see page 259], *and* when traversing γ, no poles are encountered because $0 < \mathrm{Re}(AB) < \alpha - 1$.

An Equivalency Theorem For The Field Equations For Any θ When λ(s) ≈ σ / s

Let us recall our Laplace inverse from page 329, where α and ε are real numbers with $\varepsilon \geq 0$, and θ is between 0 and π. We'll label the expression below (1) …

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\alpha + iy) \;-\; g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) \;+\; g^{u,v}(\alpha - iy)]\}\; dy \,/\sqrt{y^2 - \varepsilon^2}$$

Now from our previous research we know that if (1) equates to $g^{u,v}(\alpha)\cdot J_0(r\varepsilon)$, where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, then the form for the field equations, in the case of a *coupling* is [pp 329-30] …

$$G^{u,v} \;\approx\; \sigma[g^{u,v}(0) \;+\; 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] \qquad (2)$$

Here, the *physical* singularities associated with $\lambda(s)$ are at O and at (±1, 0). Conversely, if (2) is true, then necessarily (1) must equate to $g^{u,v}(\alpha)\cdot J_0(r\varepsilon)$, since the right-hand side of (2) *is* derived from the Laplace inverse, according to the expression below …

$$G^{u,v} \approx \kappa \int_{\gamma} e^{sr} \lambda(s) g^{u,v} ds$$

Thus, we have an equivalency here, in so much as

$$(1) = g^{u,v}(\alpha) \cdot J_0(r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true.}$$

This equivalency is almost self-evident, but it means we can look at the matter in two different ways. For example, we can choose to see it solely via (2) … or we can choose to see it via the equation

$$(1) = g^{u,v}(\alpha) \cdot J_0(r\varepsilon)$$

But regardless of which approach we take to developing a form for the field equations, in the case of a coupling, where the *physical* singularities associated with $\lambda(s)$ are at O and at $(\pm 1, 0)$; it must, in the end, be a form which is compatible with its *uncoupled* counterpart, when looking at dark energy [ξ].

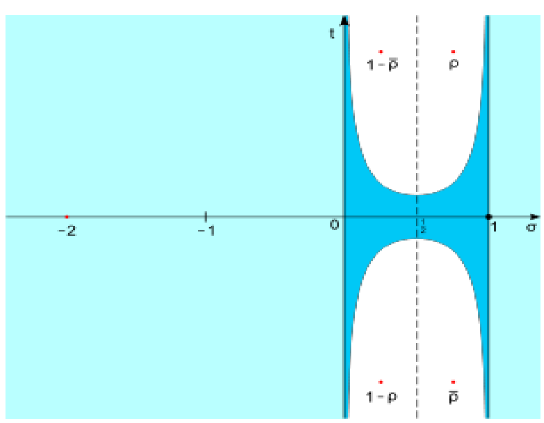
In (2) above, the dark energy component is

$$\xi = \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))] ,$$

which *exactly* matches what we see in the *uncoupled* case [pp 321-3].

Revisiting The Riemann Hypothesis – A Dialogue Between Joe and Jim

Joe develops a mathematical method $\mathcal{M}$ that shows, in the region $\mathcal{C}^*$, the critical strip *minus* the critical line, no roots of $\zeta(s)$ exist. Joe is so delighted with his proof, he decides to show it to Jim, one of his closer friends. Jim points out to Joe that roots in $\mathcal{C}^*$ can never be verified, but this doesn't seem to bother Joe much, because he says to Jim, '$\mathcal{M}$ never found any'.



A 'zero-free' region of $\zeta(s)$ shown in blue, but is it ?

Jim replies by saying, 'well Joe, had your method $\mathcal{M}$ encountered a root in $\mathcal{C}^*$, you must believe, then, it would have reported it as such, right ?' Joe agrees, and seems to have no problem with this conclusion.

But herein lies the problem. For $\mathcal{M}$ to be completely *unambiguous*, it *must* be able to report a root correctly in $\mathcal{C}^*$, should it encounter one. This $\mathcal{M}$ cannot do, according to our methodology on pages 293-6.

Thus, while it is perfectly acceptable for Joe to assert that $\mathcal{M}$ is true, Joe cannot make this assertion *without* ambiguity. For Joe could only do this if roots in $\mathcal{C}^*$ were verifiable.

So the focus here is not $\mathcal{M}$, in and of itself; but rather, what it means to *believe* $\mathcal{M}$, without *any* doubt at all. It means embracing a *false* conclusion; namely, that $\mathcal{M}$ will report roots correctly in $\mathcal{C}^*$. Since no mathematical method $\mathcal{M}$ can do this, by default, it leaves the Riemann Hypothesis in an indeterminate state; there simply is no answer, one way or the other.

Indeed, if roots in $\mathcal{C}^*$ were verifiable, it is my opinion that the Riemann Hypothesis would have been resolved long ago, perhaps by Riemann, himself …

Hidden Variables and The Field Equations For Any θ When $\lambda(s) \approx \sigma / s$

On page 307, we discussed 'hidden variables' by way of the Riemann zeta function [$\zeta(s)$], and how they might fit into the field equations of General Relativity. Here we are going to complete that picture a little more, in the case of a *coupling* between $g^{u,v}$ and $\lambda(s)$, where the *physical* singularities are at the origin O of the star, and at $(\pm 1, 0)$.

Let us begin by defining

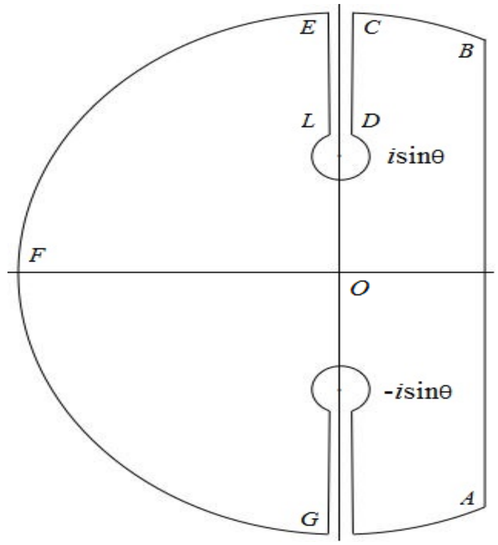
$$\mu(\beta,s) = \lambda(s)\zeta(\beta - s + \alpha\cos(\theta)) ,$$

where β is any *real* number > 1, α is associated with the location of the physical singularity, and θ is between 0 and π. Here we have a *coupling* between the dark energy density function $\lambda(s)$ and $\zeta(.)$, where $\zeta(.)$ is to be thought of as a 'form of consciousness' streaming into our universe, via the singularities associated with $\lambda(s)$.

For example, for the singularity at O, $\zeta(.)$ would not depend on θ, because of radial symmetry, and because $\alpha = 0$. Thus, the dark energy contour integral, in this case, would look like …

$$G^{u,v} \approx \kappa \int_{\gamma} e^{sr} \mu(\beta,s) g^{u,v} ds \qquad (**)$$

where $\lambda(s) \approx \sigma / s$. And the contour γ would be the one shown below, where the branching arms no longer exist, so that it is made up of AB and the large arc only.



Since $\zeta(\beta - s)$ has a *pole* at $s = \beta - 1$, we'll *exclude* it by choosing $0 < \text{Re}(AB) < \beta - 1$. The residue in (**) then computes to $\sigma g^{u,v}(0) \cdot \zeta(\beta)$, since again $\alpha = 0$ in this case.

For the *physical* singularity at (+1, 0), the *exact* form for the dark energy contour integral is

$$\kappa\sigma \int_\gamma e^{sr} g^{u,v}(s)\zeta(\beta - s + \alpha\cos(\theta)) \,/ \sqrt{s^2 - 2s\cos(\theta) + 1}\, ds \qquad (\dagger)$$

where now $\alpha = 1$, and γ includes the branching arms [see pages 221-7 for more of the details here].

And after translating s ---› s – cos(θ) [that is, let $u = s - \cos(\theta)$], (†) becomes, after mapping *u* back to *s* …

$$\kappa\sigma e^{r\cos(\theta)} \int_\gamma e^{sr} g^{u,v}(s + \cos(\theta))\zeta(\beta - s) \,/ \sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\, ds \qquad (*)$$

so that the argument to ζ(.) does not depend on θ. And again, here $0 < \text{Re}(AB) < \beta - 1$, and because of the way ζ(.) is defined in (*), we expect the integral to converge to 0 along the large arc in γ, and *also* along the small circles associated with the *branching* points, as their radii shrink to 0.

For the *physical* singularity at (–1, 0), $\alpha = -1$ in (†), and the *minus* sign in the denominator becomes a *plus* sign. After translating s ---› s + cos(θ), (*) is replaced with its counterpart, where s + cos(θ) maps to s – cos(θ), and rcos(θ) to –rcos(θ). The ζ(.) term, however, stays the same.

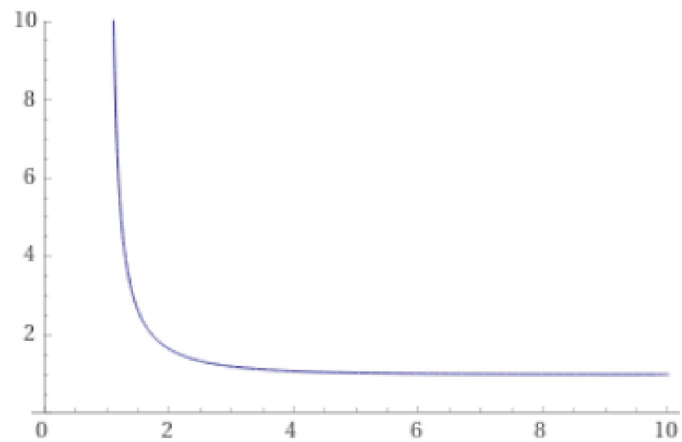
Notice that as θ ---› 0, the residue in (*) computes to $\sigma e^{r} g^{u,v}(+1) \cdot \zeta(\beta)$, and for the counterpart, the residue is $\sigma e^{-r} g^{u,v}(-1) \cdot \zeta(\beta)$. This is *highly* reminiscent of our work on pages 321-3, where we developed a form for the field equations, where there are *physical* singularities at O and at (±1, 0), associated with λ(s), in the case of a *coupling*.

Here, that form is essentially the same, but with the factor $\zeta(\beta)$ included. To wit,

$$G^{u,v} \approx \sigma \cdot \zeta(\beta)[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta))] ,$$

where $G^{u,v} = C^{u,v} - kT^{u,v}$, and $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively. Yet I wish to caution the reader that this form is a hunch, as we'll see later on.

Thus, for the case where $\beta > 1$, we see how we can incorporate a 'hidden variable' into the field equations, where the variable $\zeta(.)$ is associated, perhaps, with a form of consciousness streaming into our universe, through the *physical* singularities associated with $\lambda(s)$.



Plot of $\zeta(\beta)$ for $\beta > 1$

What is interesting to me here, in all of this, is the way in which we originally defined the argument to $\zeta(.)$. We chose $\zeta(\beta - s + \alpha\cos(\theta))$, where α is to be associated with the location of the singularity, when forming the dark energy contour integral. At O, $\alpha = 0$, so the angle θ doesn't matter, because of radial symmetry. But θ does matter when $\alpha = \pm 1$, and that is because our perspective is always *relative* to O.

The 'hidden variable', namely $\zeta(.)$, is streaming through the singularities associated with $\lambda(s)$, so that relative to O, the angle θ has a role to play in $\zeta(.)$ … when considering these singularities at $(\pm 1, 0)$. Just as it does in $\lambda(s)$, itself …

*****

We can actually create a harmonic expression for (*) on page 335, using the methods outlined on pages 316-19. After dropping the $\sigma e^{r\cos(\theta)}$ term, (*) becomes

$$\kappa \int_\gamma e^{sr} g^{u,v}(s + \cos(\theta))\zeta(\beta - s) / \sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\, ds \qquad (\S)$$

and the residue in (§) computes to $g^{u,v}(\alpha)\cdot\zeta(\beta)$, where here $\alpha = 1$, and $\theta \dashrightarrow 0$. The arms compute to

$$-2\int_0^\infty e^{iyr} g^{u,v}(\alpha + iy)\zeta(\beta - iy) \,/\sqrt{(y)(y)}\, dy \qquad 2\int_0^\infty e^{-iyr} g^{u,v}(\alpha - iy)\zeta(\beta + iy) \,/\sqrt{(y)(y)}\, dy$$

$\boldsymbol{i}\sin(\theta)$ arms $\qquad$ $\boldsymbol{-i}\sin(\theta)$ arms

and after equating the residue to the arms [pp 316-19], the following harmonic expression emerges, for *any* α, which we'll label (1) …

$$g^{u,v}(\alpha)\cdot\zeta(\beta) \;=\; 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(\alpha + iy)\zeta(\beta - iy) \;-\; g^{u,v}(\alpha - iy)\zeta(\beta + iy)] \\ + \boldsymbol{i}\sin(yr)[g^{u,v}(\alpha + iy)\zeta(\beta - iy) \;+\; g^{u,v}(\alpha - iy)\zeta(\beta + iy)]\}\, dy \,/\, y$$

It is not hard to show, that by setting the gravitational component to a constant, say $g^{u,v} = c$, the expression above reduces to

$$4\kappa\boldsymbol{i}c\int_0^\infty \{A\sin(yr) \;-\; B\cos(yr)\}dy \,/\, y \qquad (2)$$

where

$$A \;=\; \{\zeta(\beta + iy) \;+\; \zeta(\beta - iy)\}/\,2$$

$$B \;=\; \{\zeta(\beta + iy) \;-\; \zeta(\beta - iy)\}/\,2i\,.$$

From page 260, (2) computes to $c\cdot\zeta(\beta)$, since $4\kappa\boldsymbol{i} = 2/\pi$ [κ is equal to $1/2\pi i$]. And this agrees with the left side of (1).

We'll now introduce a *modified* version of (1) to suit our purposes, where ε is between 0 and 1, and we'll label this (3) …

$$2\kappa\int_\varepsilon^\infty \{\cos(yr)[g^{u,v}(\alpha + iy)\zeta(\beta - iy) \;-\; g^{u,v}(\alpha - iy)\zeta(\beta + iy)] \\ + \boldsymbol{i}\sin(yr)[g^{u,v}(\alpha + iy)\zeta(\beta - iy) \;+\; g^{u,v}(\alpha - iy)\zeta(\beta + iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

One might think, by looking at (1), and comparing to previous research notes, that the likely estimate for (3) is $g^{u,v}(\alpha)\cdot\zeta(\beta)\cdot J_0(r\varepsilon)$. There actually appears to be some evidence for this when β is somewhat larger than 1, but when β is closer to 1, discrepancies start to show up. Here is a table, where $g^{u,v} = 1$ in expression (3) above. The integration range is always from $y = \varepsilon$ to 100, and all integrations were carried out on the Wolfram website. Recall again that β is always > 1 …

value of r	value of ε	value of β	value of (3)	value of ζ(β)·J₀(rε)
1	1	3/2	.762433	1.998983
1	1	5/2	.824982	1.026503
1	1	7/2	.796810	0.862174
1	1	9/2	.778326	0.807060
1	1	11/2	.768914	0.784484
2	1	3/2	–0.068986	0.584887
2	1	5/2	0.144545	0.300347
2	1	7/2	0.197408	0.252265
2	1	9/2	0.212788	0.236139
2	1	11/2	0.218276	0.229534

The conclusion to be drawn from this data is that our estimate for (3); namely $g^{u,v}(\alpha)\cdot\zeta(\beta)\cdot J_0(r\varepsilon)$, probably isn't too far off from the truth, and seems to get better for larger β. For smaller β close to, but greater than 1, discrepancies appear, but perhaps it's not that relevant anyway, since it is the case that ζ(β) ---› ∞ as β ---› 1. Thus, ζ(β) only makes sense, *physically*, for larger β.

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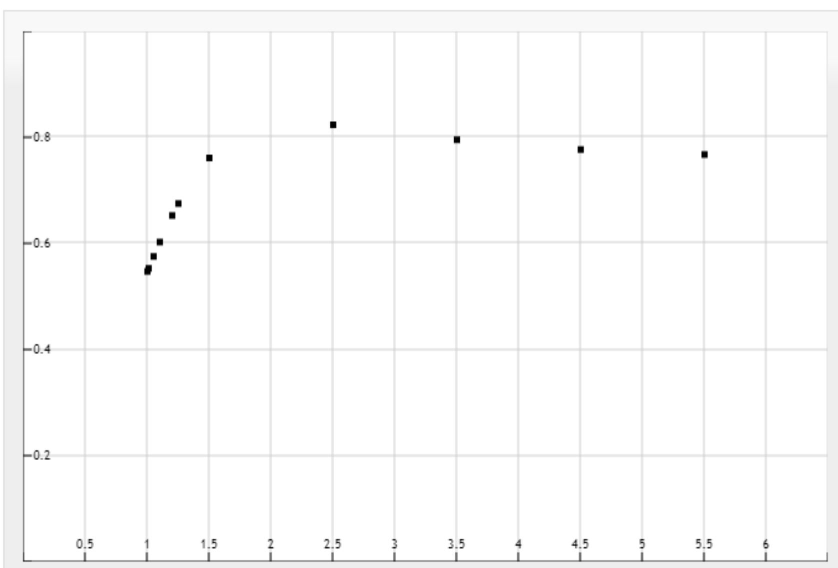
Here is a plot of the value of (3) along the *y*-axis, versus the value of β along the *x*-axis. Note that because (3) is derived from (*) on page 335, we expect that (3) will indeed converge. Thus, equating (3) to $g^{u,v}(\alpha)\cdot\zeta(\beta)\cdot J_0(r\varepsilon)$ won't match as β ---› 1, since here ζ(β) ---› ∞. This is not something we could have known by examining *only* the *physical* singularity at the origin O, but it *can* be deduced by studying the harmonics associated with the physical singularities at (±1, 0).



value of (3) versus β when r = 1 and ε = 1

The plot above demonstrates that physically *acceptable* outcomes are possible for the 'hidden variable', which we postulated to be $\zeta(\beta)$, initially. And while $\zeta(\beta)$ is still a likely candidate for *larger* β, something interesting happens as β ---› 1. The graph 'rolls over', as β ---› 1, thereby avoiding the catastrophe that otherwise would have occurred, had it simply followed $\zeta(\beta)$ out to ∞. I find this to be a rather satisfying conclusion … at this juncture …

A Harmonic Expression For $\zeta(\alpha)\cdot\zeta(\beta)$ When α and β Are Greater Than 1

Let us bring back our harmonic expression on page 337, valid for *any* α, and for $\beta > 1$, which we'll label (1), and remind ourselves here, that κ is equal to $1/2\pi i$ …

$$g^{u,v}(\alpha)\cdot\zeta(\beta) \;=\; 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(\alpha + iy)\zeta(\beta - iy) \;-\; g^{u,v}(\alpha - iy)\zeta(\beta + iy)] \\ + \boldsymbol{i}\sin(yr)[g^{u,v}(\alpha + iy)\zeta(\beta - iy) \;+\; g^{u,v}(\alpha - iy)\zeta(\beta + iy)]\}\, dy\, /\, y$$

Now let $\alpha = 0$, so that (1) becomes the following, which we'll label (2) …

$$g^{u,v}(0)\cdot\zeta(\beta) \;=\; 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(iy)\zeta(\beta - iy) \;-\; g^{u,v}(-iy)\zeta(\beta + iy)] \\ + \boldsymbol{i}\sin(yr)[g^{u,v}(iy)\zeta(\beta - iy) \;+\; g^{u,v}(-iy)\zeta(\beta + iy)]\}\, dy\, /\, y$$

Finally, let us define $g^{u,v}(s) = \zeta(\alpha - s)$, where $\alpha > 1$, so that $g^{u,v}(0) = \zeta(\alpha)$, and thus (2) becomes (3) below …

$$\zeta(\alpha)\cdot\zeta(\beta) \;=\; 2\kappa\int_0^\infty \{\cos(yr)[\zeta(\alpha - iy)\zeta(\beta - iy) \;-\; \zeta(\alpha + iy)\zeta(\beta + iy)] \\ + \boldsymbol{i}\sin(yr)[\zeta(\alpha - iy)\zeta(\beta - iy) \;+\; \zeta(\alpha + iy)\zeta(\beta + iy)]\}\, dy\, /\, y$$

Thus, we see that in the range α and $\beta > 1$, the product $\zeta(\alpha)\cdot\zeta(\beta)$ can be expressed as a Fourier transform, ostensibly along the α and β-lines in the complex plane $\mathbb{C}$, simultaneously.

As a verification here, let $\alpha = 3/2$ and $\beta = 5/2$. Then the integration on the Wolfram site reports the

$$\int_0^{300} -\frac{1}{\pi y}\, i\left(\cos(y)\left(\zeta\left(\frac{3}{2} - i\,y\right)\zeta\left(\frac{5}{2} - i\,y\right) - \zeta\left(\frac{3}{2} + i\,y\right)\zeta\left(\frac{5}{2} + i\,y\right)\right) + \right. \\ \left. i\sin(y)\left(\zeta\left(\frac{3}{2} - i\,y\right)\zeta\left(\frac{5}{2} - i\,y\right) + \zeta\left(\frac{3}{2} + i\,y\right)\zeta\left(\frac{5}{2} + i\,y\right)\right)\right) dy = 3.50402$$

value above, whilst in the image below, we see what $\zeta(\alpha)\cdot\zeta(\beta)$ actually is, again from Wolfram …

Input

$$\zeta\left(\frac{3}{2}\right)\zeta\left(\frac{5}{2}\right)$$

Decimal approximation

3.50446824141800416

The two agree to about 5 parts in 10,000, and we are satisfied that our harmonic expression (3) is, indeed, correct ! It's an interesting way to look at multiplication, for normally one would simply multiply the two parts together, and be done with it. Yet through harmonics, we see the same thing in a very different light (note that in our testing, we chose r = 1, but the choice is arbitrary here, so long as r is greater than *zero*) …

Some More on Hidden Variables and The Field Equations For Any θ When $\lambda(s) \approx \sigma / s$

On pages 334-9 we discussed 'hidden variables' when using $\zeta(.)$, and how they might be connected to the field equations of General Relativity, where there is a *coupling* between $g^{u,v}$ and $\lambda(s)$. Recall the *physical* singularities are at the origin O of the star, and at $(\pm 1, 0)$, so here we want to elaborate on things, just a little more.

Let us begin by recalling our form for the field equations, in this case, where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively [θ is between 0 and π] …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] \qquad (\dagger)$$

When ε ---› 0, we are looking to the east or to the west, relative to O, and ($\dagger$) reduces to

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r)g^{u,v}(\alpha)] \,,$$

where $\alpha = 1$. And from expression (3) on page 337, and reproduced here,

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\alpha + iy)\zeta(\beta - iy) - g^{u,v}(\alpha - iy)\zeta(\beta + iy)] + \boldsymbol{i}\sin(yr)[g^{u,v}(\alpha + iy)\zeta(\beta - iy) + g^{u,v}(\alpha - iy)\zeta(\beta + iy)]\}\, dy / \sqrt{y^2 - \varepsilon^2}$$

we infer that as ε ---› 0, the Laplace inverse is *exact*, since again, from page 337, it has already been shown that

.
.

$$g^{u,v}(\alpha)\cdot\zeta(\beta) \;=\; 2\kappa\int_0^\infty \{\cos(yr)[g^{u,v}(\alpha+iy)\zeta(\beta-iy) \;-\; g^{u,v}(\alpha-iy)\zeta(\beta+iy)] + \boldsymbol{i}\sin(yr)[g^{u,v}(\alpha+iy)\zeta(\beta-iy) \;+\; g^{u,v}(\alpha-iy)\zeta(\beta+iy)]\}\, dy / y$$

where $\beta > 1$.

Thus, we have an *exact* form for the field equations in this case, where the 'hidden variable' is $\zeta(.)$, and that form is, for *all* $\beta > 1$, and $\alpha = 1$ …

$$G^{u,v} \;\approx\; \sigma\cdot\zeta(\beta)[g^{u,v}(0) \;+\; 2\cosh(r)g^{u,v}(\alpha)]$$

Indeed, this is the *only* time things are exact, and in fact, the larger ε is [i.e. ε ---› 1], the *less* precise things become, or so it seems.

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Let us now produce some tables, as we did on page 338. We'll look at data when $r = 1$ or 2, and when $\varepsilon = 0.5$ or 1. Here are the results, where $g^{u,v} = 1$ in expression (3) on page 340 …

value of r	value of ε	value of β	value of (3)	value of $\zeta(\beta)\cdot J_0(r\varepsilon)$
1	1	3/2	.762433	1.998983
1	1	5/2	.824982	1.026503
1	1	7/2	.796810	0.862174
1	1	9/2	.778326	0.807060
1	1	11/2	.768914	0.784484
1	1	13/2	.764242	0.774385
2	1	3/2	–0.068986	0.584887
2	1	5/2	0.144545	0.300347
2	1	7/2	0.197408	0.252265
2	1	9/2	0.212788	0.236139
2	1	11/2	0.218276	0.229534
2	1	13/2	0.220505	0.226579
1	0.5	3/2	1.63471	2.451635
1	0.5	5/2	1.17966	1.258945
1	0.5	7/2	1.03203	1.057405
1	0.5	9/2	0.97692	0.989811
1	0.5	11/2	0.95349	0.962124
1	0.5	13/2	0.94283	0.949737

Notice from this tabular data that the value of (3), compared to the value of $\zeta(\beta)\cdot J_0(r\varepsilon)$, seems to be best when $\varepsilon = 0.5$, which is 30^o above the *x-axis*. And for *larger* β, the value of (3) compared to $\zeta(\beta)\cdot J_0(r\varepsilon)$, dovetails nicely.

From this we may conclude that for *larger* β, the form for the field equations,

$$G^{u,v} \approx \sigma\cdot\zeta(\beta)[g^{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] \quad (*)$$

is a reasonably good approximation that incorporates the 'hidden variable' ζ(.). For *smaller* β, close to but larger than 1, such an approximation no longer works, and the only way to compute what the hidden variable might actually be, at the *physical* singularities (±1, 0) associated with λ(s), is by performing integrations using (3). A tall order which could well be beyond our reach !

But if it is achievable, then (*) becomes, where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$ …

$$G^{u,v} \approx \sigma[\zeta(\beta)g^{u,v}(0) + 2\zeta(\beta,\theta)\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] \quad (\sim)$$

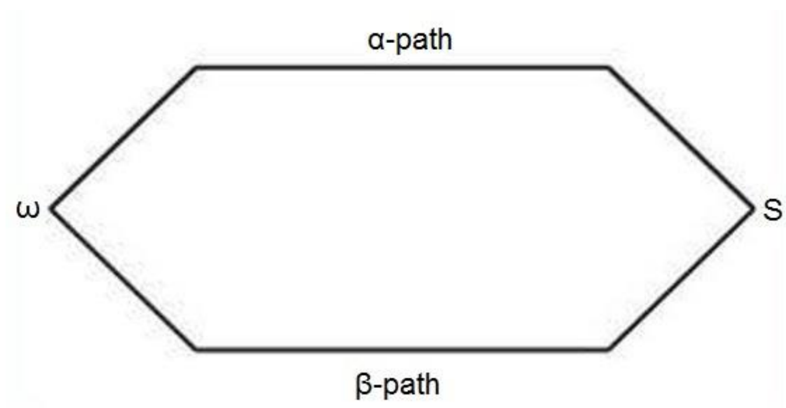
Here ζ(β) will always be associated with $g^{u,v}(0)$ because of radial symmetry [pp 307-9], and ζ(β, θ) is whatever we pick up from (3), assuming that (3) computes to

$$g^{u,v}(\alpha)\cdot\zeta(\beta,\theta)\cdot J_0(r\varepsilon)$$

Note that this implies ζ(β, θ) is *also* a function of θ now, as it should be, since the original argument to ζ(.) was $\zeta(\beta - s + \gamma\cos(\theta))$, where here γ is associated with the *physical* singularities tied to λ(s) [page 334]. Also note that when θ = 0, ζ(β, θ) = ζ(β).

Some More on The Riemann Hypothesis – A Tale of Two Frames

In the diagram below, ω is a root of ζ(s) in the *critical* strip, and ***S*** is the signal, which can be generated via the α-line *or* the β-line, as per our discussions on pages 294-6. In fact, each line contains a copy of the root ($\alpha \neq \frac{1}{2}$ and $\beta = 1 - \alpha$), so here we want to understand, in a little more detail, what it means to say that 'relative to ***S***, it is not possible to distinguish between the α-line and the β-line'.



We'll start by abbreviating the statement above, so that (*) means 'relative to ***S***, it is not possible to distinguish between the α-line and the β-line'.

Now let $\mathcal{O}$ be an observer associated with the signal's frame of reference, and let $\mathcal{O}$ be situated at a point *p anywhere* on ℓ, where ℓ is one of ℓ_α or ℓ_β. Since (*) holds true, $\mathcal{O}$ does not know which line it is on, *nor* does it know what *p* actually is. The most $\mathcal{O}$ can know, at this point, is that it is a certain distance above *or* below the *x-axis* [say ε, where $\varepsilon \geq 0$], but nothing more.

Now let *q* be the counterpart to *p*, so that if *p* happened to be $\alpha \pm i\varepsilon$, then *q* would be $\beta \pm i\varepsilon$, and $\mathcal{O}$ could just as easily be situated at *p* or *q*, without knowing what either was. Since the choice of *p* is arbitrary at this juncture, it doesn't matter what $\zeta(p)$ and $\zeta(q)$ compute to. They may be equal to one another, or they may not. Thus, the setup here, to this point, is simply a reiteration of (*), but tells us nothing about what is going on at the *point* level on ℓ, from the perspective of $\mathcal{O}$.

For $\mathcal{O}$ to make an inference at the *point* level, we need to introduce a moderator in this experiment, say $\mathcal{M}$, who is an 'all-knowing' being. $\mathcal{O}$ can ask $\mathcal{M}$ any question it has, and $\mathcal{M}$ can respond, *provided* there is no violation of (*). Now if it is the case that $\zeta(p) = \zeta(q)$ [and $\mathcal{M}$ will know if this is so], $\mathcal{M}$ can actually tell $\mathcal{O}$ that this is true, without violating (*). For now, even with this knowledge, $\mathcal{O}$ cannot distinguish between *p* and *q*, since $\zeta^{-}(a)$ maps back to *both p* and *q* simultaneously, where *a* is whatever $\zeta(p)$ and $\zeta(q)$ compute to.

On the other hand, if $\zeta(p) \neq \zeta(q)$ [and $\mathcal{M}$ will know if this is so], $\mathcal{O}$ can most certainly distinguish between *p* and *q*, by an application of ζ^{-}, *provided* $\mathcal{O}$ can obtain the information that it needs from $\mathcal{M}$ [at a minimum, what $\zeta(p)$ and $\zeta(q)$ compute to]. In this case, though, $\mathcal{M}$ will go silent and say nothing, so that (*) is not violated. Indeed, if $\mathcal{O}$ just knew via $\mathcal{M}$ that $\zeta(p) \neq \zeta(q)$, then $\mathcal{O}$ would also know that 'being situated at *p*' was not the same as 'being situated at *q*', even though $\mathcal{O}$ has no knowledge of what *p* and *q* really are. And that is because $\mathcal{O}$ can ask $\mathcal{M}$ if the value of ζ(.) has changed. In this sense, then, (*) would also be violated, which explains the moderator's silence.

Thus, our methodology $\boldsymbol{\mathcal{M}}$, first outlined on pages 294-6, validates (*) at the *point* level, whenever it is the case that $\zeta(p) = \zeta(q)$. That is to say, whenever *p* and *q* are *pseudo*-roots [pp 302-4]. The Riemann Hypothesis, then, becomes a special case where $\zeta(p) = \zeta(q) = 0$; but either way, $\boldsymbol{\mathcal{M}}$ shows us that it is *not* possible to distinguish between *p* and *q* at the *point* level, on ℓ_α and ℓ_β, if $\zeta(p)$ and $\zeta(q)$ agree.

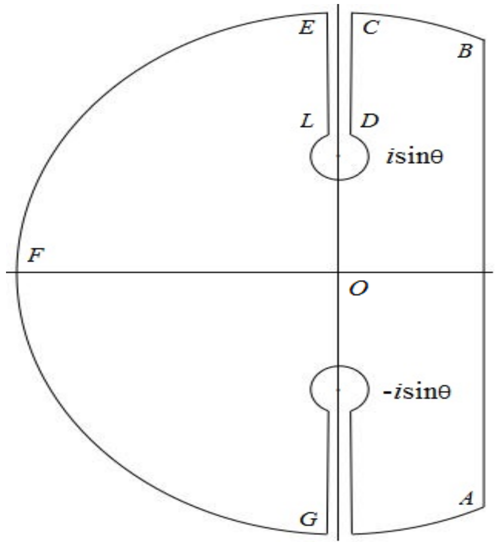
In turn, this means that if $\boldsymbol{\mathcal{M}}'$ is any other mathematical method seeking to resolve the Riemann Hypothesis, it will never be able to verify these *pseudo*-roots in $\mathcal{C}^*$, where $\mathcal{C}^*$ is the critical strip *minus* the critical line, and verify means 'prove without ambiguity'. For to do so would mean contradicting $\boldsymbol{\mathcal{M}}$, in so much as $\boldsymbol{\mathcal{M}}'$ would now be able to distinguish between *p* and *q*.

A Harmonic Expression For ζ(α)·ζ(β) When α and β Are Between 0 and 1

In this case we are in the *critical* strip, and the methods on pages 262-4 apply, where our contour integral is

$$\kappa\int_\gamma e^{sr}g(s)\,/\sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\,ds \qquad (\dagger)$$

and $g(s) = \zeta(\alpha - s)\zeta(\beta - s)$. Since α and β are between 0 and 1, traversing the contour γ, as shown below,



will pick up the residue at O, as $\theta \dashrightarrow 0$, *and also*, the residues at $\alpha - 1$ and $\beta - 1 < 0$, since these are now the *poles* of g(s). At O, the residue is $\zeta(\alpha)\cdot\zeta(\beta)$ and at the other two points, the residues are …

$$e^{r(\alpha-1)}\,\zeta(\beta-\alpha+1)/(\alpha-1) \;\text{ at }\; \alpha-1 \qquad e^{r(\beta-1)}\,\zeta(\alpha-\beta+1)/(\beta-1) \;\text{ at }\; \beta-1$$

And from page 263 [especially the expression at the top of the page], we may now conclude that [labelling this formula as (*)] …

$$\zeta(\alpha)\cdot\zeta(\beta) = 2\{e^{r(\alpha-1)}\,\zeta(\beta-\alpha+1)/(\alpha-1) + e^{r(\beta-1)}\,\zeta(\alpha-\beta+1)/(\beta-1)\} +$$

$$2\kappa\int_0^\infty \{\cos(yr)[\zeta(\alpha-iy)\zeta(\beta-iy) - \zeta(\alpha+iy)\zeta(\beta+iy)] + \boldsymbol{i}\sin(yr)[\zeta(\alpha-iy)\zeta(\beta-iy) + \zeta(\alpha+iy)\zeta(\beta+iy)]\}\,dy/y$$

Thus, the expression for $\zeta(\alpha)\cdot\zeta(\beta)$ is a doubling of the sum of the residues as shown above, added to the harmonics along the branch lines, in the contour γ. Notice, too, that when $\alpha = \beta$, the formula is not meaningful, since $\zeta(1)$ is *infinite*.

As an example, suppose $\alpha = 1/4$ and $\beta = 3/4$. Then the residue at $\alpha - 1$ computes to -1.64533 and the residue at $\beta - 1$ computes to 4.54930. The harmonic expression computes to -3.00718, so that (*) evaluates to ≈ 2.80076. By comparison, $\zeta(1/4)\cdot\zeta(3/4) \approx 2.79872$. The two agree to about 2 parts in 1000, which is very reasonable, indeed.

Here are the snapshots from the Wolfram site, where in this case we chose $r = 1$, but any $r > 0$ would suffice …

$$\int_0^{210} -\frac{1}{\pi y} i \left(\cos(y)\left(\zeta\left(\frac{1}{4} - i y\right)\zeta\left(\frac{3}{4} - i y\right) - \zeta\left(\frac{1}{4} + i y\right)\zeta\left(\frac{3}{4} + i y\right)\right) + i \sin(y) \left(\zeta\left(\frac{1}{4} - i y\right)\zeta\left(\frac{3}{4} - i y\right) + \zeta\left(\frac{1}{4} + i y\right)\zeta\left(\frac{3}{4} + i y\right)\right)\right) d y = -3.00718 + 0 i$$

$$-\frac{4\,\zeta\left(\frac{3}{2}\right)}{3\,e^{3/4}}$$

Decimal approximation

−1.64533165056556

residue at α – 1

$$-\frac{4\,\zeta\left(\frac{1}{2}\right)}{\sqrt[4]{e}}$$

Decimal approximation

4.549300940091052

residue at β – 1

$$\zeta\left(\frac{1}{4}\right)\zeta\left(\frac{3}{4}\right)$$

Decimal approximation

2.798723091545862

Similar tests were performed when α = 1/4 and β = 1/3. In this case (*) evaluated to ≈ .79135, whilst the expression ζ(1/4)·ζ(1/3) computes to ≈ .791613. Here, the two agree to about 3 parts in 10,000 !

A 'Fixed Amplitude' Theorem For Dark Energy [ξ] When λ(s) ≈ σ / s

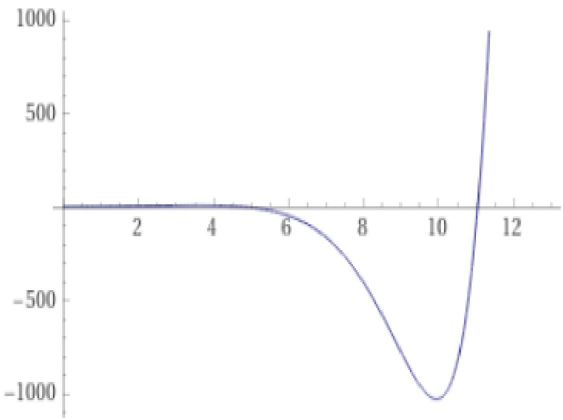
Up until now, we really haven't focused much on dark energy [ξ], along *radial* lines at different angles [θ] from the *x-axis*. Here, we're going to do a little more of that, where there are *physical* singularities at O and at (±1, 0), associated with λ(s), in the case of a *coupling*. We know already that for the *two*-dimensional case,

$$\xi = \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))]$$

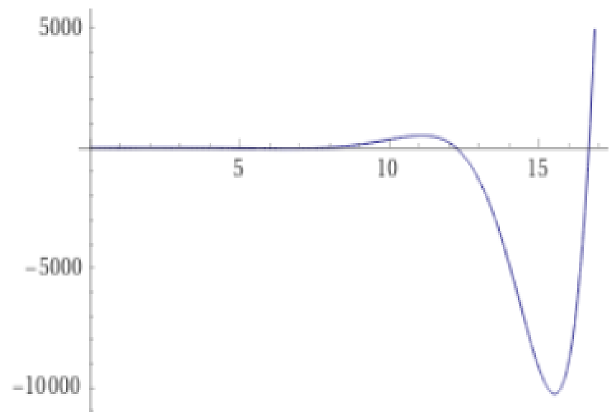
where σ is some constant [perhaps imperceptibly small], and θ is between 0 and π, say [see pages 272-3 for pictures of the evolution of the quantumlike dark energy scalar field in 3D, in the *angular* direction, for a *fixed* radius r].

Thus, when $\theta = 0$, the dark energy [ξ] is driven solely by the term $1 + 2\cosh(r)$, in accordance with the location of the *physical* singularities, where we're normalizing σ to 1 for the time being. And, when $\theta = \pi/2$, ξ is driven by the term $1 + 2J_0(r)$, where r is the radial distance from the origin O.

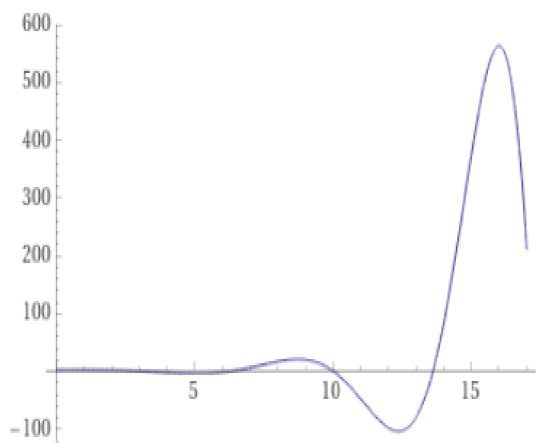
Now we'd like to show a few plots of ξ along these radial lines, when θ is between 0 and $\pi/2$, and then offer up a proposal for *detecting* ξ in the *radial* direction [*x-axis* below], in one special case.



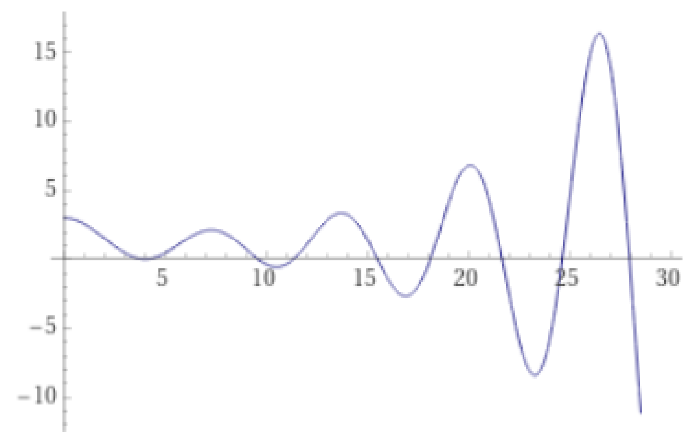
$\theta = 30^o$



$\theta = 45^o$

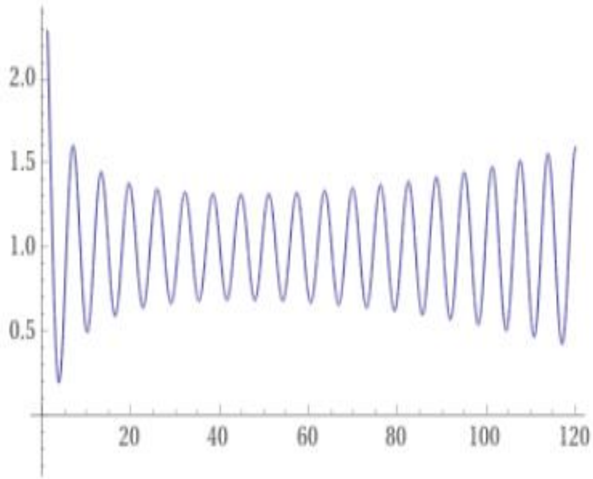


$\theta = 60^o$

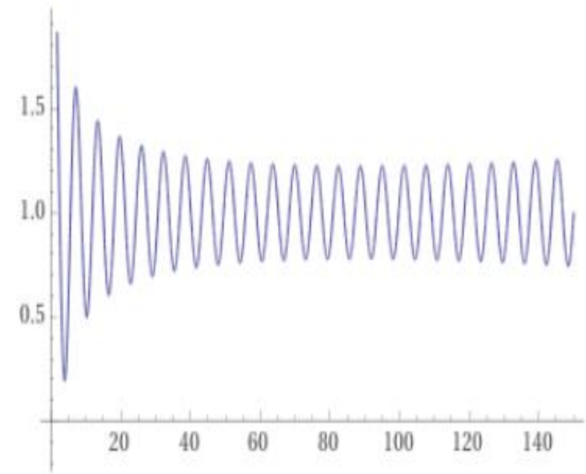


$\theta = 80^o$

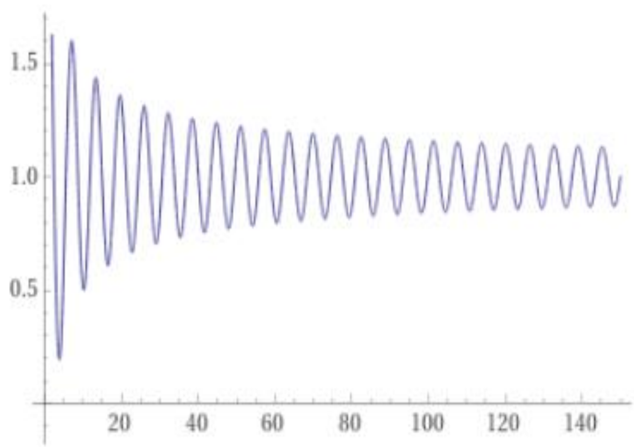
.
.
.



$\theta = 89^{o}$



$\theta = 89.5^{o}$



$\theta = 90^{o}$

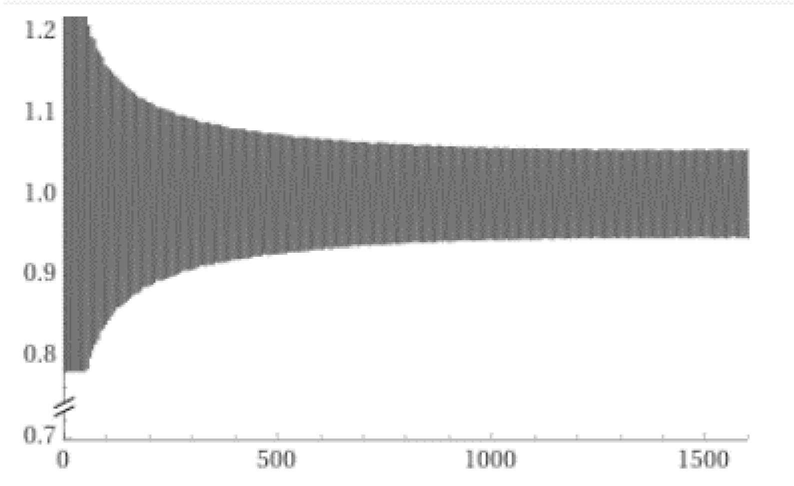
One can see from these pictures above, that the cosh(.) term in ξ has considerable influence over the dark energy waveform, in terms of ever increasing amplitudes, on *both* sides of the *x-axis*. That is, until θ approaches 90^{o}. And somewhere, *above* 89.5^{o} but *below* 90^{o}, the waveform should exhibit a *fixed* amplitude which is *not changing* in value, beyond a certain radius r, as the amplitude itself transitions from strictly *increasing* beyond a certain r, to strictly *decreasing* for *all* r.

Let's label this angle Θ, and ask ourselves the following question. Might it be possible to detect the ξ-waveform at Θ, via some *physical* experiment, since the amplitude is now *unique*, in that it isn't changing beyond a certain radius r ? If so, it might provide some evidence for dark energy, as we've described it here; which, in turn, could very well justify the existence of the dark energy density function [$\lambda(s)$]. Time will tell, I suppose, if such a thing is possible …

*****

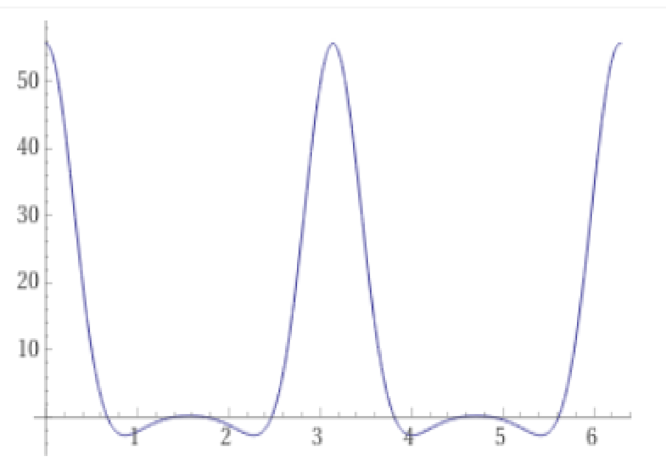
For this ξ-waveform at Θ to exist, we must assume the radius r is *bounded,* which makes sense from a *physical* perspective. Otherwise, no matter how close to 90^{o} the angle Θ actually was, the

cosh(.) term in ξ would eventually force the amplitude in this waveform to increase, by letting r tend to ∞. Here is a picture of the ξ-waveform when Θ = 89.97^{o} and r is capped at 1600 units.

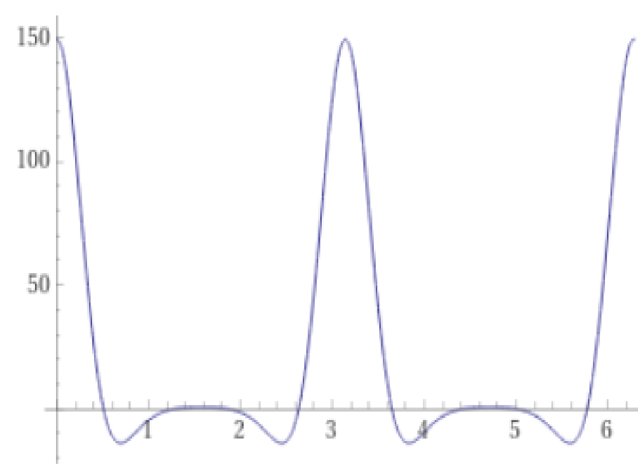


You can see from this picture, that the amplitude of the ξ-waveform is nearly *constant* beyond a certain radius r, and stays that way out to r = 1600, our upper bound (here, σ has been normalized to a value of 1).

It should be remembered, too, that we are dealing with a *quantumlike* dark energy scalar field [ξ], best understood by looking at the quantum fluctuations in the *angular* direction, along a circle r units from the origin O. Here are a few pictures that demonstrate the idea, but see also pages 278-83 for more pictures, when using a *Yukawa* density.



r = 4



r = 5

In these snapshots above, the dark energy [ξ], defined on page 345, shows the quantum fluctuations along circles of radius 4 and 5 respectively, where the *x-axis* [θ] is between 0 and 2π. Note that the *peaks* in these fluctuations are at θ = 0 and π, so that when measuring the fluctuation at these peaks in the *radial* direction, we would come up with the term 1 + 2cosh(r). We're still measuring a

fluctuation, radially, but it's less obvious that we're doing so. And similarly for any other radial line we might happen to be on, as depicted in the pictures on pages 346-7.

Now whether we can actually *detect* these quantum fluctuations, either in the angular or radial directions, is certainly debatable; but my hope is that the ξ-waveform shown at the top of page 348 might be measurable, because it is of *fixed* amplitude beyond a certain radius r. To be sure, this ξ-waveform at Θ really is a quantum fluctuation in its own right, but unique because of the constant amplitude.

An Interesting Identity From Bessel Theory Related To Dark Energy [ξ] When $\lambda(s) \approx \sigma / s$

In G.N. Watson's magnificent work titled *A Treatise on The Theory of Bessel Functions* (second edition), there is an interesting identity which applies to our representation for ξ, described on page 345, and reproduced here …

$$\xi = \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))] \quad (*)$$

The identity in Watson's book can be found in Chapter V, page 149, and is shown below, which we'll label (†) …

$$e^{z\cos\theta} J_{\nu-\frac{1}{2}}(z\sin\theta) = \frac{\Gamma(\nu)}{\Gamma(\frac{1}{2})}(2\sin\theta)^{\nu-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^{\nu+n-\frac{1}{2}}}{\Gamma(2\nu+n)} C_n^{\nu}(\cos\theta).$$

Here, z is a complex number, so we can let it be our radius r from (*) above, and C_n^{ν} is a Gegenbauer polynomial of order ν. When $\nu = ½$, C_n^{ν} reduces to a Legendre polynomial P_n, and (†) simplifies, considerably. Additionally, by replacing z with -r in (†), we obtain a second expansion, so that by adding the two expansions together, and remembering that $J_0(-x) = J_0(x)$, one obtains …

$$2\cosh(r\cos(\theta))J_0(r\sin(\theta)) = 2\cdot\Sigma\, r^n P_n(\cos(\theta)) / \Gamma(n+1)\,,\ n = 0, 2, 4 \ldots \quad (§)$$

Note that the summation on the right-hand side is over all *even* numbers, because for all *odd* numbers, the terms in the two expansions [z = r and z = -r] cancel one another.

Now (§) is most easily verified by letting r = 1, and $\theta = \pi/2$. Then the left side of (§) is $2J_0(1)$, which is approximately 1.53; and the right-hand side, out to a few terms is, approximately,

$$2(1 - 1/4 + 3/192) \approx 1.53$$

It just might be that (§) may prove to be useful, down the road, as we learn to better understand the quantumlike nature of ξ. If nothing else, it is yet another way to perceive dark energy, in this case.

*****

A Brief Note on Connections to Laplace's Equation For Dark Energy [ξ] When $\lambda(s) \approx \sigma / s$

In G.N. Watson's book, titled *A Treatise on The Theory of Bessel Functions* (second edition), we see a comment on page 155 [chapter V] related to Laplace's equation. It is shown below …

It is well known that solutions of Laplace's equation, which are analytic near the origin and which are appropriate for the discussion of physical problems connected with a sphere, may be conveniently expressed as linear combinations of functions of the type

$$r^n P_n(\cos\theta), \quad r^n P_n{}^m(\cos\theta) \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix} m\phi;$$

these are *normal solutions* of Laplace's equation when referred to polar coordinates (r, θ, ϕ).

Now from (§) on page 349, we *also* see that the dark energy component can be written as

$$\xi = \sigma[1 + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))] = \sigma[1 + 2\cdot\sum r^n P_n(\cos(\theta)) / \Gamma(n+1)], \; n = 0, 2, 4 \ldots, \qquad (§)$$

so that the inference here, from the comment above, is that ξ satisfies Laplace's equation in *polar* coordinates, where the Laplace operator in *three dimensions* reduces to the following … when ϕ is omitted.

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)$$

Early testing seems to indicate that this is true, for when I calculate $\nabla^2 f$ where $f = r^n P_n(\cos(\theta))$ and $n = 2$ or 3, $\nabla^2 f$ is indeed *zero*, providing we can calculate *meaningful* derivatives in r *and* θ.

If we are correct here, the same should be true in *three* dimensions [pp 225-6, 272-3, 323], after invoking an appropriate change of coordinates, if need be.

A Heuristical Argument For Covariance and The Field Equations When $\lambda(s) \approx \sigma / s$

Let us recall our expression for the field equations, in two dimensions, where there is a *coupling* between the gravitational tensor $g^{u,v}$ and the underlying dark energy density function λ(s) [α = cos(θ) and ε = sin(θ)] …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] \qquad (*)$$

Here, the *physical* singularities associated with λ(s) are at O and at (±1, 0), and $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively.

Now we already know that $G^{u,v}$ and $g^{u,v}$ are fully *covariant* [pp 178-80], and that in (*) above, the dark energy associated with $g^{u,v}(0)$ is σ, whilst the dark energy associated with $g^{u,v}(\alpha)$ is …

$$\xi = 2\sigma\cosh(r\cos(\theta))J_0(r\sin(\theta)).$$

Since σ is a constant, it satisfies Laplace's equation by default, and we now know that ξ *also* satisfies Laplace's equation, where the Laplace operator ∇^2 is the *three-dimensional* variant mentioned on page 350, which omits the ϕ term.

Thus, the field equations become *fully* covariant 'across the board', in so much as conservation principles apply not only to $G^{u,v}$ and $g^{u,v}$, but *also* to the dark energy components in (*). And quite frankly, we couldn't ask for a better outcome.

For here, we believe that (*) will admit *non-singular* solutions in $g^{u,v}$, and now know that dark energy, driven by the underlying dark energy density function [λ(s)], obeys Laplace.

*****

In three dimensions, the following equations apply in the case of a *coupling* [page 323], which we'll label (†) …

$$\begin{aligned} G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) &+ \\ 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) &+ \\ 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] &. \end{aligned}$$

And here, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, and the *physical* singularities associated with λ(s) are at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) .$$

Notice that a dark energy component [σ now omitted], such as $\xi = 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))$, *also* satisfies (§) on page 349, and is reproduced here, where θ has been replaced with α …

$$2\cosh(r\cos(\alpha))J_0(r\sin(\alpha)) = 2\cdot\Sigma\, r^n P_n(\cos(\alpha)) / \Gamma(n+1) , \; n = 0, 2, 4 \ldots \qquad (\S)$$

Thus, it stands to reason that ξ will also satisfy Laplace, where θ has been replaced with α in the Laplace operator on page 350. And similarly for $\xi = 2\cosh(r\cos(\phi))J_0(r\sin(\phi))$, where θ maps to ϕ.

Now as to the term $\xi = 2\cosh(r\sin(\beta))J_0(r\cos(\beta))$, in (†) above, we let cos(.) and sin(.) trade places in (§), and we replace α with β. Then Laplace is satisfied where θ has now been replaced with β in the Laplace operator, *and* sin(θ) has been replaced with cos(β) in this same operator.

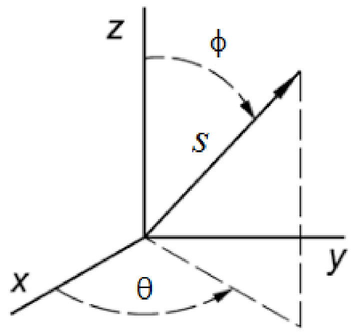
$$2\cosh(r\sin(\beta))J_0(r\cos(\beta)) = 2\cdot\Sigma\, r^n P_n(\sin(\beta)) / \Gamma(n+1) , \; n = 0, 2, 4 \ldots \qquad (\S)$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\cos\beta}\frac{\partial}{\partial\beta}\left(\cos\beta\,\frac{\partial f}{\partial\beta}\right)$$

Thus, in this sense, (†) also becomes fully covariant 'across the board', and for the time being, anyway, by a reasonable approach which justifies this conclusion, in my opinion.

And here again, we believe that (†) will admit *non-singular* solutions in $g^{u,v}$, and now know that dark energy, driven by the underlying dark energy density function [λ(s)], obeys Laplace, according to our approach above.

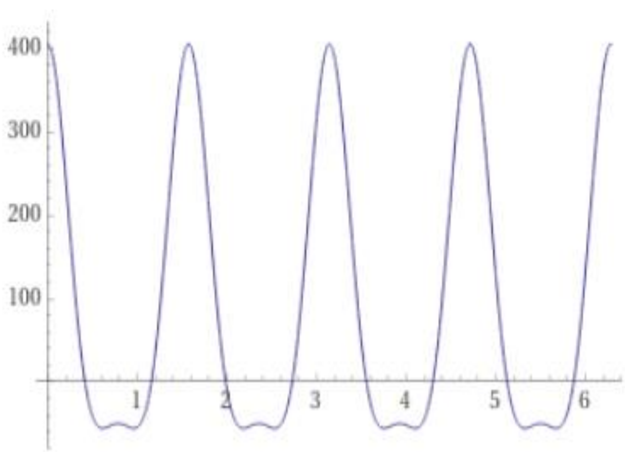
Some More Plots For a 3D Star When $\lambda(s) \approx \sigma / s$



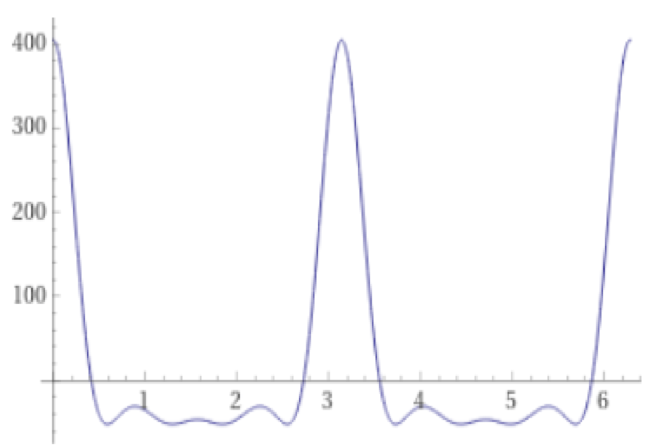
For a 3D star [pp 225-6, 323], using a *simple* inverse density function [$\lambda(s) = \sigma/s$] … with *physical* singularities at the origin [O] and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1)\ ,$$

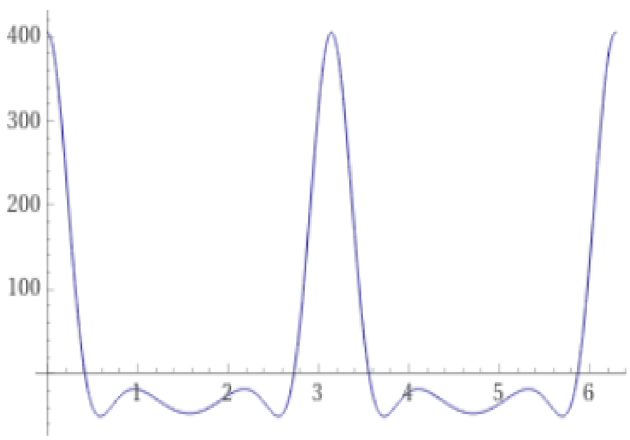
we can trace the *evolution* of our *quantumlike* dark energy scalar field, by starting with the plane at $\theta = 0^o$ to the *x*-axis, as shown above, and then rotating this plane to 30^o, 45^o, 60^o, and finally 90^o, where we reach the *y*-axis. Here are the plots for a radius r = 6; and as well, the *energy* of the quantum fluctuation along each circle has been included, which was calculated by performing integrations on the Wolfram site [ϕ is between 0 and 2π on the *x-axis* in these plots and σ = 1].



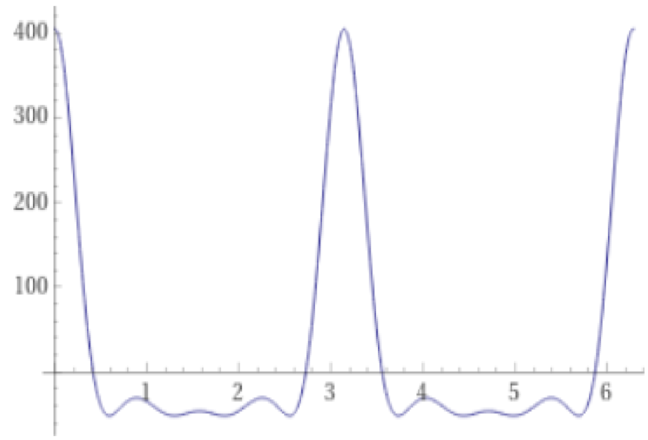
r = 6 , $\theta = 0^o$, $\phi = x$, Energy = 607 units



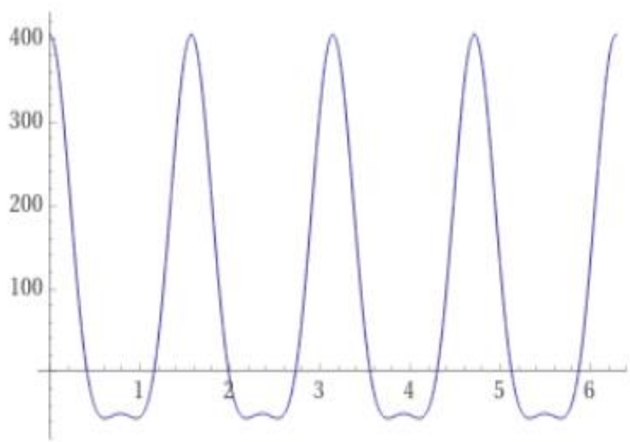
r = 6 , $\theta = 30^o$, $\phi = x$, Energy = 176 units



r = 6 , θ = 45$^{\mathbf{o}}$, ϕ = *x* , Energy = 217 units



r = 6 , θ = 60$^{\mathbf{o}}$, ϕ = *x* , Energy = 176 units



r = 6 , θ = 90$^{\mathbf{o}}$, ϕ = *x* , Energy = 607 units

One can see from these pictures that there is a symmetry about θ = 45$^{\mathbf{o}}$, and that the energy of the fluctuations is much greater at θ = 0$^{\mathbf{o}}$ and θ = 90$^{\mathbf{o}}$, than it is when θ is somewhere in between these two extremes. And this is due to the location of the *physical* singularities associated with dark energy.

Now whether there is a mean-value theorem of sorts, which associates the *average* value of the dark energy on a sphere r units from the origin O, with, say, the energy at the origin, I do not know. We do know, however, that the dark energy components satisfy Laplace, as per our previous research notes, but this may not be enough, in and of itself, to conclude that some kind of mean-value theorem does, indeed, exist (note to the reader: energy here in the diagrams above is ‘flat space’ energy; the integrations do not use a Jacobian).

*****

An Interesting Identity From Bessel Theory and Connections To Laplace

If we bring back our expression from page 349, which we'll label (†),

$$e^{z\cos\theta} J_{\nu-\frac{1}{2}}(z\sin\theta) = \frac{\Gamma(\nu)}{\Gamma(\frac{1}{2})}(2\sin\theta)^{\nu-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{z^{\nu+n-\frac{1}{2}}}{\Gamma(2\nu+n)} C_n^{\nu}(\cos\theta).$$

we see by replacing z with $\boldsymbol{i}r$, where r is our radius, that (†) becomes [after equating *real* and *imaginary* parts, with $v = ½$] …

$\cos(r\cos(\theta))I_0(r\sin(\theta)) \;=\; \Sigma\, (-1)^{n/2}\, r^n P_n(\cos(\theta)) \,/\, \Gamma(n+1)$, n = 0, 2, 4 … real part (1)

$\sin(r\cos(\theta))I_0(r\sin(\theta)) \;=\; \Sigma\, (-1)^{(n-1)/2}\, r^n P_n(\cos(\theta)) \,/\, \Gamma(n+1)$, n = 1, 3, 5 … imaginary part (2)

And here, it should be mentioned that $I_0(x) = J_0(\boldsymbol{i}x)$, where I_0 is a *modified* Bessel function of order 0, and $\boldsymbol{i}$ is $\sqrt{-1}$.

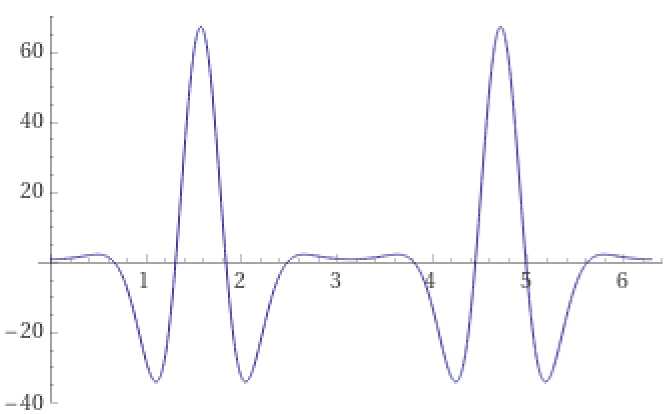
Now (1) is most easily verified by setting r = 1, and θ = π/2. Then the left side is $I_0(1)$, which is roughly 1.266, whilst the right-hand side, out to a few terms, computes to …

$$1 + 1/4 + 3/192 \;\approx\; 1.266$$

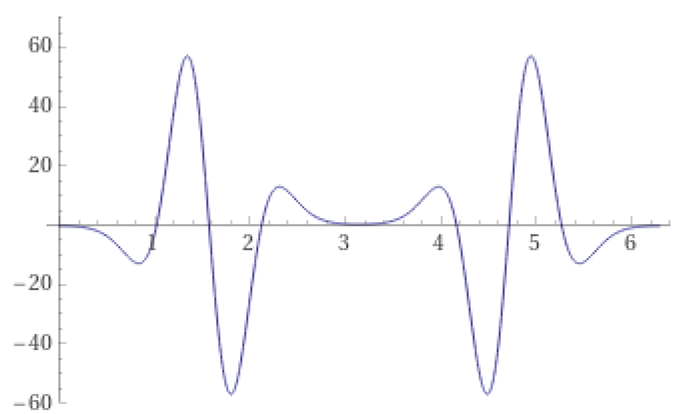
Similarly in (2), setting r = 1 and θ = 0 yields sin(1) in *radians* for the left side, which is approximately 0.841; whilst the right-hand side, out to a few terms, computes to

$$1 - 1/6 + 1/120 \;\approx\; 0.841$$

Both expressions, on the left sides of (1) and (2) obey Laplace, according to our previous research notes.



plot of (1) for r = 6, θ between 0 and 2π



plot of (2) for r = 6, θ between 0 and 2π

A Slight Diversion

As to the form for the field equations, in *three* dimensions, when $g^{u,v}$ is *coupled* to a Yukawa density [pp 237-9], with *physical* singularities associated with $\lambda(s)$ at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1)\ ,$$

this is

$$\begin{aligned} G^{u,v} \approx \sigma[g^{u,v}(0) \ + \ & 2\cosh(r\cos(\alpha))F_0(\omega\sin(\alpha))g^{u,v}(\cos(\alpha)) \ + \\ & 2\cosh(r\sin(\beta))F_0(\omega\cos(\beta))g^{u,v}(\sin(\beta)) \ + \\ & 2\cosh(r\cos(\phi))F_0(\omega\sin(\phi))g^{u,v}(\cos(\phi))]\ . \end{aligned}$$

Here, F_0 is J_0 if $r > \mu$, and F_0 is I_0 if $r < \mu$. As well, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, and $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively. And finally,

$$\text{if } r > \mu\ , \ \omega = \sqrt{r^2 - \mu^2}$$

$$\text{if } r < \mu\ , \ \omega = \sqrt{\mu^2 - r^2}$$

where $\lambda(s) = \sigma e^{-\mu s}/s$.

We are mentioning this now, because at first glance, there doesn't appear to be a way to show that the dark energy components in the equation above obey Laplace. There is nothing in (†) on page 354 that we can use here [unless $\mu = 0$], but perhaps down the road a method will be found.

A Possible Interpretation For The (r, α, β) Layout When $\lambda(s) \approx \sigma / s$

Let us bring back our expression for the field equations, in the case of a *coupling* [page 323], which we'll label (†) …

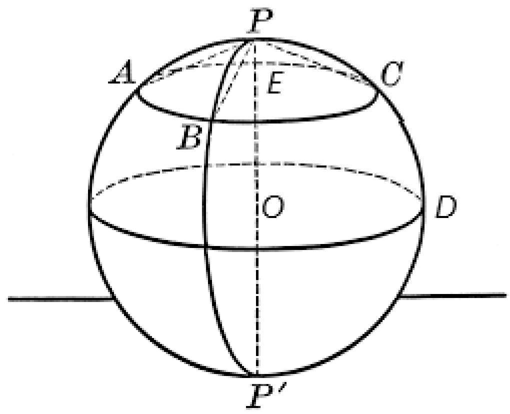
$$\begin{aligned} G^{u,v} \approx \sigma[g^{u,v}(0) \ + \ & 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) \ + \\ & 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) \ + \\ & 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))]\ . \end{aligned}$$

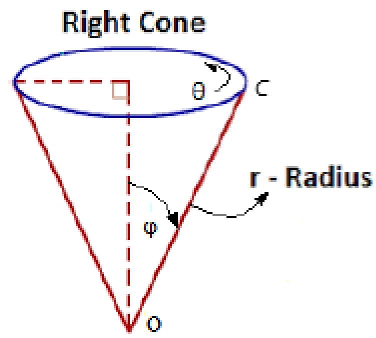
And here, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, and the *physical* singularities associated with $\lambda(s)$ are at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1)\ .$$

Now we already know, from previous research notes, that the dark energy components associated with $g^{u,v}$ satisfy Laplace, so here we want to offer up a possible interpretation for the angle α, so that relative to our coordinate system (r, α, β), we will know how to interpret Laplace.

In the left diagram below, we see a circle AECB traversing a sphere of radius r, at say, a latitude of 45^{o} relative to the equator. The right diagram shows this same circle in blue.





Thus, in an (r, θ, ϕ) layout, ϕ is the 45^{o} arc from P to C along the sphere, so that we have …

$$\cos(\alpha) = (\sqrt{2}/2)\cos(\theta). \qquad (*)$$

Now when $\phi = 90^{o}$, we are in the *x-y* plane π_{xy}, where OD is the *x-axis*, and of course, $\alpha = \theta$ (the *y-axis* is perpendicular to the page). Here the vector V_θ associated with θ, always lies on the *tangent* line to the circle [equator] centered at O, in π_{xy}, so that at D, for example, V_θ would be (0, 1). Note that V_θ is always perpendicular to V_r, the *radius* vector, as θ moves from 0 to 2π, around the equator.

At C in the left diagram above, let π_c be the *tangent* plane to the sphere at C, where C is in the *x-z* plane. Then the *radius* vector V_r is $(\sqrt{2}/2, 0, \sqrt{2}/2)$, emanating from O, so that we have a few possible choices for V_α. One is (0, 1, 0), which would be *tangential* to the circle AECB at C, in the horizontal plane π_{aecb} containing this circle; but this would essentially return us to the (r, θ, ϕ) layout, which we are *not* dealing with right now.

The correct choice, in my opinion, is that vector V_α, which is 45^{o} *above* or *below* the tangent line containing (0, 1, 0) at C, as measured *within* π_c. We make this conclusion because from (*), it is the case that $\alpha = 45^{o}$ if $\theta = 0^{o}$. Note that V_α is still perpendicular to V_r, so that it *emanates* from C *in* π_c.

Now let $\theta = 90^{o}$, turning counter-clockwise, relative to OD. Then (*) tells us $\alpha = 90^{o}$, and here the radius vector V_r is $(0, \sqrt{2}/2, \sqrt{2}/2)$. The most likely choice for our vector V_α, perpendicular to V_r, is the one which is now 90^{o} *above* or *below* the tangent line containing (1, 0, 0) at E, as measured *within* π_e – the *tangent* plane to the sphere at E. And again, V_α is perpendicular to V_r, so that it *emanates* from E *in* π_e.

Thus, we build up a 'field of vectors' on the circle AECB, at differing angles, according to the methodology above, as the angle α varies in accordance with θ.

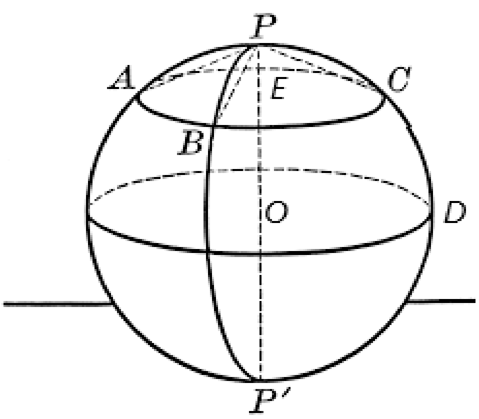
Now we can interpret Laplace in a more meaningful way, as per the previous research notes. For the dark energy components *f* = 2cosh(rcos(α))J_0(rsin(α)) or *f* = 2cosh(rcos(ϕ))J_0(rsin(ϕ)), *f* satisfies Laplace below where we replace θ by α or by ϕ, accordingly. In the case of ϕ, we would be looking at great circles running from the north pole to the south pole on our sphere, at different angles to the *x-axis*, say. And here, V_ϕ would be the *tangent* vector at any point on the circle, but in the plane of the circle itself, and again, perpendicular to the radius vector V_r.

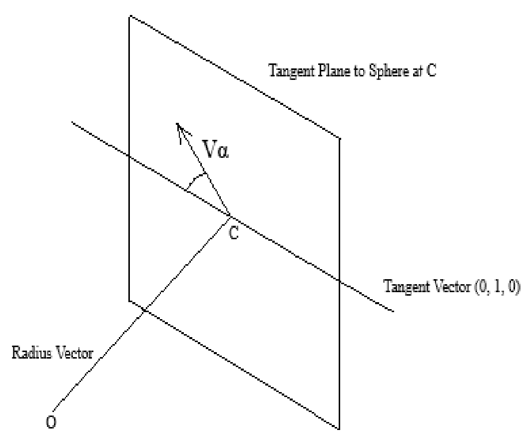
$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right)$$

If *f* = 2cosh(rsin(β))J_0(rcos(β)), the following Laplace operator applies, as per our previous research notes …

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 cos\,\beta}\frac{\partial}{\partial \beta}\left(cos\beta\;\frac{\partial f}{\partial \beta}\right)$$

Thus, $\nabla^2 f\ = 0$ in all cases, where now, perhaps, we have a more meaningful understanding of what the angle α actually is, and hence, β as well.





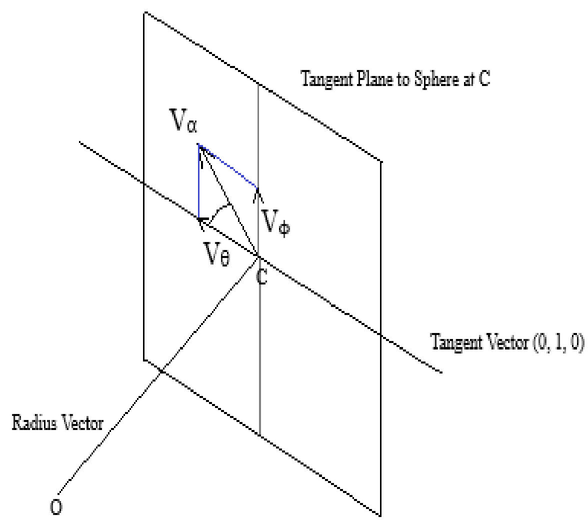
The diagram on the right shows the vector V_α in the *tangent* plane to the sphere at C, at an angle α to the tangent vector (0, 1, 0). Note that both V_α and (0, 1, 0) are perpendicular to the radius vector, and that α is measured *within* the tangent plane.

Some More On a Possible Interpretation For The (r, α, β) Layout When λ(s) ≈ σ / s

In the diagram on the right, just above, notice that V_α can be split up into *two* components – one along the tangent vector (0, 1, 0), which we'll call V_θ, since here we're moving in the direction of θ; and secondly, along the tangent vector to the great circle [APCDP′] running through C in the *x-z* plane, which we'll call V_ϕ. This second tangent vector is actually $(-\sqrt{2}/2, 0, \sqrt{2}/2)$, and emanates

from C in π_c – the *tangent* plane to the sphere at C. Notice that all of V_r, V_θ, and V_ϕ are *perpendicular* to one another, and that again, both V_θ, and V_ϕ lie in π_c, as does V_α.

Now since the *total net* flux [$\mathcal{F}$] of dark energy at C, in the (r, α) layout is *zero* [from previous research notes], it stands to reason [*physically*] that the same must be true at C in the (r, θ, ϕ) layout, where V_r is the radius vector, and V_θ and V_ϕ are the components of V_α, as described above.



This takes care of the dark energy component $f = 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))$ in (†) on page 355, and a similar argument can be made for the (r, β) layout corresponding to $f = 2\cosh(r\sin(\beta))J_0(r\cos(\beta))$, and for the (r, ϕ) layout corresponding to $f = 2\cosh(r\cos(\phi))J_0(r\sin(\phi))$. Thus, the *total net* flux of dark energy at C across *all* three components, taken together in the (r, θ, ϕ) layout, must also be *zero*.

The exercise can now be repeated for any point on the circle AECB, in the left figure on the previous page, and we should arrive at the same conclusion in (r, θ, ϕ). Indeed, anywhere on the sphere itself, since our choice of circles is arbitrary.

Thus, dark energy has no ‘sources’ or ‘sinks’ as we perceive it, in our reality, when $\lambda(s) \approx \sigma / s$. The only source for dark energy is the underlying dark energy density function [λ(s)], and while λ(s) has singularities, they are ‘erased’ at our level by way of the Laplace inverse transform, according to the following expression, in the case of a coupling …

$$G^{u,v} \approx \kappa \int_\gamma e^{sr} \lambda(s) g^{u,v} ds$$

Some More on The Riemann Hypothesis – A Tale of Two Frames, Revisited

In this section we are going to revisit our previous research note [pp 342-3] and simplify things, somewhat. The moderator $\mathcal{M}$ will still play a role here, and we'll simply consider the different possibilities. The reader is encouraged to revisit the previous note before proceeding.

Let us start with the simplest case, where the observer $\mathcal{O}$ is told by $\mathcal{M}$ that $\zeta(p) = \zeta(q)$, and $\mathcal{O}$ is situated at p or q, without knowing what either is. If now, $\mathcal{O}$ trades places with its counterpart, so that had it been at p it would now be at q [and vice versa], nothing changes. Although $\mathcal{O}$ happens to know, at the *point* level, that $\zeta(p)$ and $\zeta(q)$ agree, $\mathcal{O}$ cannot use this knowledge to distinguish between p and q. To $\mathcal{O}$, being situated at p is *exactly the same* as being situated at q, precisely because $\zeta(p) = \zeta(q)$!

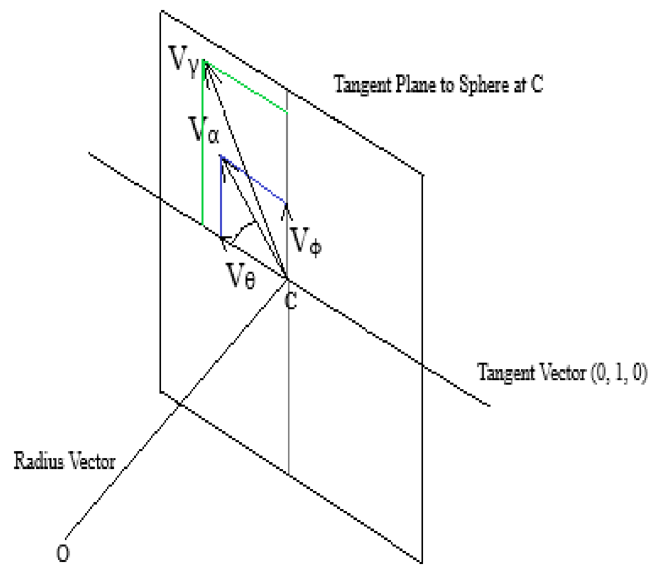
Now suppose $\mathcal{O}$ is told by $\mathcal{M}$ that $\zeta(p) \neq \zeta(q)$, even though $\mathcal{O}$ has no knowledge of what p and q really are. Once again, let $\mathcal{O}$ trade places with its counterpart, so that had it been at p it would now be at q [and vice versa]. Since $\mathcal{O}$ is aware it has traded places, and since $\mathcal{O}$ also knows that $\zeta(p) \neq \zeta(q)$, $\mathcal{O}$ can reason that being situated at p is *not the same* as being situated at q.

$\mathcal{O}$ doesn't have to ask $\mathcal{M}$ if $\zeta(.)$ changed value [p 343]; rather, it simply has to realize that trading places with its counterpart leads to the conclusion above, if $\mathcal{O}$ knows $\zeta(p)$ and $\zeta(q)$ do *not* agree. And furthermore, $\mathcal{O}$ knowing that 'being situated at p is *not the same* as being situated at q' is a contradiction of (*) on page 342, where (*) means 'relative to $\boldsymbol{S}$, it is not possible to distinguish between the α-line and the β-line'. And this is because $\mathcal{O}$ has adopted the signal's [$\boldsymbol{S}$] frame of reference, and also because p belongs to the α-line, and q to the β-line.

Thus, (*) can *only* maintain consistency at the *point* level, whenever $\zeta(p) = \zeta(q)$, and so, our original methodology $\boldsymbol{\mathcal{M}}$, first outlined on pages 294-6, validates (*) at the *point* level, whenever it is the case that $\zeta(p)$ and $\zeta(q)$ agree, but *nowhere* else. In other words, there is no mathematical method $\boldsymbol{\mathcal{M}'}$ that can verify roots or pseudo-roots of $\zeta(s)$ in $\mathcal{C}^*$, the critical strip *minus* the critical line, where verify means 'prove without ambiguity'.

Still More On a Possible Interpretation For The (r, α, β) Layout When $\lambda(s) \approx \sigma / s$

In the diagram below, recall that V_θ, and V_ϕ are the components of V_α, in the *tangent* plane to the



sphere at C, and that the *total net* flux [$\mathcal{F}$] of dark energy at C, in the (r, α) layout is *zero* [from previous research notes]. Thus, it stands to reason [*physically*] that the same must be true at C in the (r, θ, ϕ) layout, where V_r is the radius vector, and V_θ and V_ϕ are the components of V_α, as described in a previous research note [pp 357-8].

Now let's *increase* the flow across C, along both V_θ and V_ϕ, where this increase is measured by the distance between the *vertical* green and blue lines, and the *horizontal* green and blue lines, respectively. Since the *total net* flux in the (r, θ, ϕ) layout is *zero* at C, as per our comment above, it is still going to be *zero* after the increase. That is to say, flow *in* will still match flow *out*, at C.

But notice now, after the increase, that the two corresponding vectors produce a *new* vector V_γ, which is at a *different* angle γ to the tangent line (0, 1, 0), in the tangent plane at C. And since *total net* flux at C hasn't changed, it must be the case that this is *also* true in the (r, γ) layout. Thus, knowing that total net flux is *zero* in the (r, α) layout, means it must be *zero* for *any* other vector emanating from C in π_c – the *tangent* plane to the sphere at C – at *any* angle to (0, 1, 0).

We could have increased or decreased the flow along V_θ and V_ϕ, and we could have done the same for the radius vector V_r, had we wanted to. But the net result is the same; *total net* flux across C would not have changed for *any* other vector emanating from C in π_c. It would always be *zero*.

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Spherical Harmonics, Dark Energy and The Case For Laplace, When $\lambda(s) \approx \sigma / s$
– A Physical *and* Mathematical Approach –

In this note we are going to look at the *full* three dimensional Laplace operator, and take a hybrid approach to understanding Laplace. Up until now, we've looked at 'slices' in the (r, α), (r, β) and (r, ϕ) layouts, so the time has come to see if we can do the same in the (r, θ, ϕ) layout, 'in one shot' as they say.

Recall that in a *mathematical* coordinate system [pp 365-8] and labelling as (*),

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) + 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) + 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] .$$

And here, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, and the *physical* singularities associated with $\lambda(s)$ are at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) .$$

Now in *physical* coordinates, the Laplace operator under consideration is [labelling as (†)],

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} = 0$$

and we already know that Laplace [or a variant] is satisfied by our three dark energy components in (*) above, where the ϕ term in (†) is omitted, and θ is replaced by α, β or ϕ [pp 350-52].

Thus, the same must be true in the *primary* planes [*x-y*, *x-z*, *y-z*] … where dark energy is at its *strongest* [maximum value]. That dark energy is at its maximum here, is most easily seen by revisiting the diagrams on pages 352-3. As we rotate *away* from a *primary* plane, dark energy over a *great* circle of radius r, *decreases* in value and as we rotate *toward* a *primary* plane, dark energy now *increases*, reaching a maximum value in the primary plane, itself. And this is because of the location of the *physical* singularities, associated with $\lambda(s)$.

And furthermore, from our previous research [pp 349-50], we know that when looking at 'slices',

$$2\cosh(r\cos(\theta))J_0(r\sin(\theta)) = 2\cdot\Sigma\, r^n P_n(\cos(\theta)) / \Gamma(n+1) , n = 0, 2, 4 \dots , \quad (\S)$$

for an arbitrary angle θ, and that (§) is *also* true when cos(.) and sin(.) trade places. In particular, it's true for the *primary* planes, where we are dealing with a *two-dimensional* layout, by default.

Notice, now, that the series in (§) *converges* for *all* $r \geq 0$, and that, in particular, it does so in the *primary* planes, where dark energy is at a maximum. What, then, can we say about a dark energy component, such as

$$f = 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha)) ,$$

in a *three-dimensional* layout, say (r, θ, ϕ), where we might not be in a *primary* plane ? Remember, cos(α) = sin(ϕ)cos(θ), so posing the question is actually meaningful.

To answer the question, we have to borrow some mathematics from *spherical* harmonic theory. *Any* solution to Laplace in *three* dimensions [(†), page 361], using *physical* coordinates, is always a linear combination of functions of the type

$$r^n P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi$$

where P_n^m is an *associated* Legendre polynomial and $\{c_{nm}\}$ are constants. That is to say, for r ≥ 0,

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{nm} r^n P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi , \quad (§§)$$

supposing the series converges. Notice now, that when *m* is set to 0, and remains fixed at *zero*, we are summing only over the variable *n*, and recover the series (§) on the last page, where the constant is 2/Γ(n + 1), n = 0, 2, 4 … , because we are dealing with an (r, θ) layout. The angle ϕ is no longer relevant.

Thus, when traversing a great circle which is *not* in a *primary* plane, dark energy can be expressed by (§§) in an (r, θ, ϕ) layout. But does (§§) converge for *all* r ≥ 0 ? After all, it's highly unlikely we are ever going to be able to calculate the constants $\{c_{nm}\}$, which is why we offer a *physical* argument.

Since dark energy, over a great circle, is at a *maximum* in the *primary* planes, representable via (§), the series in (§§) has no choice but to converge, over any great circle *not* in a *primary* plane, and does so for *all* r ≥ 0. And this, in turn, tells us that each dark energy component in (*) [page 361] satisfies Laplace in a *three-dimensional* layout, namely, (r, θ, ϕ). No longer do we need to deal with 'slices' as we have in the last few research notes; rather, we can do it all in 'one fell swoop', according to the approach outlined in this note [see also pp 365-8 for the follow-on note].

Finally, why are we doing this ? To me the answer lies in the solution to the field equations (*) on page 361. What can we expect $g^{u,v}$ to look like ? Since the right-hand side of (*) is comprised of dark energy components that are harmonics [they satisfy Laplace], $g^{u,v}$ is likely to have a harmonic representation itself. Not unlike, perhaps, solutions to Schrodinger's equation. Will there be orbitals or other mysterious phenomena, also embedded within $g^{u,v}$? Only time will tell …

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There are several reasons for putting forth the idea that a dark energy component, such as *f* above, which is *not* in a primary plane, can be resolved via (§§). First, and perhaps foremost, we know

from previous research notes that our dark energy components satisfy Laplace in an (r, θ, ϕ) layout, even when looking at 'slices' [pp 355-8 and page 360]. Second, (§§) above reduces to (§) on page 361 when *m* is fixed at *zero*. Thus, (§) must be a 'special case' of something more general. Third, the elements

$$\{ \quad P_n^{m}(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\; m\phi \quad \}$$

form a basis for regular functions over the unit sphere, that are 'square integrable'. And finally, (§§) must converge for our dark energy component over a great circle, say, which is *not* in a primary plane, and this must be so for *all* r ≥ 0. And that is because (§) converges in a primary plane, where dark energy is at a maximum.

So there is good reason to believe (§§) holds in general for our dark energy components, which of course, is the desirable outcome. Indeed, it may allow us to solve the field equations (*) on page 361, using techniques that are already well understood for other differential equations.

An Interesting Identity Concerning $J_0(\sqrt{r^2-\mu^2}\sin(\theta))$ From Bessel Theory

The expression $J_0(\sqrt{r^2-\mu^2}\sin(\theta))$ appears in our field equations, when using a *Yukawa* density [see page 355], so here we want to write down an expansion for it, in the hopes that it will help us to better understand the dark energy components, within this context. Indeed, the goal, ultimately, is to see if these components obey Laplace.

In G.N. Watson's book titled *A Treatise on The Theory of Bessel Functions* (second edition), there is an interesting identity on page 358, which is reproduced here and labelled as (*) …

$$J_0\{\sqrt{(Z^2+z^2-2Zz\cos\phi)}\} = \sum_{m=0}^{\infty} \epsilon_m J_m(Z)\, J_m(z)\cos m\phi$$

And here, ϵ_m is 1 if $m = 0$, otherwise it is 2. Now let $\phi = \pi/2$, and replace z with iz to obtain …

$$J_0(\sqrt{Z^2-z^2}) = \sum_{m=0}^{\infty} \epsilon_m\, J_m(Z)\, i^m I_m(z)\cos(m\pi/2)$$

Recall that $i^m I_m(x) = J_m(ix)$, where I_m is a *modified* Bessel function of the first kind. Now finally, let $Z = r\sin(\theta)$ and $z = \mu\sin(\theta)$, where r is our radius, and μ appears in the Yukawa density

$$\lambda(s) = \sigma e^{-\mu s}/s.$$

Then our expression above becomes …

$$J_0(\sqrt{r^2-\mu^2}\sin(\theta)) = \sum_{m=0,2,4,\ldots} \epsilon_m\, J_m(r\sin(\theta))\, I_m(\mu\sin(\theta)) \qquad (\S)$$

We can verify this rather simply by setting r = 5, μ = 3 and θ = π/2. Then our series expansion in (§), out to a few terms, becomes …

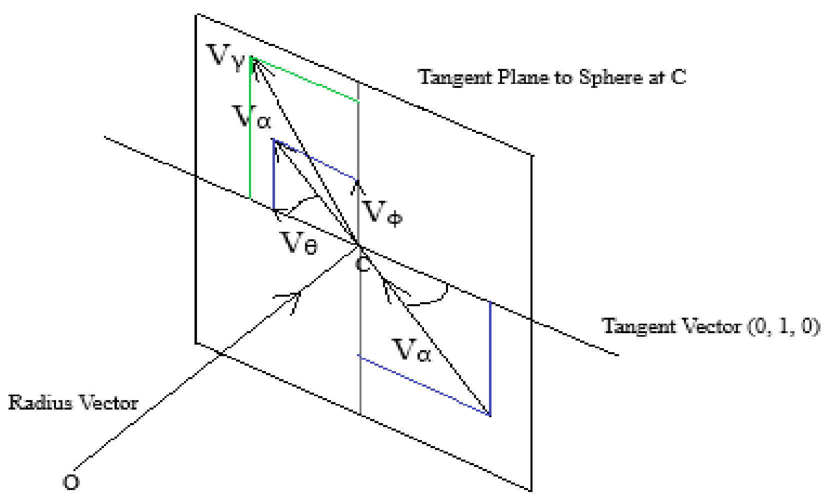
$$J_0(5)\,J_0(3\,i) - 2\,J_2(5)\,J_2(3\,i) + 2\,J_4(5)\,J_4(3\,i) - 2\,J_6(5)\,J_6(3\,i)$$

and, from the Wolfram site, this computes approximately to -0.397179. On the other hand, $J_0(4)$ is approximately -0.397149, so we have very good agreement here.

Whether the identity (§) will prove to be useful, down the road, remains to be seen. But if nothing else, we have another way of looking at the expression $J_0(\sqrt{r^2 - \mu^2}\sin(\theta))$.

Revisiting a Possible Interpretation For The (r, α, β) Layout When $\lambda(s) \approx \sigma / s$

On page 360, we put forth an argument which showed that if total net flux is *zero* in the (r, α) layout, Laplace is *also* satisfied for *any* other vector emanating from C in π_c – the *tangent* plane to the sphere at C – at *any* angle to (0, 1, 0). This argument, however, was a little thin on detail, so here we'd like to expand on things a bit.



In the diagram above, for our dark energy component ξ in (r, α), let V_r^{in} represent flow *in* at C, in the direction of V_r and let V_r^{out} represent flow *out* at C, in the direction of V_r. *Assume* $V_r^{out} = 2V_r^{in}$, and assume also $V_\alpha^{in} = 2V_\alpha^{out}$. Since we know Laplace holds at C, in the (r, α) layout for our dark energy component [ξ], it must be the case that

$$V_r^{in} + V_\alpha^{in} = V_r^{out} + V_\alpha^{out} ;$$

which means $V_r^{in} = V_\alpha^{out}$, and $V_r^{out} = V_\alpha^{in}$. Now write $V_\alpha^{in} = V_\theta^{in} + V_\phi^{in}$ [in the vectorial sense], where $V_\theta^{in} = V_\alpha^{in} \cdot \cos^2(\alpha)$ and $V_\phi^{in} = V_\alpha^{in} \cdot \sin^2(\alpha)$. Thus, with $\alpha = 45^{\mathbf{o}}$, both V_θ^{in} and V_ϕ^{in} are exactly *half* of V_α^{in}. Let's label V_ϕ^{in}, in the (r, α) layout, by the symbol ${}_\alpha V_\phi{}^{in}$.

Now let's *double* the size of V_ϕ^{in}, to *reflect* inflow of ξ in (r, γ), by assumption. Then V_γ^{in} becomes $V_\alpha^{in} + {}_\alpha V_\phi^{in} = 3/2 \cdot V_\alpha^{in}$, so that $V_\gamma^{in} \cdot \sin^2(\gamma) = 2 \cdot {}_\alpha V_\phi^{in} = V_\alpha^{in}$, since again, in (r, γ) V_ϕ^{in} has doubled. Thus, $\sin(\gamma) = \sqrt{2/3}$, so that γ is approximately 55$^\mathbf{o}$.

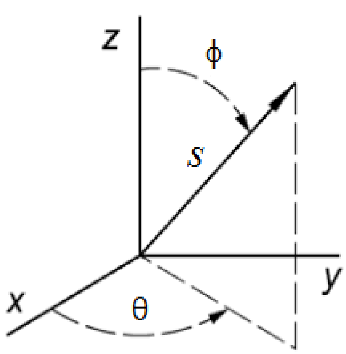
Originally, it was the case that $V_\phi^{in} = 2V_\phi^{out}$ in (r, α), so now, after the doubling, we have $V_\phi^{in} = 4V_\phi^{out}$ in (r, γ), where here $V_\phi^{out} = {}_\alpha V_\phi^{out}$. Thus, the *net* increase on *inflow* is $2V_\phi^{out}$. And, in order to preserve the *angle* of the new vector [V_γ], above *and* below the tangent line containing (0, 1, 0), because of the doubling; V_ϕ^{out} must *also* double in size in (r, γ), supposing that none of the increased outflow travels in the direction of V_θ^{out}. Thus the *net* increase on *outflow* is V_ϕ^{out}.

The net effect of doing this … is that we have an *additional* flow of V_ϕ^{out}, coming from the *net* increase on *inflow*, which has to go somewhere in (r, γ); where V_ϕ^{out} is the *original* outflow in the (r, α) layout, before its doubling. And this 'somewhere' is along V_r^{out}.

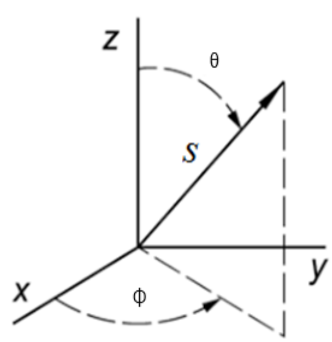
Thus, we see how we can create an inflow/outflow scenario at C for ξ, which satisfies Laplace in the (r, γ) layout, given that Laplace is satisfied by ξ in (r, α). And in this case, it was done where flow *in* didn't necessarily match flow *out* along V_r or V_α, in (r, α) at C. This was the gist of our remarks on page 360, but here we've discussed things in a little more detail.

Spherical Harmonics, Dark Energy and The Case For Laplace, When $\lambda(s) \approx \sigma / s$ – A Physical *and* Mathematical Approach, Part II –

In this note, we wish to continue the discussion on pages 361-3. To begin with, let's distinguish between *mathematical* and *physical* coordinate systems [MCS and PCS, respectively].



Mathematical Coordinates



Physical Coordinates

Most of our earlier work uses MCS, but on page 361 … the Laplacian that is shown there, and reproduced below, is actually using PCS …

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} = 0$$

This is easily remedied by letting θ and ϕ trade places, as we see in the next screenshot, which we'll label (†), and we'll call it $\nabla^2 f$ as well. It is the MCS variant of Laplace.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right)$$

In doing so, solutions to (†), which is to say $\nabla^2 f = 0$, are now of the form

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} c_{nm}\, r^n P_n^m(\cos\phi)\begin{matrix}\cos\\ \sin\end{matrix}\, m\theta\,, \qquad (\S\S)$$

where P_n^m is an *associated* Legendre polynomial and $\{c_{nm}\}$ are constants. This is our starting point for further investigation into the Laplacian nature of our dark energy components in (*), on page 361, using a *mathematical* coordinate system. Note that each term in (§§) satisfies (†) above.

Let us begin with $f = 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))$, where $\cos(\alpha) = \sin(\phi)\cos(\theta)$, in an MCS setting. Then from previous research we know that

$$2\cosh(r\cos(\alpha))J_0(r\sin(\alpha)) = 2\cdot\sum r^n P_n(\cos(\alpha)) / \Gamma(n+1)\,,\ n = 0, 2, 4 \ldots$$

$$= 2\cdot\sum r^n P_n(\sin(\phi)\cos(\theta)) / \Gamma(n+1)\,,\ n = 0, 2, 4 \ldots \qquad (\S)$$

Now if we can write the expression, just above, in the fashion of (§§), we will have succeeded in showing that f satisfies Laplace, in the (r, θ, ϕ) layout [MCS]. Note that here, P_n is just P_n^0.

The idea is to compare coefficients associated with r^n, and see if we can get some sort of agreement between them, in (§§) versus (§). When $n = 0$, which means $m = 0$, P_n^m is identically 1, so setting c_{00} to 2 gives us a match, and we are done.

Now we don't have to look at odd integers for n, but it doesn't hurt to do so. In this case, for $n = 1$, we'll 'pretend' we're summing (§) over all n ≥ 0, and the Legendre polynomials are then …

$$P_1^{-1}(x) = -\tfrac{1}{2}P_1^1(x)$$
$$P_1^0(x) = x$$
$$P_1^1(x) = -(1-x^2)^{1/2}$$

Thus, we see that $P_1^1(\cos\phi) = -\sin(\phi)$, and so in (§§), for $n = 1$ and $m = 1$, we get a match with the expression $P_1(\sin(\phi)\cos(\theta)) = \sin(\phi)\cos(\theta) = -P_1^1(\cos\phi)\cos(\theta)$, if we choose $c_{11} = -2$. The other two constants c_{10} and c_{1-1} can be set to *zero*, and hence we see that $rP_1(\sin(\phi)\cos(\theta))$ satisfies (†).

Now let's go on to $n = 2$. Here the Legendre polynomials are …

$$\begin{aligned}
P_2^{-2}(x) &= \tfrac{1}{24} P_2^2(x) \\
P_2^{-1}(x) &= -\tfrac{1}{6} P_2^1(x) \\
P_2^0(x) &= \tfrac{1}{2}(3x^2 - 1) \\
P_2^1(x) &= -3x(1 - x^2)^{1/2} \\
P_2^2(x) &= 3(1 - x^2)
\end{aligned}$$

and after a bit of algebra, we see that

$$P_2(\sin(\phi)\cos(\theta)) = -1/2 \cdot P_2{}^0(cos\phi) + 1/4 \cdot P_2{}^2(cos\phi)\cos(2\theta)$$

Thus, $r^2P_2(\sin(\phi)\cos(\theta))$ also satisfies Laplace (†), and the hope here is that in the series (§), *any* element $P_n(\sin(\phi)\cos(\theta))$ can be resolved as per (§§), using the examples above as illustrations.

If so, our dark energy components in (*), on page 361, would satisfy Laplace (†), and of course, this is our quest, ultimately … to show that this is true.

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In the case where $n = 3$, the Legendre polynomials are …

$$\begin{aligned}
P_3^{-3}(x) &= -\tfrac{1}{720} P_3^3(x) \\
P_3^{-2}(x) &= \tfrac{1}{120} P_3^2(x) \\
P_3^{-1}(x) &= -\tfrac{1}{12} P_3^1(x) \\
P_3^0(x) &= \tfrac{1}{2}(5x^3 - 3x) \\
P_3^1(x) &= \tfrac{3}{2}(1 - 5x^2)(1 - x^2)^{1/2} \\
P_3^2(x) &= 15x(1 - x^2) \\
P_3^3(x) &= -15(1 - x^2)^{3/2}
\end{aligned}$$

and here, after some mild algebra, the combination that works for us, almost *magically*, is …

$$P_3(\sin(\phi)\cos(\theta)) = 1/4 \cdot P_3{}^1(cos\phi)\cos(\theta) - 1/24 \cdot P_3{}^3(cos\phi)\cos(3\theta)$$

Thus, we see again that $r^3P_3(\sin(\phi)\cos(\theta))$ satisfies Laplace (†), and are probably at a point where we can conclude that our dark energy component $f = 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))$, also satisfies (†), by way of (§), and our approach above.

As to the dark energy component $f = 2\cosh(r\sin(\beta))J_0(r\cos(\beta))$, here $\sin(\beta) = \sin(\phi)\sin(\theta)$, so that we have …

$$2\cosh(r\sin(\beta))J_0(r\cos(\beta)) = 2\cdot\Sigma\, r^n P_n(\sin(\beta)) / \Gamma(n+1)\ ,\ n = 0, 2, 4 \ldots$$

$$= 2\cdot\Sigma\, r^n P_n(\sin(\phi)\sin(\theta)) / \Gamma(n+1)\ ,\ n = 0, 2, 4 \ldots \qquad (\sim)$$

Now the same arguments apply, for here we note that the expression below allows us to choose either cos($m\theta$) *or* sin($m\theta$).

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{nm}\, r^n P_n^{\,m}(\cos\phi) \begin{matrix}\cos\\ \sin\end{matrix}\ m\theta\ , \qquad (\S\S)$$

As an example, if $n = 1$ then $P_1^{\,1}(\cos\phi) = -\sin(\phi)$, and $P_1(\sin(\phi)\sin(\theta)) = \sin(\phi)\sin(\theta)$, and this can be written as $-P_1^{\,1}(\cos\phi)\sin(\theta)$. Thus, $rP_1(\sin(\phi)\sin(\theta))$ satisfies Laplace (†) on page 366, since it is equal to $-rP_1^{\,1}(\cos\phi)\sin(\theta)$, which is a term in (§§).

Finally, we have the dark energy component $f = 2\cosh(r\cos(\phi))J_0(r\sin(\phi))$, which as a summation, can be written as …

$$2\cosh(r\cos(\phi))J_0(r\sin(\phi)) = 2\cdot\Sigma\, r^n P_n(\cos(\phi)) / \Gamma(n+1)\ ,\ n = 0, 2, 4 \ldots\ ,$$

and here we know, from previous research, that this expression satisfies Laplace (†) on page 366, where the θ-term in this operator is omitted [see, for example, pages 355-7].

So I think at this point, we can conclude that *all* of our dark energy components in (*), on page 361, do indeed satisfy Laplace (†) on page 366, in an (r, θ, ϕ) layout [MCS], and the same will be true in a PCS setting as well, so long as we are careful to let θ and ϕ trade places, where applicable.

On The Covariant Nature of The Field Equations When $\lambda(s) \approx \sigma / s$

Let us bring back our expression for the field equations of General Relativity, from page 361, in the case of a *coupling* between the gravitational tensor $g^{u,v}$ and the underlying dark energy density function $\lambda(s) \approx \sigma / s$.

Recall that in a *mathematical* coordinate system [labelling as (*)],

$$\begin{aligned} G^{u,v} \approx \sigma[g^{u,v}(0) \ +\ & 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) \ + \\ & 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) \ + \\ & 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))]\ . \end{aligned}$$

And here, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, and the *physical* singularities associated with $\lambda(s)$ are at the origin O, and at

(1, 0, 0) (-1, 0, 0) (0, 1, 0) (0, -1, 0) (0, 0, 1) (0, 0, -1) .

As well, $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; and $C^{u,v} = R^{u,v} - \frac{1}{2}Rg^{u,v}$, where $R^{u,v}$ is the Ricci tensor and R the Ricci scalar.

Now from pages 178-80, we already know that $C^{u,v}$, $T^{u,v}$ and $g^{u,v}$ are fully covariant, and now have enough evidence to conclude that the dark energy components in (*), on page 368, exhibit covariance as well, in so much as each of them obeys Laplace (†) on page 366.

Thus, we have complete consistency in the field equations above (*), from a *physical* perspective, in so much as it is a fully covariant model, ‘across the board’, as they say.

And finally, the *quantumlike* field equations (*) represent a *coupling* between $g^{u,v}$ and $\lambda(s)$, so we expect they will lead to *non-singular* solutions for $g^{u,v}$, which will also be quantumlike, involving harmonics, among other things. And this is due to the harmonic and quantumlike nature of the dark energy components, which we now believe obey Laplace.

Some Additional Testing

In the case where $n = 4$, the Legendre polynomial is …

$$P_4^0(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

and omitting the factor 1/8, we want to know if $f = r^4P_4(\sin(\phi)\cos(\theta))$ satisfies Laplace (†) on page 366. It should, especially if it can be decomposed as per (§§) on page 368. The good news is it does, but the expressions are complicated, and are listed below for the reader. These are the first, second, and third terms in Laplace (†) on page 366 for f (again, omitting the factor 1/8) …

$$20r^2(35\sin^4(\phi)\cos^4(\theta) - 30\sin^2(\phi)\cos^2(\theta) + 3)$$

$$20r^2(-7\sin^2(\phi)\cos^4(\theta) + 21\sin^2(\phi)\cos^2(\theta)\sin^2(\theta) + 3\cos^2(\theta) - 3\sin^2(\theta))$$

$$20r^2\cos^2(\theta)(-7\sin^4(\phi)\cos^2(\theta) + 28\sin^2(\phi)\cos^2(\phi)\cos^2(\theta) + 3\sin^2(\phi) - 6\cos^2(\phi))$$

After adding these terms together … and using the appropriate trigonometric identities, where applicable, the sum of these terms reduces to …

$$-6\cos^2(\theta) + 6\cos^2(\theta) - 3 + 3 = 0$$

Revisiting an Interpretation For $g^{u,v}(\cos(\theta))$ In The Coupled Case When $\lambda(s) \approx \sigma / s$

On pages 323-5, we put forth an interpretation for $g^{u,v}(\cos(\theta))$, in the case of a *coupling* for our field equations. Here, we’d like to say just a bit more.

Since the argument $\cos(\theta)$ to $g^{u,v}$ is actually a *radial* measure, which we can always make greater than or equal to *zero*, by looking at its absolute value; the same must be true for the field equations in (*) on page 368.

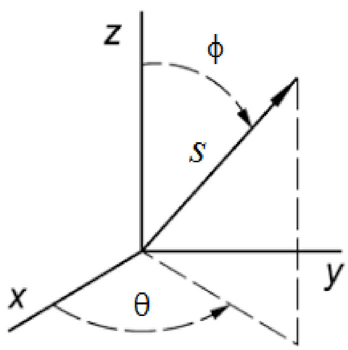
Thus, in a *mathematical* coordinate system, $g^{u,v}(\cos(\alpha))$ should be interpreted as $g^{u,v}(\cos(\alpha), \theta, \phi)$, where $\cos(\alpha) = \sin(\phi)\cos(\theta)$. Similarly, $g^{u,v}(\sin(\beta))$ should be interpreted as $g^{u,v}(\sin(\beta), \theta, \phi)$, where $\sin(\beta) = \sin(\phi)\sin(\theta)$.

And finally, $g^{u,v}(\cos(\phi))$ is to be written as $g^{u,v}(\cos(\phi), \theta, \phi)$, since a great circle running from the north pole to the south pole on our sphere of radius r, could be at any angle θ to the *x-axis*.

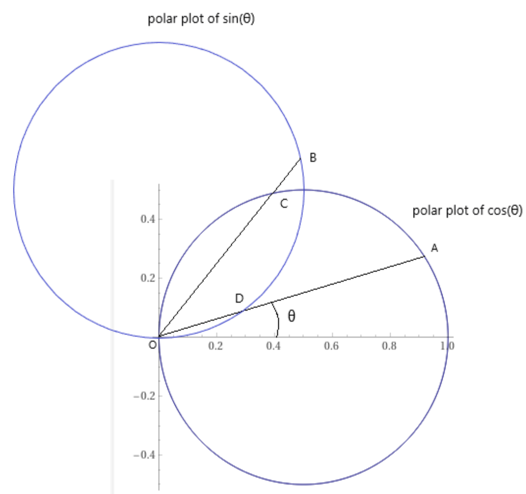
And again, the *physical* singularities associated with $\lambda(s)$ are at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1)\ ,$$

in this case [pp 368-9].



Mathematical Coordinates



In the diagram above, on the right, are the *polar* plots of $\cos(\theta)$ and $\sin(\theta)$. As θ increases in value from 0 to $\pi/2$, the *radial* segment OA, which is $\cos(\theta)$, *decreases* … and the radial segment OD [$\sin(\theta)$] *increases*.

For our *coupled* equations in *two* dimensions, and reproduced here … with *physical* singularities at the origin O and at $\{(\pm 1, 0), (0, \pm 1)\}$; $g^{u,v}(\cos(\theta))$ is a *measure* of how much the singularities at $(\pm 1, 0)$ influence the gravitational tensor, as we move away from them or toward them, relative to the dark energy component $f = 2\cosh(r\cos(\theta))J_0(r\sin(\theta))$. And similarly for $g^{u,v}(\sin(\theta))$ and $(0, \pm 1)$.

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta)) + 2\cosh(r\sin(\theta))J_0(r\cos(\theta))g^{u,v}(\sin(\theta))]$$

As to the *left*-hand side of the field equations, be it in the expression above or (*) on page 368, this is vintage Einstein. Everything is to be calculated in the usual way, as though, as it were, we had set σ to *zero*.

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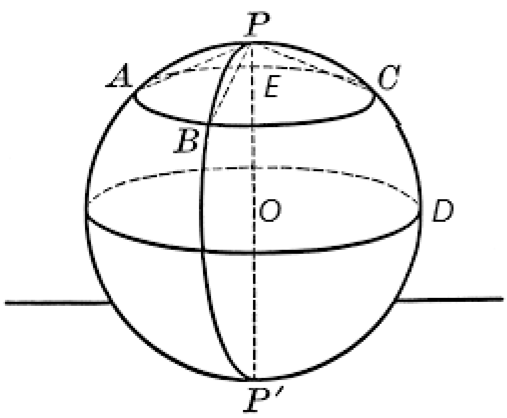
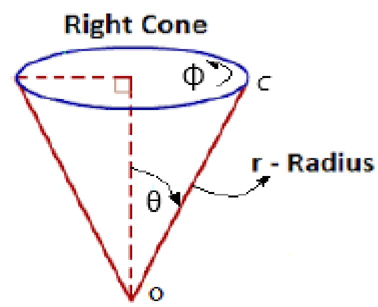
Does $e^{r\cos(\theta)}J_1(r\sin(\theta))\cos(\phi)$ Satisfy Laplace in Spherical Coordinates ?

On page 125 of G.N. Watson's book, titled *A Treatise on The Theory of Bessel Functions* (second edition), he notes that expressions of the type [labelling as (*)]

$$e^{kz}\,{\cos \atop \sin}\, m\phi \,.\, J_m(k\rho)$$

satisfy Laplace in *cylinder* coordinates [ρ, ϕ, z]. Thus, one could ask the question if the same is true in *spherical* coordinates, using a *physical* coordinate system, as shown below.





In the left diagram above, let $\mathcal{C}$ be the cylinder in the sphere, with height OE = *z* and diameter AC, so that its radius ρ is just ½ · AC. The *x*-axis, we should say, is in the direction OD.

Let us now rotate the ρ-z axes by $90^{o} - \theta$, so that the vector V_z now becomes V_θ at C – the tangent vector to the great circle APCDP' running through C. Similarly, V_ρ becomes V_r at C – the radius vector in the direction OC. And as to V_ϕ at C, nothing changes.

Now since this is a simple rotation, Laplace is *still* satisfied by (*) in *cylinder* coordinates, but by performing this rotation, we are now in an (r, θ, ϕ) layout, so that (*) should satisfy Laplace in *spherical* coordinates. Let us see if this is so, where in (*) we let $k = 1$, $m = 1$, $z = r\cos(\theta)$ and $\rho = r\sin(\theta)$, so that our function *f*, for the case of $\cos(\phi)$, is just

$$f = e^{r\cos(\theta)}J_1(r\sin(\theta))\cos(\phi).$$

Now let us bring back our expression from page 349, as shown below, which we'll label (†) …

$$e^{z\cos\theta}\, J_{\nu-\frac{1}{2}}(z\sin\theta) = \frac{\Gamma(\nu)}{\Gamma(\frac{1}{2})}(2\sin\theta)^{\nu-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^{\nu+n-\frac{1}{2}}}{\Gamma(2\nu+n)}\, C_n^{\nu}(\cos\theta).$$

And here in (†), we'll set $v = 3/2$ and z = r, so that f is the right-hand side of (†), multiplied by the term cos(ϕ). We'll call this g. Now when $n = 0$, the *relevant* term in g [meaning we omit any constants] is t = rsin(θ)cos(ϕ), since $C_0^v = 1$ for any choice of v.

In *physical* coordinates, any term in the series [pp 361-3]

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{nm} r^n P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\ m\phi\ , \qquad (\S\S)$$

always satisfies Laplace, where the Laplace operator is the one shown on page 361, and reproduced here …

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} = 0$$

And since t can be written as $-rP_1^1(\cos\theta)$cos(ϕ) [pp 365-8], we see that t satisfies Laplace above.

Now let's go on to $n = 1$. In this case, the *relevant* term in g is t = r^{2}sin(θ)cos(θ)cos(ϕ), after examining $C_1^{3/2}$, and this can be written as $-\,1/3 \cdot r^2 P_2^1(\cos\theta)$cos(ϕ). Thus, we see again that t obeys Laplace, just as it does when $n = 0$.

We could look at $n = 2$, but if we did, I can confirm for the reader that the *relevant* term in g matches $r^3P_3^1(\cos\theta)$cos(ϕ), up to some constant, after examining $C_2^{3/2}$.

And so, we have enough evidence now, to conclude that f does indeed satisfy Laplace above, and we'd expect the same if f was equal to

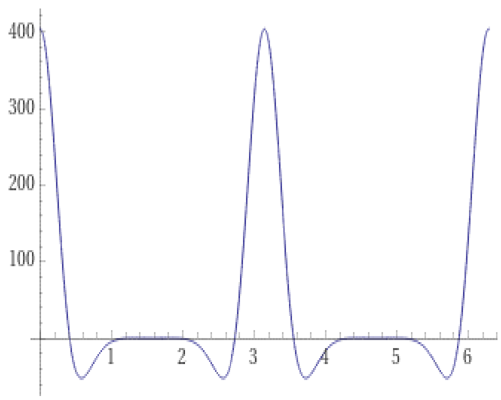
$$f = e^{r\cos(\theta)} J_1(r\sin(\theta))\sin(\phi).$$

Indeed, the choice of m shouldn't matter either, except that when matching with the associated Legendre polynomials, we'd be looking at P_n^m. And finally, the constant k was set to 1 in our studies, but it too can be arbitrary.

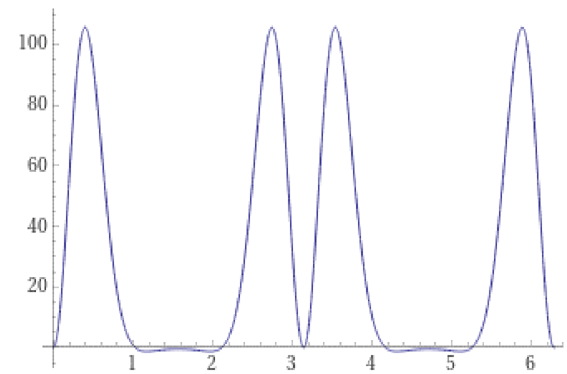
Some Plots of $f = 2\cosh(r\cos(\theta))J_m(r\sin(\theta))\cos(m\phi)$ For Even Values of m

For *even* values of m, we can generate the function f just above, by summing (†) on page 371 for both z = r and $z = -r$, and then adding the two expansions together. And because J_m is an even function when m is even, we arrive at f for m = 0, 2, 4, … all of which, obey Laplace.

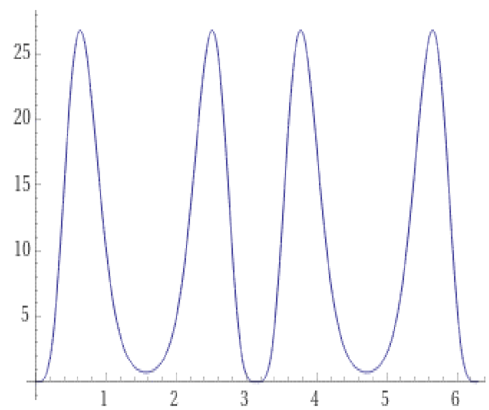
Note that when $m = 0$, we are dealing with a fundamental dark energy component f_0, in a *primary* plane [*x-y*, *x-z*, *y-z*], so that for even $m > 0$, we might be looking at 'echoes' of f_0 … in an (r, θ, ϕ) layout. It is just speculation at this point, but because f does obey Laplace for all even m, in particular, the echoes, if we wish to call them that, may have some physical significance, whatever that turns out to be. Here are the plots, using *physical* coordinates …



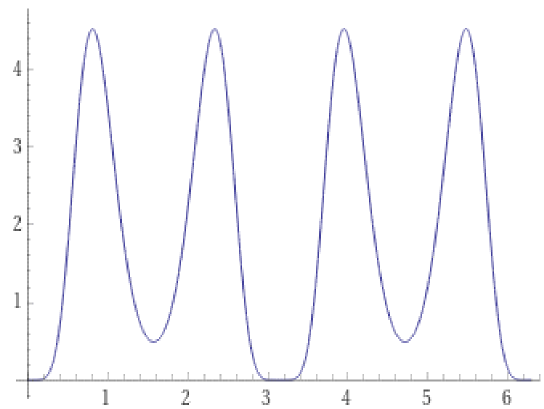
m = 0, ϕ = 0, r = 6, θ = x



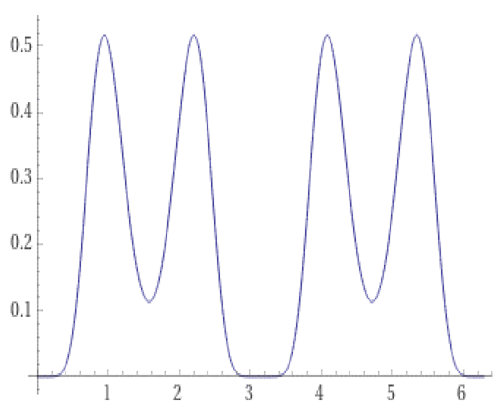
m = 2, ϕ = 0, r = 6, θ = x



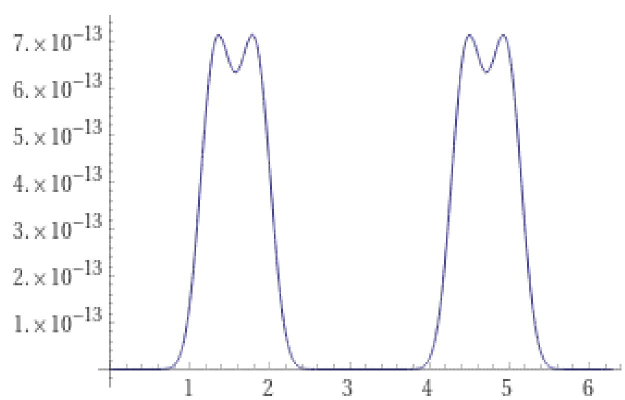
m = 4, ϕ = 0, r = 6, θ = x



m = 6, ϕ = 0, r = 6, θ = x



m = 8, ϕ = 0, r = 6, θ = x



m = 24, ϕ = 0, r = 6, θ = x

*****

For our dark energy component f = 2cosh(rcos(α))J_0(rsin(α)), we saw on pages 365-8 that for the case of n = 2, f could be resolved as (omitting r) … …

$$P_2(\sin(\phi)\cos(\theta)) \;=\; -1/2 \cdot P_2^{\,0}(\cos\phi) \;+\; 1/4 \cdot P_2^{\,2}(\cos\phi)\cos(2\theta)$$

For our dark energy component $f = 2\cosh(r\sin(\beta))J_0(r\cos(\beta))$, we looked at the case $n = 1$, but did not look at the case $n = 2$. In this case ($n = 2$), f can be resolved as (omitting r) …

$$P_2(\sin(\phi)\sin(\theta)) \;=\; -1/2 \cdot P_2^{\,0}(\cos\phi) \;-\; 1/4 \cdot P_2^{\,2}(\cos\phi)\cos(2\theta)$$

Thus, we have still more evidence that our dark energy components do, indeed, satisfy Laplace …

A Mean Value Theorem For Dark Energy When $\lambda(s) \approx \sigma / s$

Recall that in a *mathematical* coordinate system [labelling as (*)], our field equations are …

$$\begin{aligned} G^{u,v} \;\approx\; \sigma[g^{u,v}(0) \;+\; & 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) \;+ \\ & 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) \;+ \\ & 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] \;. \end{aligned}$$

And here, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, and the *physical* singularities associated with $\lambda(s)$ are at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1)\;.$$

Now the dark energy component [ξ], associated with the first gravitational component $g^{u,v}(0)$, is simply σ, so that over a sphere's surface [S] of radius r centered at the origin O, the *average* value of ξ *on* S is also going to be σ. And this, in turn, is equal to the value of ξ at the origin O.

For the dark energy component $\xi = 2\sigma\cosh(r\cos(\alpha))J_0(r\sin(\alpha))$, associated with $g^{u,v}(\cos(\alpha))$, we know from previous research [pp 351-2] this can be written as …

$$g = 2\sigma\cdot\sum r^n P_n(\sin(\phi)\cos(\theta)) \,/\, \Gamma(n+1)\;,\; n = 0, 2, 4 \ldots \qquad (\S)$$

If we now integrate ξ over S, where ϕ ranges from 0 to π, and θ ranges from 0 to 2π, and recall that our Jacobian J is $r^2\sin(\phi)$, then this is the equivalent of integrating $g \cdot J$ over S.

When $n = 0$, $P_n(\sin(\phi)\cos(\theta))$ is equal to 1, so our integrand $\mathcal{J}$ becomes $2\sigma \cdot J$, and the integration over S now computes to $2\sigma \cdot 4\pi r^2$, exactly.

When $n = 2$, $P_2(\sin(\phi)\cos(\theta))$ is equal to $½ \cdot (3\sin^2(\phi)\cos^2(\theta) - 1)$, so that the *relevant* term in $g \cdot J$ (meaning we omit any constants and also r) is $\mathcal{J} = 3\sin^3(\phi)\cos^2(\theta) - \sin(\phi)$. If we now integrate $\mathcal{J}$ over S, the value is 0. And similarly for any other choice of $n > 0$, be it *even* or *odd*. The value of the integration, for the relevant term, will always be *zero*.

Thus, the only term that matters is $n = 0$, so that the *average* value of ξ over S is, in fact, $2\sigma \cdot 4\pi r^2$ divided by the *area* of the sphere S, which gives us 2σ. And this is also equal to ξ at the origin O, meaning as r ---› 0 in ξ.

A similar exercise can be carried out for the other dark energy components [ξ] in (*), and we will come to the same conclusion; namely that the *average* value of ξ over S is equal to the value of ξ at the origin O. And this value is σ for the first component, and 2σ for the remaining components.

Thus, the *average* value of *all* of the dark energy components over S, taken together, is 7σ, which is equal to the *sum* of the dark energy components at O.

An Interesting Identity From Bessel Theory

Let us recall our expression from page 349, reproduced below …

$$e^{z\cos\theta} J_{\nu-\frac{1}{2}}(z\sin\theta) = \frac{\Gamma(\nu)}{\Gamma(\frac{1}{2})}(2\sin\theta)^{\nu-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^{\nu+n-\frac{1}{2}}}{\Gamma(2\nu+n)} C_n^{\nu}(\cos\theta).$$

If we set *v* = 1 in this expression, and use the fact that

$$J_{1/2}(r) = \sqrt{2/\pi r}\,\sin(r)\,,$$

then it is not too hard to show that

$$e^{r\cos(\theta)}\sin(r\sin(\theta)) \,/\, r\sin(\theta) = \Sigma\, r^n C_n^1(\cos(\theta)) \,/\, \Gamma(n+2)\,,\ n = 0, 1, 2, 3 \ldots$$

On The Evaluation Of a Certain Definite Integral

From our expression just above, we may write

$$e^{r\cos(\theta)}\sin(r\sin(\theta))\sin(\theta) = \Sigma\, r^{n+1} C_n^1(\cos(\theta))\sin^2(\theta) \,/\, \Gamma(n+2)\,,\ n = 0, 1, 2, 3 \ldots \qquad (*)$$

Now from the literature, it is the case that if $n \neq m$, then the following is true for Gegenbauer polynomials …

$$\int_{-1}^{1} C_n^{(\alpha)}(x) C_m^{(\alpha)}(x)(1-x^2)^{\alpha-\frac{1}{2}}\,dx = 0.$$

Thus, with α = 1 and $m = 0$, it must be the case that for all $n > 0$, with $x = \cos(\theta)$, we have

$$\int_0^{\pi} C_n^1(\cos(\theta))\sin^2(\theta)\,d\theta = 0\,,$$

and so, upon integrating the left-hand side of (*) above, from 0 to π, only the *first* term ($n = 0$) on the right-hand side of (*) matters. And this computes to πr/2, since $C_0^1(x) = 1$.

Thus, it is the case that

$$\int_0^{\pi} e^{r\cos(\theta)}\sin(r\sin(\theta))\sin(\theta)\,d\theta \;=\; \pi r/2\,,$$

for any choice of $r \geq 0$.

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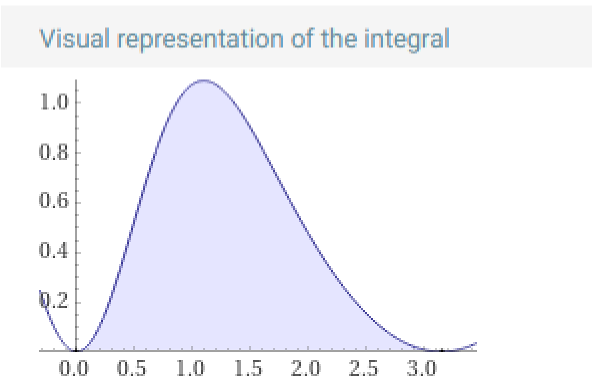
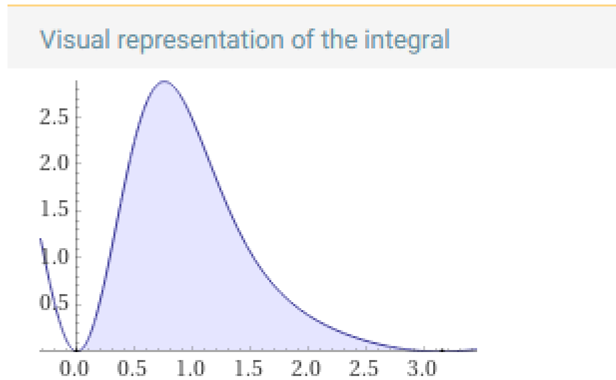
$$\int_0^{\pi} e^{\cos(x)}\sin(x)\sin(\sin(x))\,dx = 1.5708$$



$$\int_0^{\pi} e^{2\cos(x)}\sin(x)\sin(2\sin(x))\,dx = 3.14159$$



And there are many other techniques that can be used as well, for evaluating definite integrals, some of which you will find in Chapter 12 of G.N. Watson's magnificent work titled *A Treatise on The Theory of Bessel Functions* (second edition).

Undecidability, The Riemann Hypothesis and The Continuum Hypothesis, Part I

The Continuum Hypothesis (CH) is an *undecidable* statement within the framework of Zermelo-Fraenkel set theory, that incorporates the axiom of choice. Specifically, CH asserts that there are *no* sets whose cardinality lies between the integers and the real numbers (often referred to as *intermediate* cardinality). From the work of Cantor, and then later on Godel and Cohen, it was shown that ultimately CH cannot be decided.

Let $\mathcal{S}$ be the set of all sets with intermediate cardinality, and assume for the moment that CH is indeed undecidable (we know it is, but to form an equivalency here, which is what we're after, we'll proceed by making this assumption). Then there is no mathematical method $\mathcal{M}'$ that can verify any member T of $\mathcal{S}$, where verify means 'prove without ambiguity'. Thus, if CH is undecidable, then T is unverifiable.

Conversely, suppose T is any member of $\mathcal{S}$ and that T is unverifiable. Then $\mathcal{M}'$ will still conclude that $\mathcal{S}$ is empty [$\mathcal{S} = \{\Phi\}$], even though this is really not the case. In turn, this leaves CH in an

indeterminate state because $\mathcal{M}'$ is no longer trustable. $\mathcal{M}'$ would be telling us the truth if $\mathcal{S}$ was really empty, but $\mathcal{M}'$ would *not* be telling us the truth if $\mathcal{S}$ was really not empty. Thus, if T is unverifiable, then CH is undecidable. And so we have the following equivalency …

CH is undecidable if and only if T is unverifiable, where T belongs to $\mathcal{S}$ (*)

Now replace CH with RH, where RH is the Riemann Hypothesis, and replace T by a root or pseudo-root [ω] in $\mathcal{C}^*$ – the critical strip *minus* the critical line. Then from (*) above, we can conclude that since the undecidability of CH is *equivalent* to the unverifiability of T, it must be the case that the undecidability of RH is *equivalent* to the unverifiability of ω in $\mathcal{C}^*$.

RH is undecidable if and only if ω is unverifiable, where ω belongs to $\mathcal{C}^*$ (†)

Thus it follows that if we can show ω is unverifiable in $\mathcal{C}^*$, we will have demonstrated indeterminacy in RH. And this we did, starting with our methodology on pages 294-6, and going forward from there [see for example, pp 342-3 and page 359].

So by using CH as our guide, so to speak, we realize from our earlier work on RH, that the Riemann Hypothesis has no answer; it too is undecidable, if we accept the methods first outlined on pages 294-6. And I think the methods there are acceptable, despite the fact that undecidable arguments, in general, carry with them a fair degree of controversy, no matter the audience …

Undecidability, The Riemann Hypothesis and The Continuum Hypothesis, Part II

In this brief note, we're going to look a little more closely at the two statements that are connected to the Continuum Hypothesis (CH), and then tie this back to the Riemann Hypothesis (RH). Let $\mathcal{S}$ be the set of all sets with intermediate cardinality and let T be a member of $\mathcal{S}$. Then these statements are, within the context of ZFC (Zermelo-Fraenkel set theory, plus the axiom of choice) …

(1) there is no way to prove that T exists and
(2) there is no way to prove that T does not exist

If now $\mathcal{M}'$ is some mathematical method seeking to resolve CH, and based on ZFC, then by (1) $\mathcal{M}'$ will conclude that $\mathcal{S}$ is empty. However, by (2) $\mathcal{M}'$ can't conclude (unequivocally) that $\mathcal{S}$ is empty, which means $\mathcal{M}'$ is *not* trustable. $\mathcal{M}'$ could very well be telling us the truth if $\mathcal{S}$ was really empty, but $\mathcal{M}'$ would *not* be telling us the truth if $\mathcal{S}$ was really not empty. And this is the *same* conclusion we came to in the previous note above, because of the unverifiability of T.

Thus CH is undecidable and for the *same* reasons, RH as well. If $\mathcal{T}$ is the set of all roots and pseudo-roots of ζ(s) in $\mathcal{C}^*$ – the critical strip *minus* the critical line – then any method $\mathcal{M}'$ based on ZFC principles, that is seeking to resolve RH, one way or the other, will not be able to do so. And that is because any ω in $\mathcal{T}$ is unverifiable, if we accept our methodology, first outlined on pages 294-6.

Undecidability, The Riemann Hypothesis and The Continuum Hypothesis, Part III

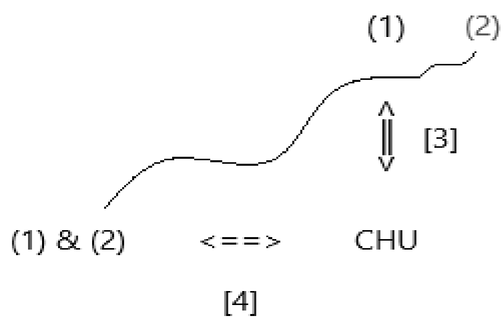
Building on Parts I and II above, we see from Part I that the undecidability of CH (which we'll call CHU), is *equivalent* to statement (1), whereas in Part II, CHU is equivalent to *both* statements (1) and (2). Again, these statements are, where T is a member of ***S***, the set of all sets of intermediate cardinality …

(1) there is no way to prove that T exists and
(2) there is no way to prove that T does not exist (unequivocally)

Thus in Part I, we use (1), but *infer* (2) must hold, whereas in Part II, *both* (1) and (2) are actually used, together.

(3) CHU is equivalent to (1), Part I
(4) CHU is equivalent to (1) and (2), Part II

Now since (3) and (4) hold, it must be the case by *equivalency* that statement (2) is actually *inherited* in (3) from (4), even if *prima facie* this isn't obvious. Thus, a sufficient test for the *undecidability* of CH is really statement (1), where statement (2) is *tacitly* present. We infer statement (2) in (3), as we said, but now learn that it was really *part of* (3), all along !



Now the proof of (2) above is due to Cohen, but based on what we've just said, it seems to have a broader meaning. In the case of CH and the set ***S***, it appears in (3) as a *silent* partner, so to speak, *because* of the unverifiability of T. Indeed, statement (2) really isn't specific to CH at all, in my opinion; rather it surfaces in ZFC (Zermelo-Fraenkel set theory, plus the axiom of choice) because the elements of ***S*** cannot be verified. Thus, statement (2) is really commentary on the limitations of ZFC, itself, within the context of unverifiable sets.

If we are on the right track here, the same will be true for RH, the Riemann Hypothesis. If $\mathcal{T}$ is the set of all roots and pseudo-roots of $\zeta(s)$ in $\mathcal{C}^*$ – the critical strip *minus* the critical line – then $\mathcal{T}$ is a set of unverifiable elements [pp 294-6]. Thus, by what we've just said, statement (1) is a sufficient test for the undecidability of RH because statement (2) is implicit – it is there by default, so to speak, within the ZFC framework. And so, we don't actually have to go out and prove (2) directly,

for RH, as Cohen did for CH [here, of course, T becomes a root or pseudo-root of ζ(s) in $\mathcal{C}^*$, in these statements, and $\mathcal{S}$ becomes $\mathcal{T}$].

In other words, because the elements of $\mathcal{T}$ are unverifiable, it is not necessary to prove 'there is *no* mathematical method $\mathcal{M}'$ that can conclude [unequivocally] $\mathcal{T}$ is empty'. The limitations of ZFC will do this for us, within the context of unverifiable sets … …

On The Evaluation Of a Certain Definite Integral, Part II

If we differentiate *n* times under the integral sign [d/dr], in the expression below from Part I [pp 375-6] …

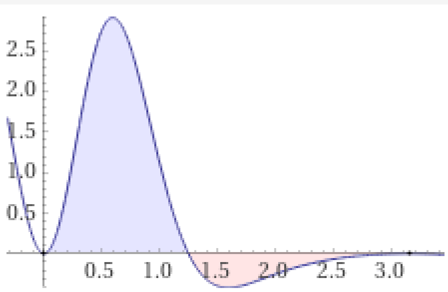
$$\int_0^{\pi} e^{r\cos(\theta)} \sin(r\sin(\theta)) \sin(\theta)\, d\theta \;=\; \pi r/2\,,$$

then it is not too hard to show that for any *real* r and integer *n* …

$$\begin{aligned}\int_0^{\pi} e^{r\cos(\theta)} \sin(r\sin(\theta) + n\theta) \sin(\theta)\, d\theta \;&=\; \pi r/2, \text{ if } n = 0\\ &=\; \pi/2\,, \text{ if } n = 1\\ &=\; 0 \quad\;, \text{ if } n > 1\end{aligned}$$

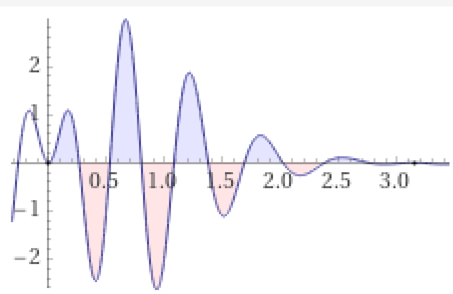
$$\int_0^{\pi} e^{2\cos(x)} \sin(x) \sin(x + 2\sin(x))\, dx = 1.5708$$

Visual representation of the integral



$$\int_0^{\pi} e^{2\cos(x)} \sin(x) \sin(2\sin(x) + 10\,x)\, dx = 0$$

Visual representation of the integral



Undecidability and Ghosts In The Room

Let GH be the ‘ghost hypothesis’ which asserts that there are no ghosts in the room R. And let $\mathcal{D}'$ be an apparatus designed to detect ghosts, and built according to some set of principles. And define the following statements thusly, where G is a ghost that belongs to the set of all ghosts $\mathcal{G}$ in R …

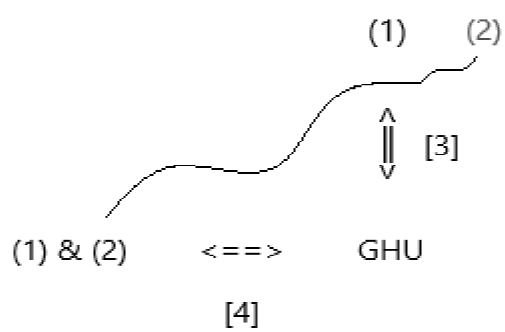
(1) there is no way for $\mathcal{D}'$ to show that G exists and
(2) there is no way for $\mathcal{D}'$ to show that G does not exist (unequivocally)

Following our reasoning in Part I [pp 376-7] we see that if GH is *undecidable* [which we’ll label GHU] then (1) holds, and conversely. And, following our reasoning in Part II [page 377], we see that if GH is *undecidable*, then (1) & (2) hold, and conversely. Thus,

(3) GHU is equivalent to (1) above
(4) GHU is equivalent to (1) and (2) above

To expand on things a bit, suppose GHU holds. Then $\mathcal{D}'$ cannot detect any ghost in R, so (1) follows. Conversely, if (1) holds, $\mathcal{D}'$ will conclude there are no ghosts in R, even if this isn’t so. Thus $\mathcal{D}'$ is *not* trustable, and so (3) follows.

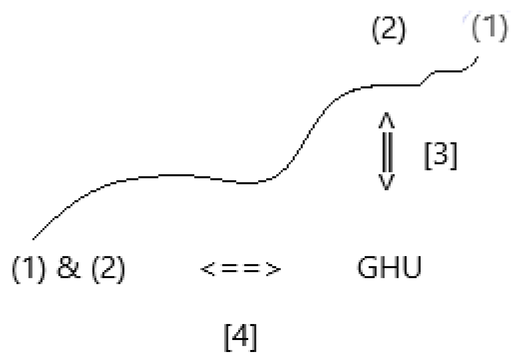
Now if again GHU holds, then clearly (1) and (2) above are true. Conversely, if (1) & (2) hold, then by (1) $\mathcal{D}'$ will conclude there are no ghosts in R, but by (2) $\mathcal{D}'$ cannot say this, *unequivocally*. Thus, $\mathcal{D}'$ is again *not* trustable and (4) follows.



we can never know for sure that there are
ghosts in the room

So just as in Part (III) [pp 378-9], it must be the case that statement (2) is inherited in (3) from (4), because of *equivalency*, which means (1) is a sufficient test for the undecidability of GH. And indeed, it is not hard to show that (2) is *also* a sufficient test for the undecidability of GH, making one wonder if statements (1) and (2) really are two sides of the same coin …

To wit, if GHU holds, then $\mathcal{D}'$ cannot say there are no ghosts in R, *unequivocally*, which is (2). Conversely, if (2) holds – meaning $\mathcal{D}'$ cannot say, *definitively*, there are no ghosts in R – then $\mathcal{D}'$ cannot confirm there really are *no* ghosts in R, if, in fact, this is the case. In turn, this makes GH undecidable, and so (2) is equivalent to GHU, and thus (1) is *inherited* in (3) from (4), by equivalency … …



we can never know for sure that there are
no ghosts in the room

OTHER CONSIDERATIONS

Let us bring back our two statements from page 380, where $\mathcal{D}'$ is our detector and G is a ghost in the room R …

(1) there is no way for $\mathcal{D}'$ to show that there are ghosts in R
(2) there is no way for $\mathcal{D}'$ to show that there are no ghosts in R (unequivocally)

Now suppose it is the case that (2) holds but (1) doesn't, so that $\mathcal{D}'$ has the ability to verify ghosts in R. Let there be $n > 1$ ghosts in R, and let them begin to leave, one by one. After the first ghost leaves, $\mathcal{D}'$ will know there are $n - 1$ ghosts remaining, because it now has the ability to verify them.

Eventually, there will only be *one* ghost G left in R, and once it leaves, $\mathcal{D}'$ will say it couldn't verify that G left, because if it could, $\mathcal{D}'$ would be able to say, *unequivocally*, that there were *no* ghosts in R. And this is a violation of (2). So on the one hand $\mathcal{D}'$ can verify that the first $n - 1$ ghosts have left the room, but on the other, it can't verify that the last ghost has left the room (in fact $\mathcal{D}'$ will continue to report $n = 1$ even though the true value of n is now 0).

Well, you can't have it both ways. Either $\mathcal{D}'$ can verify these ghosts have left, or it can't. If it can, we run into a contradiction. Therefore, if (2) holds, (1) must hold as well.

On the other hand, suppose (1) holds but (2) doesn't, so that $\mathcal{D}'$ can say, *unequivocally*, there are no ghosts in R. Now let ghosts start to enter the room, one by one. Initially, there are no ghosts in R, and since (2) no longer holds, $\mathcal{D}'$ can say, *unequivocally*, that $n = 0$. But as another ghost G enters

R, $\mathcal{D}'$ will *not* be able to verify G by (1), and so concludes that n is *still* 0. And indeed, it won't matter how many ghosts there are in the room – $\mathcal{D}'$ will continue to report that $n = 0$. Thus, $\mathcal{D}'$ can't say, *unequivocally*, that $n = 0$, only when there really are no ghosts in the room. Our assumption that (2) doesn't hold produces a contradiction, so that if (1) holds, (2) must hold also.

Putting the two results together suggests that (1) and (2) are actually *equivalen*t to one another. If we have (1) then we must have (2), and vice-versa. Thus, (1) and (2) appear to be two sides of the same coin, as we said earlier.

In turn, this means that *either* (1) *or* (2) is a sufficient test for the undecidability of GH, our 'ghost hypothesis' (page 380). If we opt for (1), we don't need to prove (2), because it will be true by default, and if we opt for (2), we don't need to prove (1), for the same reason.

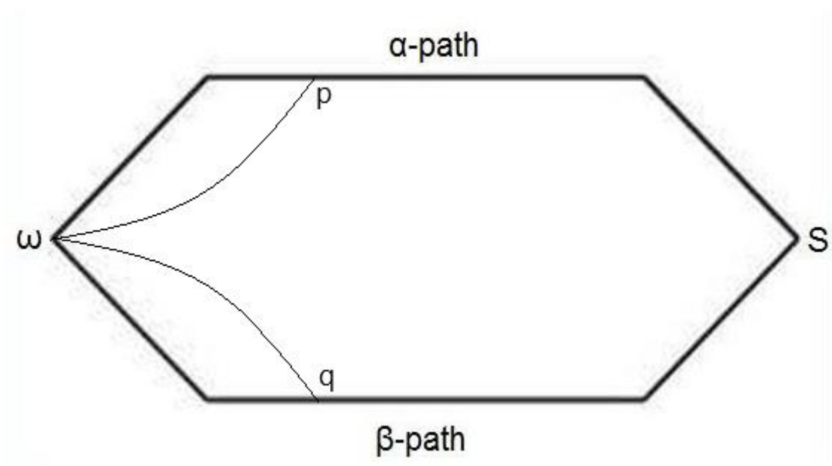
And indeed, this will be true for the Riemann Hypothesis (RH) as well. If $\mathcal{T}$ is the set of all roots and pseudo-roots of ζ(s) in $\mathcal{C}^*$ – the critical strip *minus* the critical line – then these statements are, where $\mathcal{M}'$ is some mathematical method …

(1) there is no way for $\mathcal{M}'$ to show that $\mathcal{T}$ is not empty
(2) there is no way for $\mathcal{M}'$ to show that $\mathcal{T}$ is empty (unequivocally)

And from what we've just said concerning GH, we can demonstrate *undecidability* in RH by choosing *either* (1) *or* (2). In the essay, we chose (1), as per the methods first outlined on pages 294-6 …

Undecidability and The Riemann Hypothesis – A Brief Summary

Initially, we showed by the use of Fourier transforms, that we could generate a signal $\boldsymbol{S}$ from *either* the α-line or the β-line in $\mathcal{C}^*$ – the critical strip *minus* the critical line [pp 294-6].



Here, it is the case that ω is a root or pseudo-root of ζ(s), lying on *both* the α-line and the β-line, so that $\zeta(p) = \zeta(q)$ [1], where $p = \alpha \pm i\varepsilon$, and $q = \beta \pm i\varepsilon$, for some $\varepsilon \geq 0$ and $\beta = 1 - \alpha$. Thus, relative to

$\boldsymbol{S}$, it is not possible to distinguish between the α-line and the β-line $^{(*)}$, which means that if (1) holds, then relative to $\boldsymbol{S}$, it is not possible to distinguish between p and q [pp 342-3 and page 359].

To wit, if an observer $\mathcal{O}$, in the *signal's* frame of reference, is situated at p or q, then $\mathcal{O}$ is 'blind to the line' and so, won't know if it's really at p or at q. If $\mathcal{O}$ now knows that (1) holds, then even with this knowledge, $\mathcal{O}$ cannot distinguish between p and q, which means that to $\mathcal{O}$, there is absolutely *no* difference between being situated at p versus at q. The two perspectives are, in fact, identical.

On the other hand, if $\mathcal{O}$ knows $\zeta(p) \neq \zeta(q)$ $^{(2)}$, then $\mathcal{O}$ will know that being situated at p is *not* the same as being situated at q. $\mathcal{O}$ is still 'blind to the line' but now knows there is a *difference* between ℓ_α and ℓ_β, at p versus at q, because of (2). However, this is a violation of (*) above, since relative to $\boldsymbol{S}$, it should *never* be possible to detect such a difference between ℓ_α and ℓ_β, at these two points.

Thus, our methodology $\mathcal{M}$, first outlined on pages 294-6, validates (*) at the *point* level, whenever it is the case that $\zeta(p) = \zeta(q)$, but *nowhere* else. In turn, this means that if $\mathcal{M}'$ is any other mathematical method seeking to resolve the Riemann Hypothesis, it will never be able to verify these roots or *pseudo*-roots in $\mathcal{C}^*$, where $\mathcal{C}^*$ is the critical strip *minus* the critical line, and verify means 'prove without ambiguity'. For to do so would mean contradicting $\mathcal{M}$, in so much as $\mathcal{M}'$ would now be able to distinguish between p and q.

And finally, based on our previous research note, titled *Undecidability and Ghosts In The Room* [pp 380-2], we only need to prove the unverifiability of $\mathcal{T}$ – the set of all roots and pseudo-roots of $\zeta(s)$ in $\mathcal{C}^*$ – in order to show that the Riemann Hypothesis is undecidable.

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$

Recall that in a *mathematical* coordinate system [labelling as (*)], the *coupled* field equations are ...

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) + 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) + 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] \,.$$

And here, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, and the *physical* singularities associated with $\lambda(s)$ are at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) \,.$$

As well, $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; and $C^{u,v} = R^{u,v} - \frac{1}{2}Rg^{u,v}$, where $R^{u,v}$ is the Ricci tensor and R the Ricci scalar.

Now if, for the time being, we omit all of the dark energy components, except for the one at the origin [O], where there is a dark energy singularity, then we have …

$$G^{u,v} \approx \sigma g^{u,v}(0) \quad (\dagger)$$

And since (†) is going to be symmetric in all directions, we'll suppose that $g^{1,1} = g^{rr}$ and $g^{4,4} = -g^{tt}$ (the radial and time components, respectively), are a function of r only. Furthermore, for a *perfect* star, we'll presume that only pressure and density are present in $T^{u,v}$, along the diagonal, so that in contravariant form, $T^{4,4} = -\rho g^{4,4}$, where ρ is density.

The expression for $C^{4,4}$ is $-(1/r^2) \cdot g^{4,4} \cdot (r(1 - g^{1,1}))'$, where ′ means d/dr, so that in this case, (†) becomes …

$$-(1/r^2) \cdot g^{4,4} \cdot (r(1 - g^{1,1}))' = -k\rho g^{4,4} + \sigma g^{4,4}(0) ,$$

or, upon rearranging terms,

$$(r(1 - g^{1,1}))' = k\rho r^2 - \sigma r^2 \cdot g^{4,4}(0) / g^{4,4}(r) \qquad (\S)$$

Now since we believe $g^{4,4}$ is well-behaved at O, it must be the case that the right-hand side of (§) tends to 0 as r ---› 0. Thus, we may conclude that

$$\lim_{r \to 0} \left(r \cdot [g^{1,1}]' + g^{1,1}\right) = 1 ,$$

so at the center of the star [O], where the dark energy singularity resides, $g^{1,1}$ is actually *finite* and tends to 1, if $[g^{1,1}]'$ is finite at O. The singularity is probably a gateway into some reality beyond our own.

In general, though, it won't be this easy to formulate $C^{u,v}$ and $T^{u,v}$, when *all* the dark energy components in (*) are included. Since the right-hand side of (*) is an (r, θ, ϕ) layout, so must be the left-hand side, which means that the gravitational tensor itself, on the left-hand side, is also a function of (r, θ, ϕ). The components must match for a given choice of *u* and *v*, left side to right side, before a solution is found, and admittedly, it's a tall order, no matter how you look at it.

But after looking at the methodology that was developed for the Schwarzschild metric, in the case of a perfect star, it doesn't seem to me that we can use that approach. For there, a solution is developed *outside* the star, and used to interpolate *inside* the star. Here, in our case, when looking at (*), the right-hand side contains, among other things, the term $\sigma g^{u,v}(0)$. This alone implies a new approach is probably in order …

OTHER CONSIDERATIONS

In the case where we are dealing *strictly* with the quantumlike dark energy components, (*) becomes

$$\begin{aligned} G^{u,v} \approx \sigma[&2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) + \\ &2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) + \\ &2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] \qquad (1) \end{aligned}$$

And here, we believe a *harmonic* solution $\mathcal{H}^{u,v} = g^{u,v}$ exists, that is valid for *all* $r \geq 0$. In the other case, discussed earlier, where we *only* have a dark energy singularity at the origin [O],

$$G^{u,v} \approx \sigma g^{u,v}(0) \qquad (2)$$

and here we believe another solution $\mathcal{N}^{u,v} = g^{u,v}$ exists, that is also valid for *all* $r \geq 0$. Thus, providing it's *physically* acceptable, solving (*) on page 383 might best be done by breaking it up into two pieces; by solving (1) and (2) separately, and then adding the two solutions together, so that the total solution $\mathcal{T}^{u,v}$ is …

$$\mathcal{T}^{u,v} = \mathcal{H}^{u,v} + \mathcal{N}^{u,v}$$

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part II

Using the same setup as in Part I [pp 383-5], we're now going to calculate an expression for $g^{4,4}(0)$. Here, we'll start with $C^{1,1}$, which computes to …

$$(2/r) \cdot \{[g_{t,t}]' / 2g_{t,t}\} \cdot [g^{r,r}]^2 - (1/r^2) \cdot g^{r,r} \cdot (1 - g^{r,r})$$

The $T^{1,1}$ component is $pg^{1,1}$, where p is pressure in the star, so for the *coupled* scenario; that is to say …

$$G^{u,v} \approx \sigma g^{u,v}(0)\,, \qquad (\dagger)$$

it is the case that …

$$(2/r) \cdot \{[g_{t,t}]' / 2g_{t,t}\} \cdot [g^{r,r}]^2 - (1/r^2) \cdot g^{r,r} \cdot (1 - g^{r,r}) = kpg^{r,r} + \sigma g^{r,r}(0)\,. \qquad (\S)$$

Multiplying each side of this equation by r, and using the fact that $g^{r,r}$ tends to 1 as r ---› 0, (§) reduces to (as r ---› 0) …

$$[g_{t,t}]' / g_{t,t} \approx (1 - g^{r,r})/r\,.$$

Remembering from Part I that $g^{4,4} = -g^{tt}$, and moving the left side of the equation, just above, into its contravariant form, so that $g^{t,t} = 1/g_{t,t}$, we have (as r ---› 0) …

$$[g^{4,4}]' / g^{4,4} \approx (g^{1,1} - 1)/r\,.$$

Expanding $g^{1,1}$ on the right-hand side as a Taylor series, in the equation just above, gives us, to first order …

$$g^{1,1}(r) \approx g^{1,1}(0) + r \cdot [g^{1,1}(0)]'$$

And finally, knowing that $g^{1,1}(0) = 1$, we see that for the *coupled* case, in the limit, as r ---› 0 …

$$g^{4,4}(0) = [g^{4,4}(0)]' / [g^{1,1}(0)]'$$

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part III

In doing the calculations above, in Parts I and II, I took my information from Stefan Waner's book, titled *Introduction to Differential Geometry and General Relativity* [pp 116-123, Sixth Printing, May 2014].

Here, in looking for a solution to $G^{u,v} \approx 0$, Waner begins with the basic assumption that the gravitational tensor, in covariant form $[g_{u,v}]$ is

$$\begin{bmatrix} g_{rr} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -g_{tt} \end{bmatrix}$$

Waner also mentions that the *inner* block, containing the terms r^2 and $r^2\sin^2(\theta)$, must be this way in order to preserve spherical symmetry [p 117]. So if we accept this layout for $g_{u,v}$, then the curvature tensors used in Parts I and II; namely $C^{1,1}$ and $C^{4,4}$ in contravariant form, are correct.

Thus, our conclusions are also correct; namely, that for the *coupled* equation in contravariant form,

$$G^{u,v} \approx \sigma g^{u,v}(0) , \quad (\dagger)$$

we have, from Parts I and II above [pp 383-5] …

$$g^{1,1}(0) = g_{1,1}(0) = 1 \ ; \ g^{4,4}(0) = [g^{4,4}(0)]' / [g^{1,1}(0)]'$$

And, of course, the presumption here is that $g^{4,4}(0)$ is *also* well-behaved at the origin O of the star, as is its covariant counterpart, $g_{4,4}(0)$, relative to (†).

Thus, in covariant form, a non-singular solution $[g_{u,v}]$ to (†) at O might be possible, since we only need to find g_{rr} and g_{tt} in the picture above. And these are the covariant equivalents of $g^{1,1} = g^{rr}$ and $g^{4,4} = -g^{tt}$, obtained from (†).

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part IV

Let us bring back our expression from Part II [p 385] and rewrite it, so that for the *coupled* case …

$$[g^{1,1}(0)]' = [g^{4,4}(0)]' / [g^{4,4}(0)] .$$

Thus, as r ---› 0, we have …

$$[g^{1,1}(r)]' \approx [g^{4,4}(r)]' / [g^{4,4}(r)] = [\log g^{4,4}(r)]'$$

which means that up to some constant, say *C*, it is the case that for *small* r …

$$g^{1,1}(r) \approx \{\log g^{4,4}(r)\} + C$$

And so, for small r ---› 0, with $\mathcal{C} = exp(-C)$ …

$$g^{4,4}(r) \approx \mathcal{C}\cdot exp(g^{1,1}(r)) \quad (*)$$

We can actually confirm this formula (*), in a *weak* sense, by noting that for the Schwarzschild metric, where *m* is the mass of the star [Waner, page 123, Sixth Printing] …

$$g_{r,r} = r/(r - 2m), \text{ so that } g^{r,r} = g^{1,1} = (r - 2m)/r\ .$$

Similarly,

$$g_{t,t} = (r - 2m)/r, \text{ so that } g^{t,t} = r/(r - 2m) = -g^{4,4}\ .$$

Thus, as r ---› 0, the first Schwarzschild component $g^{1,1} = (r - 2m)/r$, tends to $-\infty$, so that the right-hand side of (*) is 0. On the other hand, the fourth Schwarzschild component $g^{4,4} = -r/(r - 2m)$, *also* tends to 0, so that *both* sides of (*) agree. But it is a *weak* confirmation, because in the case of Schwarzschild, the components are only valid *outside* of the star.

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part V

In Bernard Schutz' book titled *A First Course in General Relativity* [Second Edition], the covariant form for $C_{2,2}$ is listed on page 260 as follows …

$$C_{\theta\theta} = r^2 e^{-2\Lambda}[\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r],$$

where $e^{-2\Lambda}$ is defined to be $g^{r,r}$ and $e^{-2\Phi}$ is $g^{t,t}$. And here $T_{2,2} = pr^2$, and $g_{2,2} = r^2$ … so that the *covariant* form for our *coupled* equation is now [with $g_{2,2}(0) = 0$] …

$$C_{2,2} = kpr^2 + \sigma g_{2,2}(0) \quad (\S)$$

Cancelling the term r^2, which is common to both sides of (§), and then multiplying each side by r, leaves us with …

$$re^{-2\Lambda}\,[\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r] = kpr$$

Now as r ---› 0, all terms in the bracketed expression just above, will vanish on the left-hand side, save for the *third* and *fifth*; and the right-hand side will also vanish, so that for *small* r ---› 0,

$$\Phi' \approx \Lambda'$$

Thus, Φ and Λ behave in *exactly* the same way, up to some constant, as r ---› 0, and we now have, from the expression just above …

$$[g_{t,t}]' / g_{t,t} \approx [g_{r,r}]' / g_{r,r}$$

And since $g_{r,r}(r)$ ---› 1 as r ---› 0, the expression above can be written as …

$$[g_{r,r}]' \approx [g_{t,t}]' / g_{t,t} \approx [\log g_{t,t}(r)]'$$

Thus, just as in Part IV, for some *positive* constant $\mathscr{D}$, with *small* r ---› 0 and $g_{1,1} = g_{r,r}$ … we have

$$g_{t,t}(r) \approx \mathscr{D} \cdot exp(g_{1,1}(r)) . \quad (\sim)$$

Now let's compare this to (*) in Part IV, where we found that

$$g^{4,4}(r) \approx \mathscr{C} \cdot exp(g^{1,1}(r)) . \quad (*)$$

The reader may see an apparent contradiction here, since both $\mathscr{C}$ and $\mathscr{D}$ are *positive* constants. For recall that $g^{4,4} = -g^{t,t}$, so that a *sign* change emerges between the two expressions, just above. But is it a conflict ?

Not really, for in Part IV, we saw that

$$[g^{1,1}(r)]' \approx [g^{4,4}(r)]' / [g^{4,4}(r)] = [\log g^{4,4}(r)]'$$

and this can *also* be written as …

$$[g^{1,1}(r)]' \approx [g^{4,4}(r)]' / [g^{4,4}(r)] = [\log -g^{4,4}(r)]'$$

Similarly, we may write …

$$[g_{r,r}]' \approx [g_{t,t}]' / g_{t,t} \approx [\log g_{t,t}(r)]' = [\log -g_{t,t}(r)]'$$

so that ultimately, the sign of $g^{t,t}$ is arbitrary, from this analysis, as r ---› 0. It could be *positive* or *negative*, but either way, we still maintain consistency with our previous results.

Said another way, if $g^{4,4}$ is positive in (*) above, then we would use $-g_{t,t}$ in our analysis that led to (~). And if $g_{t,t}$ is positive in (~) above, then we would use $-g^{4,4}$ in our analysis that led to (*).

Thus, we have demonstrated that no inconsistencies arise when analyzing $C_{2,2}$ … and comparing our results here with what we found in Parts I through IV.

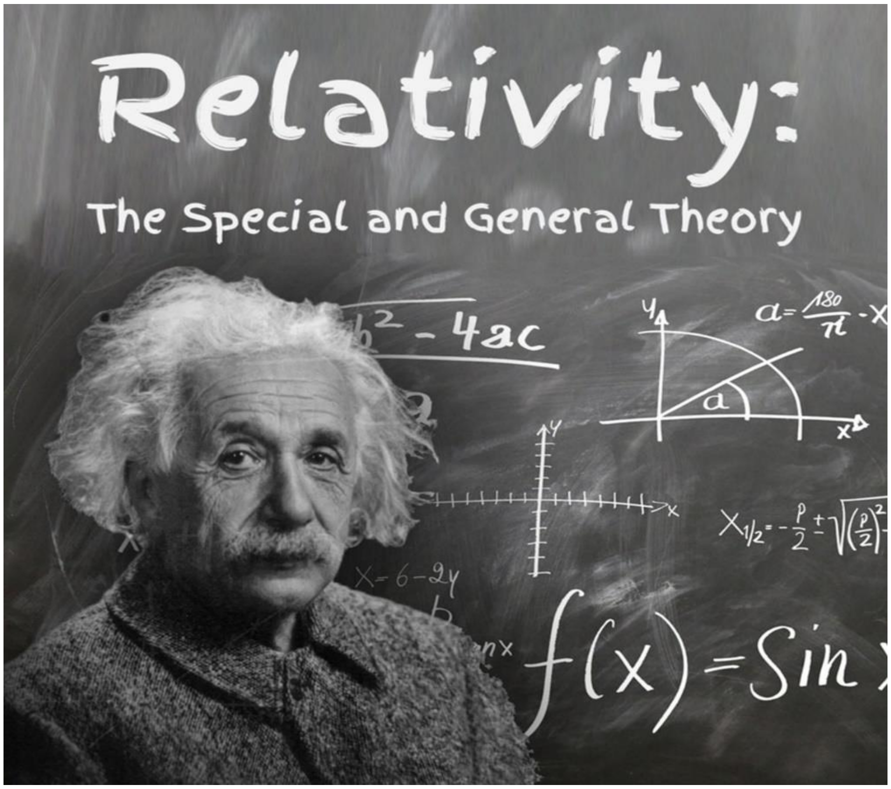
*****

Now since $\Phi' \approx \Lambda'$ as r ---› 0 [page 387], it suggests to us that for *small* r, $\Phi \approx \Lambda + \kappa$, where κ is some constant. And since we know $g_{r,r}(0) = 1$, Λ tends to 0 as r ---› 0, implying that $g_{t,t}$ is *positive* as r ---› 0 [recall that $\Phi = ½ \cdot \log(g_{t,t})$].

*****

Dr. Einstein

'I have a *general* interest in special relativity and a *special* interest in general relativity', to which Dr. Einstein replied, 'it is good that you are both a generalist and a specialist, for you need both at any point in space-time, to understand the theories' …



Albert Einstein March 1879 – April 1955

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part VI

In our previous research above [pp 383-8], we analyzed a few curvature tensors and developed various results as r ---› 0. In particular, we saw that

$$[g^{1,1}(0)]' = [g^{4,4}(0)]' / [g^{4,4}(0)] ,$$

where, for example, $[g^{1,1}(0)]'$ means 'take the derivative of $g^{1,1}(r)$, and evaluate at the origin O'. But we could ask the question, 'what happens if $[g^{1,1}(0)]'$ is actually *zero* ?'

In such a case, $[g^{4,4}(0)]'$ is also *zero,* so that in the expression above, $[g^{4,4}(0)]$ is actually undefined. And so, the expressions (~) and (*) on page 388 can still be used, *even* if $[g^{1,1}(0)]' = 0$.

Now let's bring back our formula (§) from page 384; namely,

$$g^{4,4} \cdot (r(1 - g^{1,1}))' \;=\; k\rho r^2 g^{4,4} \;-\; \sigma r^2 g^{4,4}(0)\,, \qquad (\S)$$

and let us assume that near O (r ---› 0), we can use the following Taylor series expansion, where again, ' means take the derivative of $g^{1,1}(r)$ and evaluate at O …

$$g^{1,1}(r) \;\approx\; g^{1,1}(0) \;+\; r \cdot [g^{1,1}(0)]'\,.$$

Then since $g^{1,1}(0) = 1$, (§) becomes

$$-2g^{4,4} \cdot [g^{1,1}(0)]' \cdot r \;=\; k\rho r^2 g^{4,4} \;-\; \sigma r^2 g^{4,4}(0)\,,$$

or, upon rearranging terms,

$$-2g^{4,4} \cdot [g^{1,1}(0)]' \;=\; k\rho r g^{4,4} \;-\; \sigma r g^{4,4}(0) \qquad (1)$$

Now as r ---› 0, the right-hand side of (1) will be 0, and so on the *left* side, either $g^{4,4}(0) = 0$, or $[g^{1,1}(0)]' = 0$. Since we *don't* believe $g^{4,4}(0)$ is equal to 0, it must be the case that $[g^{1,1}(0)]' = 0$, and hence $[g^{4,4}(0)]'$ is also *zero*, based on our previous comments.

Thus, let us try the following Taylor series expansion near O, where '' means 'take the second derivative of $g^{1,1}(r)$ and evaluate at O' …

$$g^{1,1}(r) \;\approx\; g^{1,1}(0) \;+\; r^2/2 \cdot [g^{1,1}(0)]'' \qquad (2)$$

If we now insert (2) into (§), we obtain …

$$-(3/2) \cdot g^{4,4} \cdot [g^{1,1}(0)]'' \cdot r^2 \;=\; k\rho r^2 g^{4,4} \;-\; \sigma r^2 g^{4,4}(0)\,,$$

and, after cancelling the r^2 term on *both* sides, we have …

$$-(3/2) \cdot g^{4,4} \cdot [g^{1,1}(0)]'' = \; k\rho g^{4,4} \;-\; \sigma g^{4,4}(0)\,.$$

Now letting r ---› 0, we can replace $g^{4,4} = g^{4,4}(r)$ with $g^{4,4}(0)$, so that the equation just above becomes …

$$[g^{1,1}(0)]'' \;=\; (2/3) \cdot (\sigma \;-\; k\rho)\,.$$

And so, our Taylor series expansion near O, for *small* r ---› 0 is …

$$g^{1,1}(r) \;\approx\; 1 \;+\; (1/3) \cdot (\sigma \;-\; k\rho) \cdot r^2$$

And thus, from (*) on page 388, assuming $g^{t,t}$ is *positive* as r ---› 0, we have (for *small* r ---› 0) …

$$-g^{4,4}(r) \;\approx\; \mathcal{C} \cdot exp(g^{1,1}(r))\,,$$

since $g^{4,4} = -g^{t,t}$. And here, $\mathcal{C}$ is some positive constant [see Part V, pages 387-8].

OTHER CONSIDERATIONS

Suppose $[g^{1,1}(0)]' = \varepsilon > 0$, for some arbitrary choice of ε, however small. Then the analysis that led to (*) and (~) on page 388, and reproduced here, applies …

$$g_{t,t}(r) \approx \mathscr{D} \cdot exp(g_{1,1}(r)) . \quad (\sim)$$

$$g^{4,4}(r) \approx \mathscr{C} \cdot exp(g^{1,1}(r)) . \quad (*)$$

Now let ε ---› 0. Then in the limit, (*) and (~) still hold, and we define $g_{t,t}(r)$ and $g^{4,4}(r)$ as above, for small r ---› 0, *even* when $[g^{1,1}(0)]' = [g^{4,4}(0)]' = 0$.

*****

The cosmological constant Λ is roughly 10^{-52}, and Einstein's gravitational constant k is about 10^{-43}, in standard units. Our expression for $g^{1,1}(r)$ for *small* r ---› 0 is …

$$g^{1,1}(r) \approx 1 + (1/3) \cdot (\sigma - k\rho) \cdot r^2 \quad (\dagger)$$

and I don't think σ is going to be very much different than Λ. So in the expression above [†], I wonder now if *small* r ---› 0 really is a *quantumlike* effect, at very small scales, that is described by parabolic behavior, as we move closer and closer to O, where the dark energy singularity resides.

*****

If we attempt a similar analysis for $g_{4,4}$, by looking at $C_{4,4}$, then (§) on page 390 becomes …

$$g_{4,4} \cdot (r(1 - g^{1,1}))' = k\rho r^2 g_{4,4} - \sigma r^2 g_{4,4}(0) . \quad (\S)$$

And this is because the tensors are 'lowered' by operating on them with $g_{4,4} \cdot g_{4,4}$.

However, nothing new can be gained from this equation above … because the piece that *doesn't* change is

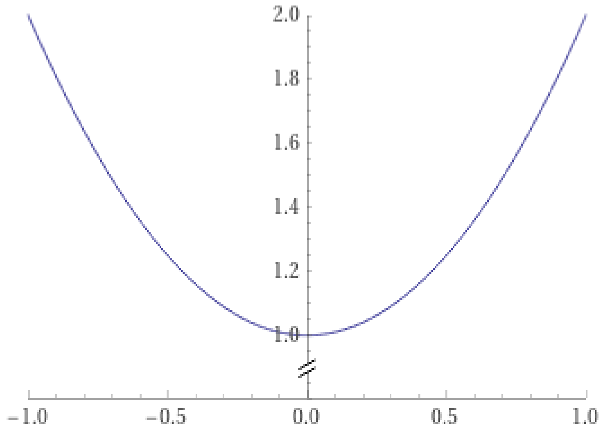
$$(r(1 - g^{1,1}))' .$$

Thus, if we want an estimate for $g_{1,1}$, for *small* r ---› 0, it is best done by taking the inverse of $g^{1,1}(r)$ in (†) above, since this is normally what we would do for a *diagonal* $[g^{u,v}]$, anyway. In so doing, we have, for *small* r ---› 0 and $Q(r) = (1/3) \cdot (\sigma - k\rho) \cdot r^2$ …

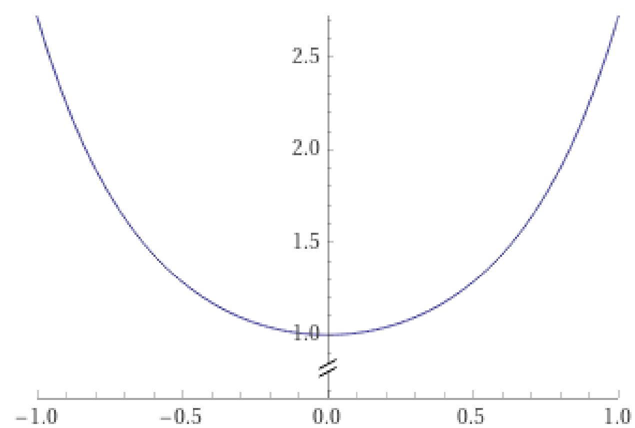
$$g_{1,1}(r) \approx 1 / (1 + Q(r)) \quad ; \quad g_{t,t}(r) \approx \mathscr{D} \cdot exp(g_{1,1}(r)) ,$$

where $\mathscr{D}$ is some *positive* constant. And so, our metric ds^2 for *small* r ---› 0 looks like …

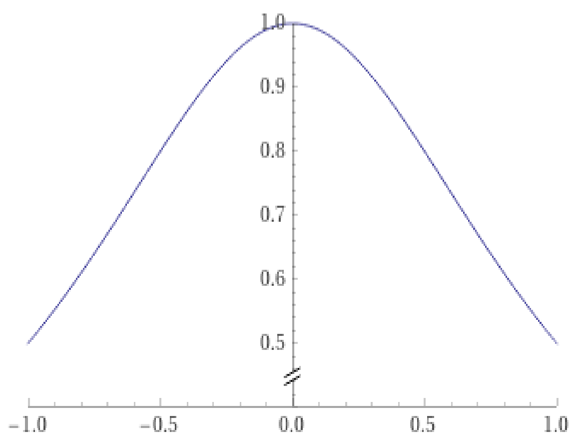
$$ds^2 \approx g_{1,1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 - g_{t,t}(r)dt^2$$



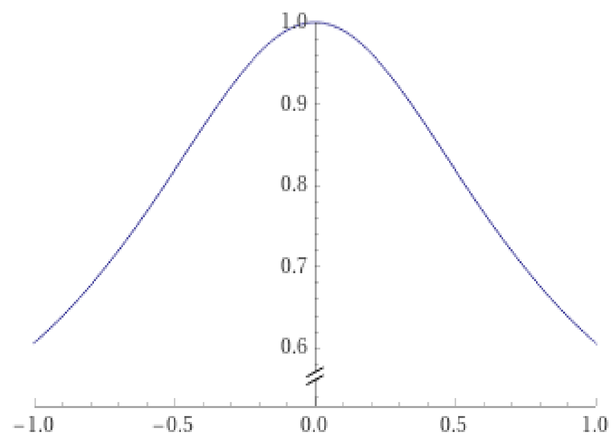
$1 + r^2$ akin to $g^{1,1}(r)$ [A]



$(1/e)\,exp(1 + r^2)$ akin to $g^{t,t}(r)$ [B]

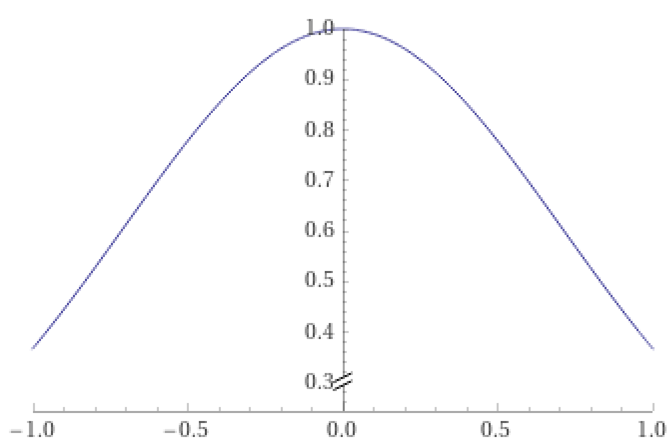


$1 / (1 + r^2)$ akin to $g_{1,1}(r)$ [C]



$(1/e)\,exp(1 / (1 + r^2))$ akin to $g_{t,t}(r)$ [D]

In pictures B and D we have chosen our constant to be $(1 / e)$, so that at r = 0, $g^{t,t}(r) \cdot g_{t,t}(r) = 1$, which we expect, *exactly*. Now compare [D] to the picture below, where we simply *invert* B.



The inversion of B amounts to plotting $e \cdot exp(-(1 + r^2))$, and while we could take this expression for $g_{t,t}(r)$, I believe it is more correct to use the expression in [D], deduced from Parts I to V [pp 383-8] in our research.

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part VII

Let us bring back our expression from page 385, namely …

$$(2/r) \cdot \{[g_{t,t}]' / 2g_{t,t}\} \cdot [g^{r,r}]^2 - (1/r^2) \cdot g^{r,r} \cdot (1 - g^{r,r}) = kpg^{r,r} + \sigma g^{r,r}(0)\,, \quad (\S)$$

and multiply each side by r^2, so that

$$2r \cdot \{[g_{t,t}]' / 2g_{t,t}\} \cdot [g^{r,r}]^2 - g^{r,r} \cdot (1 - g^{r,r}) = kpr^2 g^{r,r} + \sigma r^2 g^{r,r}(0) \quad (\S\S)$$

Since we already know that $[g^{1,1}(0)]' = [g^{4,4}(0)]' = 0$ (as are the covariant equivalents), we will now insert the following Taylor series expression into (§§), and then let r ---› 0, recalling as well that $g^{r,r}(0) = g^{1,1}(0) = 1$. It is the $(1 - g^{r,r})$ term above where the insertion takes place …

$$g^{1,1}(r) \approx g^{1,1}(0) + r^2/2 \cdot [g^{1,1}(0)]''$$

And so as r ---› 0, (§§) reduces to …

$$r^2 / 2 \cdot [g^{1,1}(0)]'' = kpr^2 + \sigma r^2$$

Cancelling the term r^2 on *both* sides now gives us …

$$[g^{1,1}(0)]'' = 2(\sigma + kp)\,,$$

and hence

$$g^{1,1}(r) \approx 1 + (\sigma + kp) \cdot r^2$$

Now this must agree with our expression (†) on page 391, and reproduced here,

$$g^{1,1}(r) \approx 1 + (1/3) \cdot (\sigma - k\rho) \cdot r^2 \quad (\dagger)$$

so that

$$3(\sigma + kp) = (\sigma - k\rho)\,. \quad (\dagger\dagger)$$

Now something rather fascinating emerges from this result. First, if σ = 0, then for a *perfect* star in the *static* case (no flow of star material), with *no* dark energy singularity at its origin O, it is the case that

$$3p + \rho = 0\,,$$

where p and ρ are pressure and density in the perfect star (essentially a perfect fluid). On the other hand, if σ is *not zero*, then from (††),

$$\sigma \;=\; -\,(k/2)\cdot(3p \;+\; \rho) \qquad (**)$$

In this case, there is a dark energy singularity $\boldsymbol{S}$ at the center [O] of the star, and the underlying dark energy density function associated with $\boldsymbol{S}$ is $\lambda(s) \;\approx\; \sigma / s$. The dark energy, itself, which permeates space-time is *also* σ, since it is the Laplace inverse of λ(s), and here we traverse the dark energy *contour* integral in counterclockwise fashion.

But seen as the constant associated with λ(s), it appears as though σ is intimately connected to the Einstein constant k, as well as pressure and density in the star. This is true when the singularity $\boldsymbol{S}$ is at O, and consequently, there seems to be no connection to the cosmological constant Λ in this case; at least based on current estimates of what Λ might be. Yet these estimates are controversial, to say the least …

A SIMPLE CASE STUDY

In our equation of state, where σ = 0, and there is *no* dark energy singularity at the center [O] of the perfect star, viz.,

$$3p \;+\; \rho \;=\; 0\,, \qquad (*)$$

let us suppose density [ρ] in the star is an *inward* force with *positive* sign. Then this must be exactly balanced by pressure [p], an *outward* force with *negative* sign, according to the formula above, for the *static* case (no flow of star material).

Now if σ is *not zero*, so that dark energy is pouring into the star at O via the singularity there, then there will be a slight increase in outward pressure, making p even more negative. At the same time, it is possible that the density ρ might decrease slightly, because we would expect the star to expand by an almost imperceptible amount, symmetrically, in all directions.

The net effect is that (*) would become *negative*, making σ *positive*, from (**) above. Here σ is to be interpreted as the constant associated with the underlying dark energy density function,

$$\lambda(s) \;\approx\; \sigma / s\,. \qquad (\sim)$$

Now as we said before, the dark energy, itself, which permeates space-time is *also* σ, in this case, because it is the Laplace inverse of λ(s). And if we calculate this inverse from the dark energy *contour* integral, using a standard counterclockwise orientation, then dark energy [σ] will *also* be *positive*, which guarantees us consistency.

On the other hand, if we were to reverse our sign conventions, then from (**) σ would become *negative* in (~); and we would compute a *negative* value for dark energy, itself, via the Laplace inverse of λ(s), again using a counterclockwise orientation, also guaranteeing us consistency.

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part VIII

In this note, we want to calculate the volume of a *very small* sphere [S] at the origin O of the star, where our dark energy singularity $\mathcal{S}$ resides. Here, the radius of S is much less than 1, so that our metric is …

$$ds^2 \approx g_{1,1}(r)dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 - g_{t,t}(r)dt^2$$

Recall that $Q(r) = (1/3) \cdot (\sigma - k\rho) \cdot r^2$, and that

$$g_{1,1}(r) \approx 1 / (1 + Q(r)) \quad ; \quad g_{t,t}(r) \approx \mathcal{D} \cdot exp(g_{1,1}(r)) ,$$

where $\mathcal{D}$ is some *positive* constant. And, since dark energy [σ] is pouring into the star, we expect S to expand ever so slightly. Let's see what happens …

Now with spherical symmetry, our Jacobian [J] for a given radius r will be $r^2\sin(\phi)$, so that the integration becomes, in a *mathematical* coordinate system …

$$\int_0^r \int_0^{2\pi} \int_0^{\pi} r^2\sin(\phi)\sqrt{g_{1,1}}\, d\phi d\theta dr .$$

And this is because when we integrate in the direction of r, we must do so according to our metric above. The integration thus becomes …

$$\int_0^r \int_0^{2\pi} \int_0^{\pi} r^2\sin(\phi)\sqrt{1 / (1 + Q(r))}\, d\phi d\theta dr .$$

Now because we expect the sphere to expand, ever so slightly, it stands to reason that Q(r) must be slightly *negative*, and so $(\sigma - k\rho) < 0$. And since from (††) on page 393, and reproduced here …

$$3(\sigma + kp) = (\sigma - k\rho) , \quad (††)$$

it must *also* be true that $\sigma + kp < 0$. Thus, for *positive* σ, it must be the case that

$$0 < \sigma < min(k\rho , -kp) .$$

Recall, too, that for an *outward* pressure p, we have chosen it to be of *negative* sign, and for an *inward* density ρ, it is of *positive* sign.

Now since $Q(r) = (1/3) \cdot (\sigma - k\rho) \cdot r^2$ is slightly negative, let us define it to be $-\alpha^2 r^2$, so that our integral above becomes …

$$\int_0^r \int_0^{2\pi} \int_0^{\pi} r^2 \sin(\phi)\sqrt{1/(1 - \alpha^2 r^2)}\, d\phi d\theta dr \,.$$

If we use a Taylor series expansion here, realizing now that r is *very small*, the expression just above becomes, approximately …

$$\int_0^r \int_0^{2\pi} \int_0^{\pi} r^2 \sin(\phi)(1 + \tfrac{1}{2} \cdot \alpha^2 r^2) d\phi d\theta dr \,.$$

The integration is now straightforward, and for $\alpha^2 = -(1/3) \cdot (\sigma - k\rho) > 0$, computes to …

$$4\pi\,[\tfrac{1}{3} \cdot r^3 + \tfrac{1}{10} \cdot \alpha^2 r^5]$$

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part IX

In Part VII [pp 393-4] we developed an equation for σ; namely,

$$\sigma = -(k/2) \cdot (3p + \rho) \qquad (**)$$

based on an analysis of (§§) below. In that analysis, we let r ---› 0, and in the process, omitted the *first* term in (§§) on the left side, to arrive at the result that we did. And that is because we knew $[g_{t,t}(0)]' = 0$. Here we'd like to refine the calculation, just a bit now.

$$2r \cdot \{[g_{t,t}]' / 2g_{t,t}\} \cdot [g^{r,r}]^2 - g^{r,r} \cdot (1 - g^{r,r}) = kpr^2 g^{r,r} + \sigma r^2 g^{r,r}(0) \qquad (§§)$$

From page 391 we know that with $Q(r) = (1/3) \cdot (\sigma - k\rho) \cdot r^2$, it is the case that

$$g_{1,1}(r) \approx 1 / (1 + Q(r)) \quad ; \quad g_{t,t}(r) \approx \mathscr{D} \cdot exp(g_{1,1}(r)) \,,$$

where $\mathscr{D}$ is some positive constant. Thus, near the origin O of the star we may write …

$$[g_{t,t}]' / g_{t,t} = [\log g_{t,t}]' = [g_{1,1}(r)]' \approx -Q(r)' = -(2/3) \cdot (\sigma - k\rho) \cdot r$$

If we now insert this into (§§), along with our Taylor series expansion below, and let r ---› 0,

$$g^{1,1}(r) \approx g^{1,1}(0) + r^2/2 \cdot [g^{1,1}(0)]''$$

we obtain, with $g^{1,1}(0) = 1$ …

$$-(2/3) \cdot (\sigma - k\rho) \cdot r^2 + r^2 / 2 \cdot [g^{1,1}(0)]'' = kpr^2 + \sigma r^2 \,.$$

And cancelling the r^2 term on *both* sides leaves us with …

$$-(2/3)\cdot(\sigma - k\rho) + (1/2)\cdot[g^{1,1}(0)]'' = kp + \sigma$$

Our original coefficient in the first Taylor series expansion, from page 390, was

$$[g^{1,1}(0)]'' = (2/3)\cdot(\sigma - k\rho)\,,$$

so if we insert this into the equation, just above, we have

$$-(2/3)\cdot(\sigma - k\rho) + (1/3)\cdot(\sigma - k\rho) = kp + \sigma\,.$$

And so our new equation of state is …

$$-(1/3)\cdot(\sigma - k\rho) = kp + \sigma\,,$$

or equivalently,

$$\sigma = -(k/4)\cdot(3p - \rho)\,.$$

And if we adhere to our previous sign conventions, where density [ρ] is a *positive* inward force, whilst pressure [*p*] is a *negative* outward force, then the expression above becomes …

$$\sigma = -(k/4)\cdot(3p + \rho)\,.$$

Comparing to our earlier result in Part VII, and reproduced here, with sign conventions in place,

$$\sigma = -(k/2)\cdot(3p + \rho)\,,$$

and we see that the two expressions only differ by a factor of 2 in the denominator …

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part X

Let us bring back our expression from page 385; namely …

$$(2/r)\cdot\{[g_{t,t}]' / 2g_{t,t}\}\cdot[g^{r,r}]^2 - (1/r^2)\cdot g^{r,r}\cdot(1 - g^{r,r}) = kpg^{r,r} + \sigma g^{r,r}(0)\,. \qquad (\S)$$

And let us multiply each side by $[g_{r,r}]^3$, so that we are ‘lowering indices’. This give us …

$$(2/r)\cdot\{[g_{t,t}]' / 2g_{t,t}\}\cdot g_{r,r} - (1/r^2)\cdot g_{r,r}\cdot(g_{r,r} - 1) = kp[g_{r,r}]^2 + \sigma[g_{r,r}]^3 g^{r,r}(0)\,.$$

Now since we already know that $[g_{r,r}(0)]' = [g_{t,t}(0)]' = 0$, let us insert the following Taylor series expansion into the equation, just above, remembering that $g_{r,r}(0) = 1$ …

$$g_{r,r}(r) \approx g_{r,r}(0) + r^2/2\cdot[g_{r,r}(0)]''\,.$$

This gives us

$$(2/r) \cdot \{[g_{t,t}]' / 2g_{t,t}\} \cdot g_{r,r} \; - \; (1/r^2) \cdot g_{r,r} \cdot (r^2/2 \cdot [g_{r,r}(0)]'') \; = \; kp[g_{r,r}]^2 \; + \; \sigma[g_{r,r}]^3 g^{r,r}(0) \, .$$

Now multiply each side by r^2, yielding …

$$r \cdot \{[g_{t,t}]' / g_{t,t}\} \cdot g_{r,r} \; - \; g_{r,r} \cdot (r^2/2 \cdot [g_{r,r}(0)]'') \; = \; kp[g_{r,r}]^2 r^2 \; + \; \sigma[g_{r,r}]^3 r^2 g^{r,r}(0) \, . \qquad (\S\S)$$

As r ---› 0, $g_{r,r}$ ---› 1, and if we omit the *first* term on the *left* side, the expression above reduces to

$$-(r^2/2) \cdot [g_{r,r}(0)]'' \; = \; kpr^2 \; + \; \sigma r^2$$

Cancelling the r^2 term, which is common to both sides, now gives us …

$$[g_{r,r}(0)]'' \; = \; -2(\sigma \; + \; kp)$$

Thus, for *small* r ---› 0, our Taylor series expansion becomes …

$$g_{r,r}(r) \; \approx \; 1 \; - \; (\sigma \; + \; kp) \cdot r^2$$

And from (††) on page 393, and reproduced here, our equation of state is ..

$$3(\sigma \; + \; kp) \; = \; (\sigma \; - \; k\rho) \, , \qquad (\dagger\dagger)$$

and thus, we see that

$$g_{1,1}(r) \; = \; g_{r,r}(r) \; \approx \; 1 \; - \; (1/3) \cdot (\sigma \; - \; k\rho) \cdot r^2 \qquad (*)$$

Now originally, with $Q(r) = (1/3) \cdot (\sigma \; - \; k\rho) \cdot r^2$, we found that [p 391]

$$g^{1,1}(r) \; \approx \; 1 \; + \; (1/3) \cdot (\sigma \; - \; k\rho) \cdot r^2 \; = \; 1 \; + \; Q(r) \, ,$$

and actually used its *inverse* for $g_{1,1}(r)$; that is to say, we let

$$g_{1,1}(r) \; \approx \; 1 \, / \, (1 \; + \; Q(r)) \, .$$

But to a first order approximation, because Q(r) is *very small*, this can be written as ….

$$g_{1,1}(r) \; \approx \; 1 \, / \, (1 \; + \; Q(r)) \; \approx \; 1 \; - \; Q(r) \, ,$$

which agrees *exactly* with our calculation for $g_{1,1}(r)$ in (*) above !

*****

If we repeat the exercise, but *include* the first term on the left side in (§), then (§§) above becomes, with r ---› 0 …

$$-(2/3) \cdot (\sigma - k\rho)r^2 - (r^2/2) \cdot [g_{r,r}(0)]'' = kpr^2 + \sigma r^2$$

Cancelling the r^2 term on *both* sides gives us

$$-(2/3) \cdot (\sigma - k\rho) - (1/2) \cdot [g_{r,r}(0)]'' = kp + \sigma$$

Bringing back our equation of state for this case [p 397], which is …

$$-(1/3) \cdot (\sigma - k\rho) = kp + \sigma ,$$

the equation now becomes

$$-(2/3) \cdot (\sigma - k\rho) - (1/2) \cdot [g_{r,r}(0)]'' = -(1/3) \cdot (\sigma - k\rho) .$$

And so,

$$[g_{r,r}(0)]'' = -(2/3) \cdot (\sigma - k\rho) ,$$

whence our Taylor series expansion is now …

$$g_{1,1}(r) = g_{r,r}(r) \approx 1 - (1/3) \cdot (\sigma - k\rho) \cdot r^2 .$$

And this agrees *exactly* with (*) on the last page; that is to say, our Taylor series expansion where we omitted the *first* term on the left side in (§§). Thus, whether we *do* or *do not* include the *first* term on the *left* side in the curvature tensor, makes *no* difference at all when calculating $g_{r,r}(r)$ very close to the origin [O] of the star. Said another way, $g_{r,r}(r)$ remains *invariant* under such an operation.

But what does change is the equation of state, as it should, giving us confidence that we have done things correctly in our analysis of things.

OTHER CONSIDERATIONS

Initially, our form for the *coupled* equations, with a dark energy singularity at the origin of the star, was taken to be

$$G^{u,v} \approx \sigma g^{u,v}(0) , \quad (**)$$

in *contravariant* mode. But what if we arbitrarily decide we want to work in *covariant* form ? It's easy enough to lower indices in $G^{u,v}$ using a diagonal $[g_{u,v}]$, since everything here is going to be a function of r and θ. But if we lower the left side in (**) via $[g_{u,v}]$, what should we do with the right side ? It *isn't* a function of r or θ at all.

Rather, it's a constant, and from a philosophical standpoint, I could just as easily have started with the covariant form, originally,

$$G_{u,v} \approx \sigma g_{u,v}(0) \,,$$

and asked the same question in reverse, if I wanted to move to contravariant form. To me, the only consistent way of thinking about this is to say, broadly, that ‘if I raise or lower indices on the left side, I must do the same on the right side, within the *intended* context’.

Since the right side is a constant, in *either* form, I would then lower $g^{1,1}(0)$, for example, in a diagonal $[g^{u,v}]$, by taking its inverse. Or, if you like,

$$g_{1,1} \cdot g_{1,1} \cdot g^{1,1} \,,$$

evaluated at 0.

Said another way, if I start with

$$G_{u,v} \approx \sigma g_{u,v}(0) \,,$$

and solve for $[g_{u,v}]$, will I get a result that is consistent with the solution $[g^{u,v}]$ I would have got, had I started with

$$G^{u,v} \approx \sigma g^{u,v}(0) \,.$$

Based on the conclusions in this research note, I’m optimistic that this is indeed the case …

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part XI

Let us bring back our *first* equation of state, from page 393; namely,

$$3(\sigma + kp) = (\sigma - k\rho) \,, \qquad (\dagger\dagger)$$

where, in an analysis of (§) below, the *first* term on the left side was omitted …

$$(2/r) \cdot \{[g_{t,t}]' / 2g_{t,t}\} \cdot [g^{r,r}]^2 - (1/r^2) \cdot g^{r,r} \cdot (1 - g^{r,r}) = kpg^{r,r} + \sigma g^{r,r}(0) \,. \qquad (\S)$$

Since we already know $\sigma - k\rho < 0$ [p 395], and since from (††)

$$\sigma = -(k/2) \cdot (3p + \rho) \,,$$

it must be the case that

$$\sigma - k\rho = -(k/2) \cdot (3p + \rho) - k\rho = -(3k/2) \cdot (p + \rho) < 0 \,.$$

Thus, $p > -\rho$, where here, density $[\rho]$ in the star is an *inward* force with *positive* sign, and pressure $[p]$, is an *outward* force with *negative* sign. That is to say, p is a *negative* value and ρ is a *positive* value.

And since here, dark energy is pouring into the star at its center, we believe the star will expand, ever so slightly, so that $3p + \rho < 0$, and hence the following inequality arises …

$$\rho / 3 < -p < \rho . \qquad (\sim)$$

If we *include* the first term on the left side of (§) in our analysis, then the equation of state, *without* respect to sign conventions, is

$$-3(\sigma + \mathrm{k}p) = (\sigma - \mathrm{k}\rho) ,$$

so that

$$\sigma = -(\mathrm{k}/4) \cdot (3p - \rho) . \qquad (\dagger)$$

Since $\sigma - \mathrm{k}\rho < 0$ still holds, it must be the case that

$$\sigma - \mathrm{k}\rho = -(\mathrm{k}/4) \cdot (3p - \rho) - \mathrm{k}\rho = -(3\mathrm{k}/4) \cdot (p + \rho) < 0 ,$$

so that again, $p > -\rho$. If we now adopt our sign conventions as above, then (~) will *also* hold in this case, since equilibrium is reached in (†) when $3p$ is exactly balanced by ρ, and $\sigma = 0$.

*****

In the first case, where the *first* term in (§) is *excluded* in our analysis, we showed on page 395 that

$$0 < \sigma < \mathit{min}(\mathrm{k}\rho , -\mathrm{k}p) ,$$

and thus, from (~) above,

$$0 < \sigma < -\mathrm{k}p < \mathrm{k}\rho .$$

In the second case, where the *first* term in (§) is *included* in our analysis, our equation of state, *without* respect to sign conventions, is again …

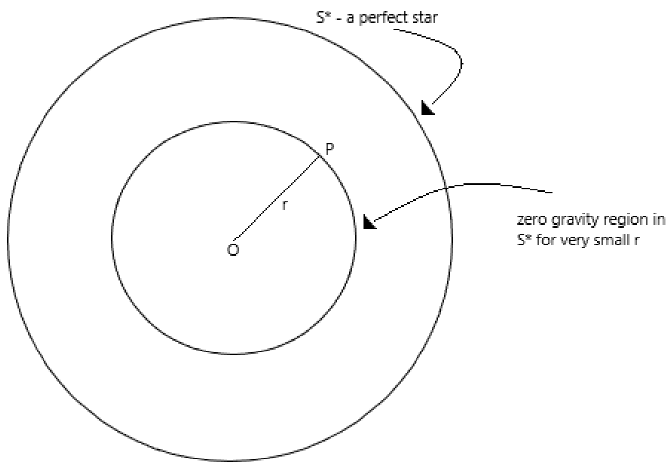
$$-3(\sigma + \mathrm{k}p) = (\sigma - \mathrm{k}\rho) .$$

Since $\sigma - \mathrm{k}\rho < 0$ still applies, it must be the case from the equation above, that $\sigma > -\mathrm{k}p$. And if we now invoke our sign conventions, as above, then $\sigma > -\mathrm{k}p$, where now p is a *negative* value and ρ is *positive*. Thus, the following inequality arises, in this case, according to our interpretation of things …

$$0 < -\mathrm{k}p < \sigma < \mathrm{k}\rho .$$

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part XII

In the diagram below, let S* be our *perfect* star and define $\mathcal{G}$ to be the (near) zero-gravity region, out to some *very small* radius r. Let this be called the F_0 frame, since here there is no dark energy.

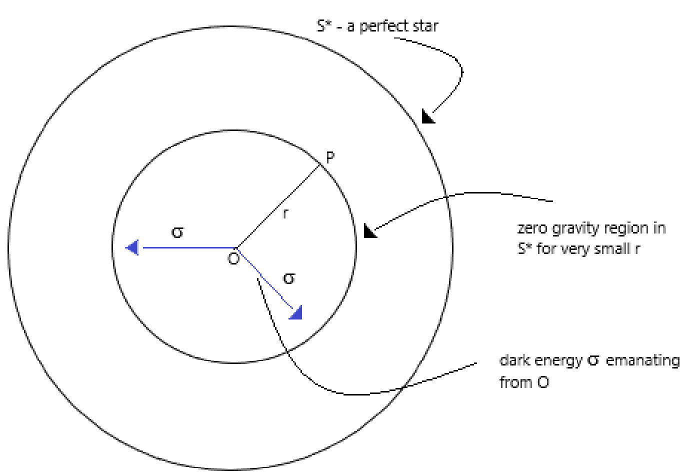


Because $\mathcal{G}$ is essentially zero-gravity, the standard metric applies, which becomes, in *physical* coordinates …

$$ds^2 \approx dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 - dt^2$$

Now imagine a ray of light $\mathcal{R}$ emanating from O in S*, and travelling at a speed *c* in this medium. Then because $g_{r,r}$ and $g_{t,t}$ are both 1 in $\mathcal{G}$, from ds^2 above, it must be the case that $\mathcal{R}$ will arrive at P in time $t = r / c$, at least in the stationary frame. The 'ruler tick' and the 'clock tick' are *fixed* now, because $g_{r,r}$ and $g_{t,t}$ are both 1 in F_0.

In the diagram below, suppose there is a dark energy singularity at the center of S*, with dark energy pouring into the star, according to the density function $\lambda(s) \approx \sigma / s$. Let this be the $F\sigma$ frame.



Again, imagine a ray of light $\mathcal{R}$ emanating from O in S*, and travelling at a speed *c* in this medium, through $\mathcal{G}$. And let us now take the bold step of assuming that *c* is *not* affected by the dark energy [σ], itself. The light's energy [E = h*v*, where h is Planck's constant] might be affected by σ, but not *c*.

Then it stands to reason that since $\mathcal{G}$ is essentially zero-gravity, $\mathcal{R}$ will arrive at P in time $t' = r' / c$, since ds^2 now changes *because* of σ. Indeed, from previous notes we know that in this case

$$ds^2 \approx g_{1,1}(r)dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 - g_{t,t}(r)dt^2 ;$$

and with $Q(r) = (1/3) \cdot (\sigma - k\rho) \cdot r^2$, where $0 < \sigma < k\rho$, that

$$g_{1,1}(r) \approx 1 - Q(r) \quad ; \quad g_{t,t}(r) \approx \mathcal{D} \cdot exp(g_{1,1}(r)) ,$$

where $\mathcal{D}$ is some *positive* constant.

However, because $r' / r = t' / t$, the 'ruler tick' in the σ-frame [Fσ], relative to its counterpart in F_0, is equal to the 'clock tick' in Fσ, relative to its counterpart in F_0. And since $g_{r,r}$ and $g_{t,t}$ are both 1 in F_0, it seems to suggest that $g_{1,1}(r)$ and $g_{t,t}(r)$, in ds^2 above, must therefore, *agree* in Fσ. And this is due to the constancy of *c* in both frames.

Now if we're on the right track here, then to a first order approximation, for *very small* r …

$$g_{t,t}(r) \approx \mathcal{D} \cdot exp(g_{1,1}(r)) = \mathcal{D} \cdot e \cdot exp(-Q(r)) \approx \mathcal{D} \cdot e \cdot (1 - Q(r)) \approx \mathcal{D} \cdot e \cdot g_{1,1}(r) ,$$

so that necessarily $\mathcal{D} \cdot e = 1$, and thus $\mathcal{D} = 1 / e$.

OTHER CONSIDERATIONS

If, in the first case [F_0], we believe the correct metric for ds^2 can be obtained, simply by setting σ = 0, then here

$$ds^2 \approx g_{1,1}(r)dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 - g_{t,t}(r)dt^2 ;$$

and now, with $Q(r) = (1/3) \cdot (-k\rho) \cdot r^2$, where $0 < k\rho$, that

$$g_{1,1}(r) \approx 1 - Q(r) \quad ; \quad g_{t,t}(r) \approx \mathcal{D} \cdot exp(g_{1,1}(r)) ,$$

where $\mathcal{D}$ is some *positive* constant.

If we let ρ ---› 0, so that pressure *p* in the perfect star S^* vanishes as well [from the equation of state, pp 400-1], then S* ceases to exist, so that we are now in a truly gravity-free region of space, and thus the metric above becomes [since Q(r) is now zero] …

$$ds^2 \approx dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 - \mathcal{D} \cdot e \cdot dt^2$$

If we want this to agree with the standard Schwarzschild metric, free of any matter, then necessarily $\mathcal{D} \cdot e = 1$.

*****

On pages 402-3, the 'ruler tick' and 'clock tick' in F_0 can always be normalized to 1, so that in turn, the speed of light *c* becomes 1. This may make it easier to follow the line of reasoning there. For example, if in F_0 the beam of light $\mathcal{R}$ travels from O to P [a distance of r units], we can simply refer to this as one 'ruler tick' in the 'spacelike' sense. And we can arbitrarily assign one 'clock tick' to this event, if we decide to measure in the 'timelike' sense.

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part XIII

In this note, we want to take a look at the *interior* Schwarzschild metric [$\mathcal{M}$], and compare its solution, near the origin O of our perfect star [S*], to the metric discussed in the last research note on page 403, under 'Other Considerations'.

From the Wiki page, $\mathcal{M}$ takes the form, in *physical* coordinates …

$$c^2 d\tau^2 = -\frac{1}{4}\left(3\sqrt{1-\frac{r_s}{r_g}} - \sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2 c^2 dt^2 + \left(1-\frac{r^2 r_s}{r_g^3}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

where *c* is the speed of light, r_s is the Schwarzschild radius, and r_g is the coordinate radius of S*.

Now because S* is a perfect star, just as in our previous research, the density [ρ] of the star is always constant, because we're dealing with an incompressible fluid. And we'll assume here that *c* has been normalized to 1 and that the mass of S* is M.

Then it is the case that, with G being the gravitational constant …

$$r_s = 2GM / c^2 = 2GM = 2G \cdot (4/3)\pi r_g^3 \cdot \rho ,$$

so that

$$r_s / r_g^3 = 8\pi G\rho/3 = k\rho/3 ,$$

where k is Einstein's constant, and again $c = 1$.

Thus, the coefficient associated with dr^2 above is, to first order …

$$(1 - r^2 r_s / r_g^3)^{-1} \approx 1 + r^2 r_s / r_g^3 = 1 + (k\rho/3) \cdot r^2 ,$$

and this agrees *exactly* with $g_{1,1}(r)$ on page 403, under 'Other Considerations'. Now for the harder part, which is the coefficient associated with dt^2 above …

This coefficient is a very peculiar expression, in that it is the difference of two *radical* terms, which is then *squared*. And this alone should tell us we may not get perfect agreement with $g_{t,t}(r)$, on page 403 in ‘Other Considerations’, but let’s see how close we can come … after making some assumptions.

To begin with, we’ll assume r_s / r_g is a negligible quantity, as it is for most stars, so that the expression $(1 - r_s / r_g)^{1/2}$ reduces to 1. Then the coefficient associated with dt^2 becomes …

$$(-1/4) \cdot [3 - (1 - r^2 r_s / r_g^3)^{1/2}]^2 = (-1/4) \cdot [9 - 6 \cdot (1 - r^2 r_s / r_g^3)^{1/2} + (1 - r^2 r_s / r_g^3)] .$$

Now a Taylor series expansion of the middle term, on the *right*-hand side, gives us, to first order, for *small* r …

$$(1 - r^2 r_s / r_g^3)^{1/2} \approx 1 - (1/2) \cdot r^2 r_s / r_g^3 ,$$

so that if we insert this into the first equation above, and collect terms, we obtain

$$(-1/4) \cdot [9 - 6 + 3 \cdot r^2 r_s / r_g^3 + (1 - r^2 r_s / r_g^3)] = -(1 + (1/2) \cdot r^2 r_s / r_g^3) ,$$

and this is equal to

$$-(1 + ½ \cdot (k\rho/3) \cdot r^2) . \quad (*)$$

Thus, from (*) above, $g_{t,t}(r) = 1 + ½ \cdot (k\rho/3) \cdot r^2$, and this differs from $g_{t,t}(r)$ on page 403, under the heading ‘Other Considerations’, since there $g_{t,t}(r) = 1 + (k\rho/3) \cdot r^2$, if $\mathscr{D} \cdot e = 1$. Still, the agreement is not too bad, considering that in ‘Other Considerations’ we postulated that a solution to $G^{u,v} \approx 0$ might be feasible, if we let σ ---› 0 in our approximate solution to $G^{u,v} \approx \sigma g^{u,v}(0)$, as r ---› 0.

SOME ADDITIONAL THOUGHTS

Suppose the *interior* metric $\mathcal{M}$, shown on page 404, is actually

$$c^2 d\tau^2 = -\left(2\sqrt{1-\frac{r_s}{r_g}} - \sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2 c^2 dt^2 + \left(1-\frac{r^2 r_s}{r_g^3}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) ,$$

at least near the *origin* of S*. Then even at the *edge* of S*, with $r = r_g$, we see that $g_{t,t}(r)$ agrees with its counterpart in the *exterior* metric. And of course, the same is true with $g_{1,1}(r)$.

But in this case, if $g_{t,t}(r)$ is expanded as it was above, making the same assumptions, then we find that $g_{t,t}(r) = 1 + (k\rho / 3) \cdot r^2$, and this agrees *exactly* with $g_{t,t}(r)$ on page 403, under ‘Other Considerations’ [for our own sun, $r_s / r_g \approx 0.000004$].

Should we accept the interior metric above as being closer to the truth, than the one on page 404, at least near the origin of S* ? No, not necessarily, but if we do, we achieve remarkable agreement with what we see on page 403 in ‘Other Considerations’. In fact, the agreement is *exact* !

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part XIV

In this note, we want to review our work and see if there are any alternatives to $g_{t,t}(r)$, that might be compatible with the original *interior* Schwarzschild metric $\mathcal{M}$, as outlined on pages 404-5. We know that with

$$g_{1,1}(r) \approx 1 - (1/3) \cdot (\sigma - k\rho) \cdot r^2 ,$$

the agreement with its counterpart in $\mathcal{M}$ is *exact* for *small* r, and $r_s / r_g \approx 0$, if $\sigma = 0$.

For some motivation here, let's look at the Friedmann acceleration equation, for an isotropic universe filled with a *perfect* fluid.

$$3\frac{\ddot{a}}{a} = \Lambda - 4\pi G(\rho + 3p)$$

And here, Λ [which is akin to σ] is the cosmological constant, and $\ddot{a}$ is a kind of acceleration parameter, which is related to the size of the universe, denoted by *a*.

*****

Now if Λ is *zero* and the universe is *static*, it's rather like our perfect star S*, where $\sigma = 0$, and there is no flow of material within S*. In such a case, $\ddot{a} = 0$, so that $3p + \rho = 0$ (*), and we should expect the same of our 'equation of state' in S*. Thus, we are looking for alternatives to

$$g_{t,t}(r) \approx 1 - (1/3) \cdot (\sigma - k\rho) \cdot r^2 , \quad (\sim)$$

which *don't* violate (*).

Since we believe the first term in (~) is acceptable, all that remains is the second term [$g_{t,t}(r)' = 0$ at the origin O], and we'll suppose this can be written as a multiple of what we see in (~); that is to say,

$$g_{t,t}(r) \approx 1 - (\alpha/3) \cdot (\sigma - k\rho) \cdot r^2 , \quad (\dagger)$$

where α is some constant.

If we now recover our equation of state from page 397, and factor in α, we then have …

$$-(2\alpha /3) \cdot (\sigma - k\rho) + (1/3) \cdot (\sigma - k\rho) = kp + \sigma .$$

And setting $\sigma = 0$, we find that $(2\alpha - 1)\rho = 3p$, so that necessarily $\alpha = 1$, in order to attain the equation of state, without respect to sign conventions. Thus, $g_{t,t}(r)$ in (~) is the correct choice here.

*****

We could also choose $\alpha = 0$, but if we did, we'd lose the parabolic component and $g_{t,t}(r)$ would evaluate to 1.

*****

When developing equations of state, such as we did on pages 393-4 and pages 396-7, we always have to remember that the analysis was done within the context of an 'in the limit' argument, as the radius r ---› 0.

For example, on page 393, the equation of state we developed was …

$$\sigma = -(k/2)\cdot(3p + \rho), \qquad (\dagger\dagger)$$

where σ is the dark energy pouring into the perfect star [S*], at its origin O, according to the density function $\lambda(s) \approx \sigma / s$.

Now if $\sigma = 0$, then necessarily $3p + \rho = 0$ in (††), and for an *incompressible* fluid, the density ρ is *not* changing in S*, so we can arbitrarily choose the value of ρ at O, if we like. For pressure p in S*, it may be a little more subtle, since it's not clear to me if perfect fluids $\mathscr{F}$ exist for S*, where p is constant everywhere in the star, itself.

If such an $\mathscr{F}$ does exist, we can choose any point in S* when referencing p, just like we could for ρ, otherwise we'd probably want to reference p at O, relative to the equation (††) above. And that is because all of our analysis, starting on page 383 and going forward from there, uses a lim r ---› 0 argument …

*****

In a paper on the Internet, titled '*On Schwarzschild's Interior Solution and Perfect Fluid Star Model*', by Barletta et al (2020), the authors revisit the original publication by Schwarzschild, and along the way, develop an equation for the pressure $p(r)$, for $r \geq 0$. This equation is [from page 11 of their article] …

$$\frac{p_0}{c^2} + \rho_0 = 2\rho_0 \left| 3 - \frac{1}{r_0}\left(\frac{r_0}{2m} - 1\right)^{-1/2} \left(\frac{r_0^3}{2m} - r^2\right)^{1/2} \right|^{-1}$$

Here, r_0 is the radius of the perfect star [S*] … and p_0 and ρ_0 are pressure and density in the star, for $r \geq 0$. As well, because they are dealing with an *incompressible* fluid, ρ_0 is constant everywhere in S*. And finally, the authors define m to be GM / c^2, where M is the mass of the star.

Now when r = 0, the expression above becomes

$$p_0 / c^2 + \rho_0 = 2\rho_0 \,|\, 3 - (1 / r_0)\cdot(r_0 / 2m - 1)^{-1/2}\cdot(r_0^3 / 2m)^{1/2} \,|^{-1},$$

and this can be written as …

$$p_0 / c^2 \; + \; \rho_0 \; = \; 2\rho_0 \,|\, 3 \; - \; (1 / r_0) \cdot (r_0 / 2m)^{-1/2} \cdot (1 \; - 2m / r_0)^{-1/2} \cdot (r_0^3 / 2m)^{1/2} \,|^{-1} . \qquad (\&)$$

And with a *very small* Schwarzschild radius [$2m$], so that $2m / r_0$ is negligible, the *right*-hand side of (&) computes to ρ_0. Indeed, in this case we can make $2m / r_0$ *arbitrarily small*, if we wish. Thus, at r = 0, (&) reduces to

$$p_0 / c^2 \; + \; \rho_0 \; = \; \rho_0 \, ,$$

so that pressure p_0, at the center [O] of the perfect star, would be *zero* in this *hypothetical* case.

Clearly this is *not* the true value of p_0 at O, but if we want an 'apples to apples' comparison with the *same* assumption made on pages 404-5; namely that r_s / r_g is a negligible quantity, where r_s is the Schwarzschild radius and r_g the radius of the star, then the conclusion above is 'comparatively fair'.

Now on page 406, we asked if there were any *other* possibilities for $g_{t,t}(r)$, by introducing α into the equation below,

$$g_{t,t}(r) \; \approx \; 1 \; - \; (\alpha/3) \cdot (\sigma \; - \; k\rho) \cdot r^2 \, ;$$

and concluded that our equation of state [$3p \; + \; \rho = 0$ $^{(*)}$] could only hold if $(2\alpha - 1)\rho = 3p$ was true, without respect to sign conventions, when σ = 0. This forced us to choose α = 0 or 1. However, notice here that *if* α = ½, then necessarily $p = 0$ at the center of S*, just as it is for p_0 at O above, in the hypothetical setting.

And by choosing α = ½, our $g_{t,t}(r)$ component now agrees *exactly* with what we calculated for its counterpart in the original *interior* Schwarzschild metric [pp 404-5], where assumptions *coincide*; that is to say, for σ = 0 …

$$g_{t,t}(r) = 1 \; + \; ½ \cdot (k\rho / 3) \cdot r^2 \, .$$

But in order to get this agreement, we have to sacrifice our equation of state (*), and instead accept a value of *zero* for p at O, which I'm not sure we want to do.

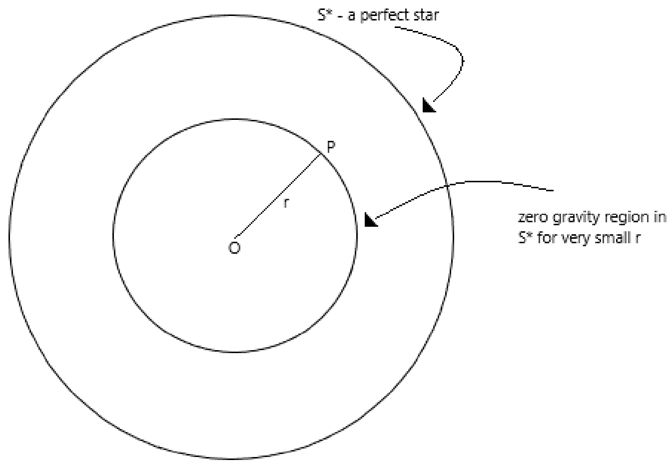
Some Notes On $G^{u,v} \; \approx \; \sigma g^{u,v}(0)$ When $\lambda(s) \; \approx \; \sigma / s$, Part XV

Let us bring back our diagram from page 402, where here σ = 0 in the F_0 frame, and our metric becomes, from 'Other Considerations' on page 403 …

$$ds^2 \; \approx \; g_{1,1}(r)dr^2 \; + \; r^2 d\theta^2 \; + \; r^2\sin^2(\theta)d\phi^2 \; - \; g_{t,t}(r)dt^2 \, .$$

And now, with $Q(r) = (1/3) \cdot (-k\rho) \cdot r^2$, where $0 \; < \; k\rho$, and $\mathscr{D}$ is a *positive* constant, we have …

$$g_{1,1}(r) \; \approx \; 1 \; - \; Q(r) \quad ; \quad g_{t,t}(r) \; \approx \; \mathscr{D} \cdot exp(g_{1,1}(r)) \, .$$



In the diagram above, let the ‘distance’ from O to P be the smallest possible ‘length’ or ‘clock tick’, available to us at the quantum or sub-quantum level, so that nothing shorter than this exists in a *measurable* sense of the word. Let us refer to this ‘distance’ as ε.

If we now move from O to P, we must do so by way of a ‘quantum hop’, and since there is nothing smaller than ε, a *single* hop will place us at P, whether we are looking in the spacelike or timelike direction. Indeed, there is no reason to believe that we would even be able to distinguish between these two directions, at such small scales.

To see this more clearly, suppose it takes *two* ‘quantum clock ticks’ [QCTs] to hop from O to P. Then *one* QCT must equal *one-half* a QRT [quantum ruler tick, which is ε], which, by definition, doesn’t exist. Thus the QCT must be *greater* than or *equal* to the QRT.

On the other hand, it can never take *one-half* a QCT to hop from O to P, since by definition, the QCT is the smallest possible ‘clock tick’ available to us. Indeed, if this were so, then *one* QCT would be equal to *two* QRTs, and so we may conclude the QCT must be *less* than or *equal* to the QRT. Putting the two together tells us the QCT must be *equal* to the QRT; namely ε.

*****

In the sequence below, only values for which QCT ≥ QRT are admissible, since by definition, the QRT cannot be divided any further.

$$\text{QCT} = \ldots\ 3\cdot\text{QRT},\ 2\cdot\text{QRT},\ 1\cdot\text{QRT},\ \tfrac{1}{2}\cdot\text{QRT},\ \tfrac{1}{3}\cdot\text{QRT},\ \ldots$$

Similarly, in the sequence below, only values for which QRT ≥ QCT are admissible, since by definition, the QCT cannot be divided any further.

$$\text{QRT} = \ldots\ 3\cdot\text{QCT},\ 2\cdot\text{QCT},\ 1\cdot\text{QCT},\ \tfrac{1}{2}\cdot\text{QCT},\ \tfrac{1}{3}\cdot\text{QCT},\ \ldots$$

Thus, it must be the case, that …

$$\text{QRT} \;\leq\; \text{QCT} \;\leq\; \text{QRT} \;\leq\; \text{QCT}\,,$$

so that necessarily, the QCT and QRT must be *equal*.

*****

Thus, there is *no* difference between the 'clock tick' and the 'ruler tick' at these very small scales, so that if we want to formulate a metric for ds^2 at the 'classical level', which reflects this quantum-like behavior (r ---› 0), it behooves us to search for one where

$$g_{1,1}(r) \approx 1 - Q(r) \quad ; \quad g_{t,t}(r) \approx \mathcal{D} \cdot exp(g_{1,1}(r))$$

are, to a first-order approximation, *equal* to one another. And this can be done by setting $\mathcal{D} \cdot e = 1$.

Now as we said earlier, $g_{t,t}(r)$, as shown here, differs from its counterpart in the original *interior* Schwarzschild metric [$\mathcal{M}$] by a factor of ½, in the parabolic component [pp 404-5]; but I would submit to the reader that this difference is due to quantumlike behavior, as described above, that is not reflected in $\mathcal{M}$, at very small scales near the origin [O] of the star.

If this behavior was reflected in $\mathcal{M}$, then the metric below might be closer to the truth near O, since here $g_{t,t}(r)$ agrees with its counterpart just above, if r_s / r_g is negligible, and r ---› 0 [p 405] …

$$c^2 d\tau^2 = \; -\left(2\sqrt{1-\frac{r_s}{r_g}} - \sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2 c^2 dt^2 + \left(1-\frac{r^2 r_s}{r_g^3}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Some Notes On $G^{u,v} \approx \sigma g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$, Part XVI

In our original analysis of $g = g_{t,t}(r)$, associated with the *original* interior Schwarzschild metric [pp 404-5], and shown below [labelling as (~)],

$$\frac{1}{4}\left(3\sqrt{1-\frac{r_s}{r_g}} - \sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2$$

we treated r_s / r_g as a *negligible* quantity in the first term above, but *retained* r_s / r_g^3 in the second term. In some ways, this really isn't an 'apples to apples' comparison with $h = g_{t,t}(r)$, as shown on page 403 in 'Other Considerations', and reproduced below, where the scaling factor is $\mathcal{D} \cdot e = 1$ …

$$1 + (k\rho/3) \cdot r^2\,.$$

And that is because for $0 \leq r \leq r_g$, g is an *increasing* function bounded *above* by 1, whereas h is also an *increasing* function bounded *below* by 1. But this particular problem is easily remedied by setting $\mathcal{D} \cdot e$ to $g(0)$, and then rescaling so that $f = \mathcal{D} \cdot e \cdot h = g(0) \cdot h$.

Now with this rescaling, $g(0)$ and $f(0)$ agree, and if we expand $g(0)$ as a Taylor series, for *small* values of r_s / r_g, then f becomes [remembering from earlier notes that $r_s / r_g^3 = k\rho/3$] …

$$f \approx ¼ \cdot \big(3 \cdot (1 - r_s / 2r_g) - 1\big)^2 \cdot \big(1 + (r_s / r_g^3) \cdot r^2\big) .$$

And, omitting the term $(r_s / 2r_g)^2$ in this expansion, we obtain [again for *small* r_s / r_g] …

$$f \approx \big(1 - 3r_s / 2r_g\big) \cdot \big(1 + (r_s / r_g^3) \cdot r^2\big) , \qquad (*)$$

and this is the result we are looking for in f, that tracks g for *small* r.

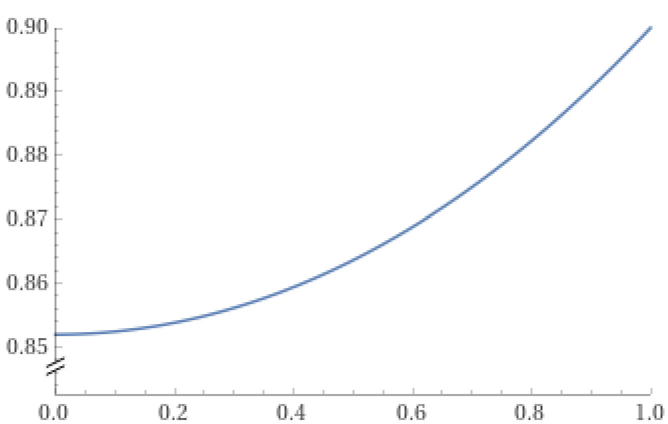
Now notice here that to first-order, f and g *do* agree at the origin O of the star [S*], and *even* when we let $r = r_g$ in (*) above, f computes to

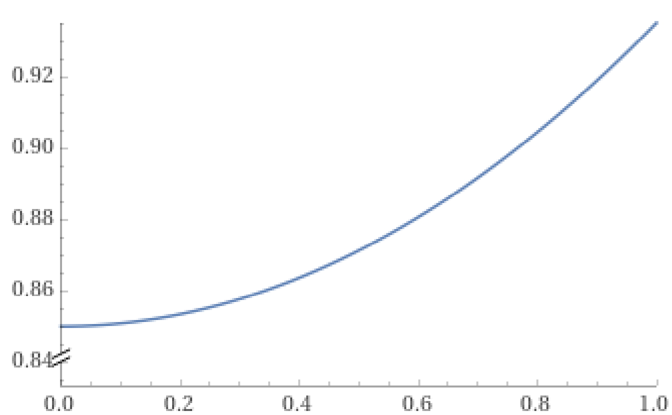
$$\big(1 - 3r_s / 2r_g\big) \cdot \big(1 + (r_s / r_g)\big) ,$$

which, to first-order, becomes $1 - r_s / 2r_g$. Comparing to g at $r = r_g$, this value is $1 - r_s / r_g$. Thus f and g are both *increasing* functions which agree at O [and thus track each other closely for *small* r], but differ by $r_s / 2r_g$ on the boundary of the star. This is a fairly decent result, since our analysis of things, beginning on page 383 and going forward from there, is largely a lim r ---› 0 argument. But to get this result, we have to rescale $\mathscr{D} \cdot e$, as we said above.

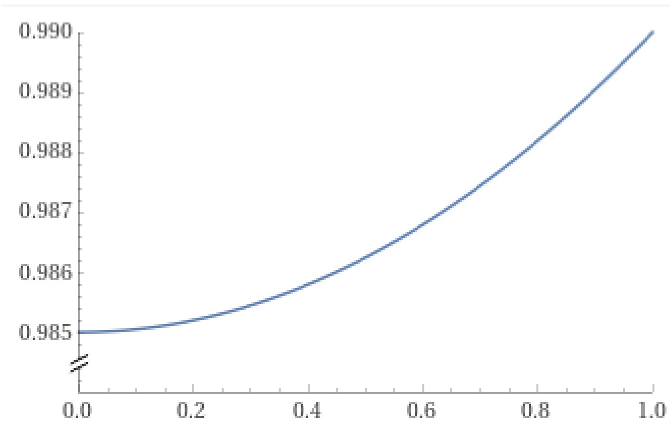
And finally, any analysis done *after* pp 404-5, should now give consideration to what we've said here in Part XVI, as I do believe rescaling is the more appropriate approach.

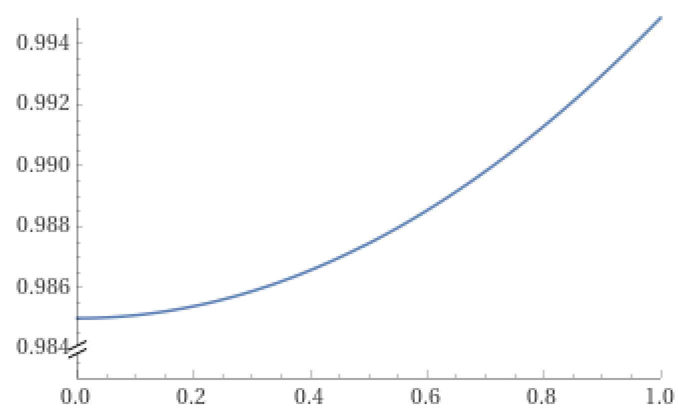
In the pictures below, $r_s = 0.1$, $r_g = 1$, and $0 \le r \le r_g$. The picture on the *left* is $g = g_{t,t}(r)$ from the *original* interior Schwarzschild metric [~] on page 410 of this research note. The diagram on the *right* is $f = g_{t,t}(r)$ and labelled (*) on this page. Notice that here, r_s / r_g is *not* small.





In the pictures below, $r_s = 0.01$, $r_g = 1$, and $0 \leq r \leq r_g$. The picture on the *left* is $g = g_{t,t}(r)$ from the *original* interior Schwarzschild metric [~] on page 410 of this research note. The diagram on the *right* is $f = g_{t,t}(r)$ and labelled (*) on page 411. In this case r_s / r_g is small, but not overly so …





At $r = 0$, the agreement is exact, and at $r = r_g$ (which is 1), $g_{t,t}(r)$ on the *left* matches $1 - r_s / r_g$, while on the *right*, $g_{t,t}(r)$ matches $1 - r_s / 2r_g$, as it should be.

*****

One could ask if there is an *optimal* $\alpha > 0$ for which $g = g_{t,t}(r, \alpha)$ below, agrees with its counterpart f, at *both* the origin *and* at $r = r_g$. Such an α does indeed exist, and we'll see that it is actually $\alpha = 1$.

$$\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}} \right)^2$$

If we perform the same analysis as we did above, then f becomes, for *small* r_s / r_g …

$$f(r, \alpha) \approx \left(1 - (\alpha + 1) r_s / r_g\right) \cdot \left(1 + (r_s / r_g^3) \cdot r^2\right), \quad (\dagger)$$

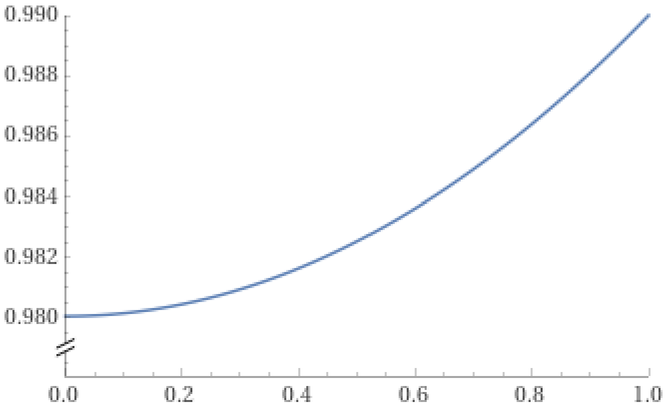
and here f and g agree at O. We saw already that with $\alpha = ½$, this was true from the analysis above.

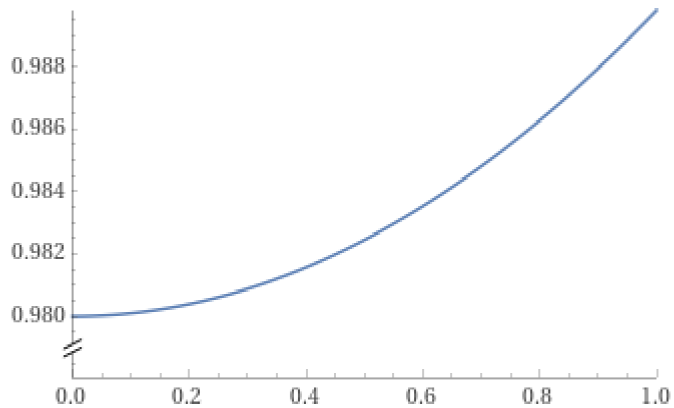
Now at $r = r_g$, $g_{t,t}(r, \alpha)$ *still* computes to $1 - r_s / r_g$, as it did above when $\alpha = ½$, and if we set $r = r_g$ in (†), and omit the term $(\alpha + 1)(r_s / r_g)^2$ in the expansion, we find that

$$f(r_g, \alpha) \approx 1 - \alpha \cdot r_s / r_g .$$

Thus, the *optimal* choice for α is indeed 1.

In the pictures below, $r_s = 0.01$, $r_g = 1$, $\alpha = 1$, and $0 \leq r \leq r_g$. The picture on the *left* is $g = g_{t,t}(r, \alpha)$ from page 412 of this research note. The diagram on the *right* is $f = f\,(r, \alpha)$ and labelled (†) on page 412. In this case r_s / r_g is small, but not overly so …





You can see from these pictures that they agree when $r = 0$ *and* when $r = r_g$.

A BRIEF PHILOSOPHICAL DISCUSSION OF THE TWO ANALYSES

On pages 404-5, we started our analysis by setting $\alpha = ½$ in the *interior* Schwarzschild component $g = g_{t,t}(r, \alpha)$ below, which is the *accepted* classical value,

$$\left((\alpha+1)\sqrt{1-\frac{r_s}{r_g}}-\alpha\sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2$$

and omitted r_s / r_g in the first term. This allowed us to 'lift' g above a *lower* bound of 1, so that we would be able to compare it to our component $h = g_{t,t}(r)$ on page 403 under 'Other Considerations'; that is to say,

$$h = 1 + (r_s / r_g^3) \cdot r^2 \,,$$

where again, $r_s / r_g^3 = k\rho/3$. And this we were able to do *without* rescaling h, and indeed, this was part of the motivation for lifting g in the first place.

Now admittedly, the analysis was largely *theoretical* in nature, and may have no *physical* value at all; but it did allow us to say some things about the *optimal* choice for α [which is 1; pp 405, 410, 412], and too, allowed us to investigate 'quantum clock ticks and ruler ticks' [pp 408-10].

And, the analysis also led to a *second* approach [pp 410-413] where we decided to rescale h, which, in turn, led to some interesting results that may have both theoretical *and* physical value. Perhaps the *most* significant conclusion was that *both* analyses concluded $\alpha = 1$ was *optimal* for g.

SEEING THE CONNECTIONS

Let us bring back our expression for $g = g_{t,t}(r, \alpha)$ below, the *interior* Schwarzschild component, and

$$\left((\alpha+1)\sqrt{1-\frac{r_s}{r_g}}-\alpha\sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2$$

recall that in the 'lifted frame' $\mathcal{L}$, the expression r_s / r_g in the *first* term is omitted, so that a *lower* bound of 1 is established in any calculations. This allows us to produce the function below,

$$h = 1 + \alpha \cdot (r_s / r_g^3) \cdot r^2$$

where h is computed from a Taylor series expansion of g, for *small* r. And recall as well that r_s / r_g^3 is equal to $k\rho / 3$. Note that when $\alpha = 1$, we recover $g_{t,t}(r)$, as shown on page 403 in 'Other Considerations', where the scaling factor is $\mathcal{D} \cdot e = 1$ and $g_{t,t}(r)$, itself, is derived from a lim r ---› 0 analysis [p 383 *ff.*]. It thus becomes our *de facto* standard.

There are three cases to consider. First, if $\alpha = 0$, then g and h agree in $\mathcal{L}$, and both are 1. And furthermore, from our expression developed on page 406, and shown below,

$$(2\alpha - 1)\rho = 3p \,, \qquad (*)$$

we see that for $\alpha = 0$, our 'equation of state', *also* developed using a lim r ---› 0 argument, is preserved *without* respect to sign conventions [pp 406-8]. Notice too, that the α variable used in (*) corresponds to the α variable in g and h, in the lifted frame $\mathcal{L}$ [again pp 406-8], and it is important to see this connection.

In the second case, if $\alpha = 1$, then g matches h in $\mathcal{L}$, using a Taylor series expansion for *small* r, and additionally our 'equation of state' is again preserved from (*), *without* respect to sign conventions. There are, however, no other choices for α that will preserve the 'equation of state'.

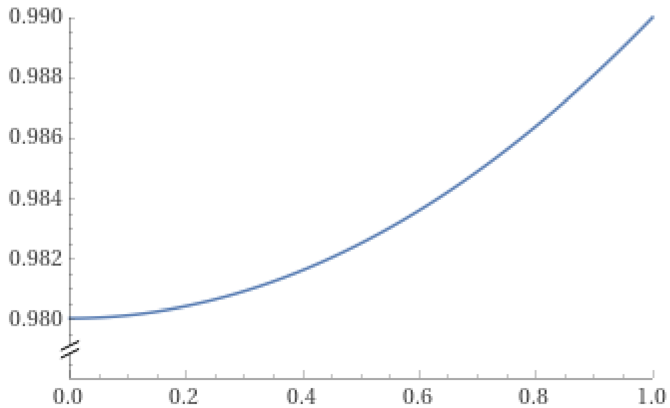
And this brings us to the last case; namely $\alpha = ½$, which is the *accepted* classical value for g, that is to say, the *interior* Schwarzschild *time* component $g_{t,t}(r, \alpha)$. In this case, if we lift g into $\mathcal{L}$, a Taylor series expansion gives us, for *small* r,

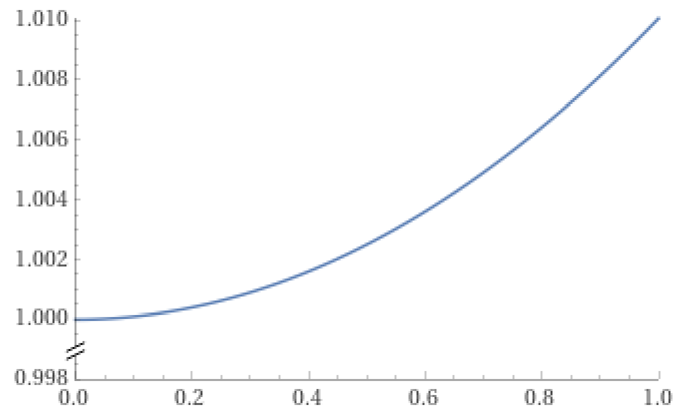
$$g = 1 + ½ \cdot (r_s / r_g^3) \cdot r^2 \,,$$

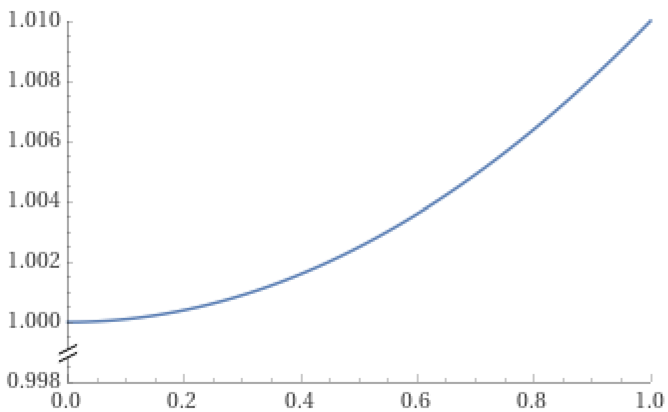
and this does agree with h. But from (*), our `equation of state' is violated now, since the pressure p in our perfect star [S*] becomes *zero*, while the density ρ in S* remains in tact. And thus, our 'equation of state'; namely $3p + \rho = 0$, *without* respect to sign conventions, no longer holds.

Thus, by using a 'lifting argument' and comparing to our lim r ---› 0 analysis [p 383 *ff.*], we see that $\alpha = ½$ is really not a viable choice for the Schwarzschild component $g = g_{t,t}(r, \alpha)$.

In the pictures below, $r_s = 0.01$, $r_g = 1$, $\alpha = 1$, and $0 \le r \le r_g$. The picture on the *left* is $g = g_{t,t}(r, \alpha)$ from page 414 of this research note. The diagram on the *right* is $g_{t,t}(r, \alpha)$ after *lifting* it into $\mathcal{L}$, and the third diagram (bottom row) is *h* from page 414. In this case r_s / r_g is small, but not overly so …







You can see from the pictures above, that the second and third diagrams look identical, as they should. By lifting *g* into $\mathcal{L}$, we can compare it to a corresponding parabolic curve *h*, which adheres to (*) on page 414, as shown below…

$$(2\alpha - 1)\rho = 3p \,. \qquad (*)$$

Only when $\alpha = 0$ or 1, does (*) conform to our 'equation of state' below, *without* respect to sign conventions, that was developed in the original analysis [p 383 *ff.*] …

$$3p + \rho = 0 \,.$$

SEEING THE CONNECTIONS, PART II

Let us bring back our expression for $g = g_{t,t}(r, \alpha)$ below, the *interior* Schwarzschild component, and

$$\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}}\,\right)^2$$

here, we wish to expand *both* terms above as a Taylor series, when r_s / r_g is *small*. This gives us, to first-order …

$$f(r, \alpha) \approx \{1 - (\alpha + 1)r_s / r_g\} \cdot \left(1 + \beta(r_s / r_g^3) \cdot r^2\right), \quad (\dagger)$$

where

$$\beta = \alpha / \left(1 - (\alpha + 1)r_s / r_g\right). \quad (\sim)$$

Thus, f is essentially a lift into $\mathcal{L}$, which is then 'pulled back' by an offset of $-(\alpha + 1)r_s / r_g$, and so, is bounded *above* by 1.

Note that when $r = 0$, both f and g agree, since g expands as a Taylor series in the *first* term for small r_s / r_g. And for *any* $\alpha \geq 0$, when $r = r_g$, both f and g agree and are equal to $1 - r_s / r_g$. This is a significant improvement over our *rescaling* analysis on pages 410-413.

Now the rescaling term $\{1 - (\alpha + 1)r_s / r_g\}$, in (†) above, does *not* affect our equation of state [pp 396-7], but the coefficient β in the parabolic component *does*. As such, the expression

$$(2\alpha - 1)\rho = 3p,$$

developed on page 406, now changes to

$$(2\beta - 1)\rho = 3p.$$

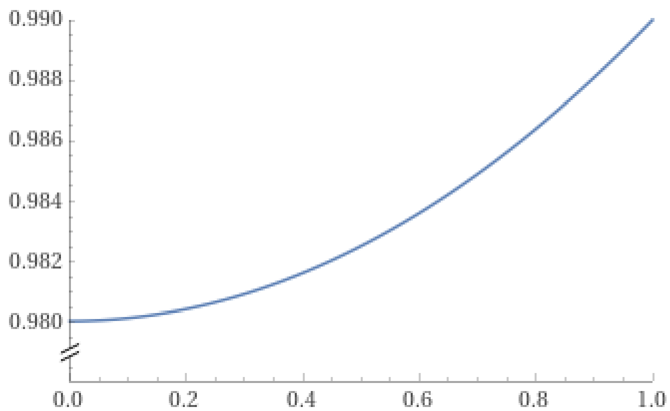
And so, our 'equation of state' is now preserved, *without* respect to sign conventions, when $\beta = 0$ or when $\beta = 1$.

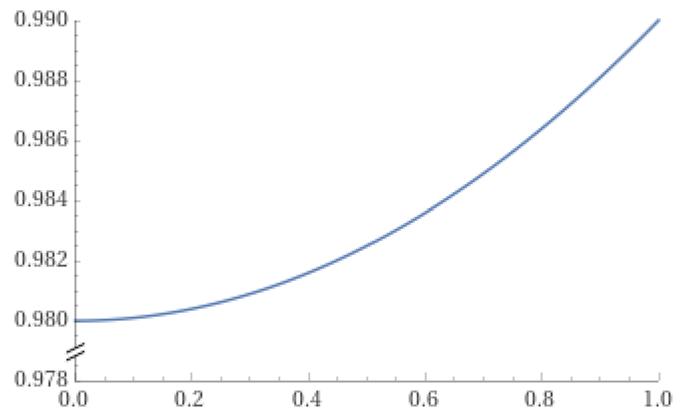
Now when $\beta = 0$, α is also 0, so that f and g agree, and are equal to $1 - r_s / r_g$, for *all* $0 \leq r \leq r_g$. On the other hand, when $\beta = 1$, (~) tells us that

$$\alpha = (1 - r_s / r_g) / (1 + r_s / r_g). \quad (*)$$

For most stars, a *small* r_s / r_g is appropriate, so that from (*) just above, $\alpha \approx 1$ is the *optimal* choice for g, according to our analysis. For the classically accepted value of $\alpha = ½$ in g, (*) now tells us that r_s / r_g is about ⅓, which would be more appropriate for something like a neutron star.

In the pictures below, $r_s = 0.01$, $r_g = 1$, $\alpha = 1$, and $0 \le r \le r_g$. The picture on the *left* is $g = g_{t,t}(r, \alpha)$ from page 416 of this research note. The diagram on the *right* is $f = f\,(r, \alpha)$ and labelled (†) on page 416. In this case r_s / r_g is small, but not overly so …





You can see from these pictures above, that they agree when $r = 0$ *and* when $r = r_g$, and that they are virtually identical …

SEEING THE CONNECTIONS, PART III

In our analysis on pages 410-13, which used *rescaling*, we concluded that $\alpha = 1$ was optimal for the *interior* Schwarzschild component $g = g_{t,t}(r, \alpha)$, and did so *without* any *direct* reference to the 'equation of state', which held true, nonetheless.

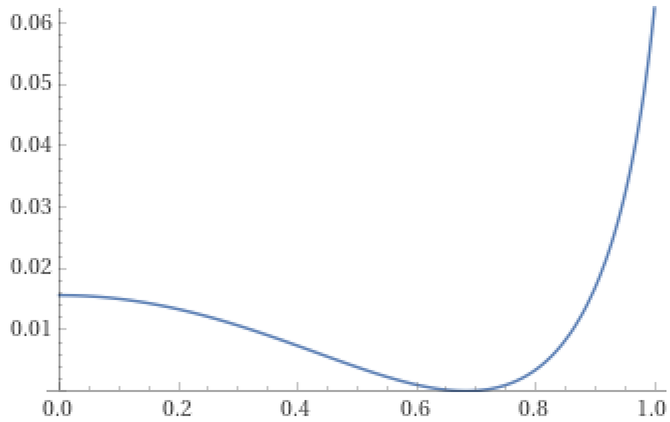
This gives us some confidence that the result (*) from page 416, and shown below, may in fact, be plausible, since for *small* r_s / r_g, α is very nearly 1 here.

$$\alpha \;=\; (1 \;-\; r_s / r_g) / (1 \;+\; r_s / r_g)\,. \qquad (*)$$

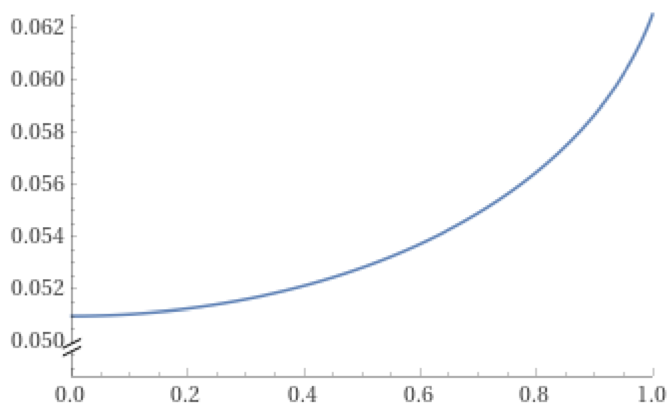
But what happens to $g = g_{t,t}(r, \alpha)$, as shown below, when r_s / r_g is *large* ? Will we still see a parabolic curve from the origin of our star [S*] to the surface, where $r = r_g$?

$$\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}} \right)^2$$

In the picture below, $r_s = 15/16$, $r_g = 1$, $\alpha = 1/2$, and $0 \le r \le r_g$. The picture is $g = g_{t,t}(r, \alpha)$, where we are using the classically accepted value [$\alpha = ½$] in the time component. If the plot was parabolic in nature, all would be well, but in fact, it isn't. Initially it falls away, bottoms out at roughly $r \approx 0.7$, with a value of about zero, and then rises sharply to $1 - r_s / r_g$, which is $1/16$. To say the least, this is anything but parabolic, but it could be made parabolic if α was much, much *smaller* than ½.



So what value of α should we choose ? Well, that's a good question, but if we use (*) above as a *very rough* guide, then $\alpha = 1/31 \approx 0.03225$ might be a starting point. And here is the plot of *g*, using this value of α, where all other parameters stay the same …



Now this is probably *not* the correct choice for α, because (*) is really intended for *small* r_s / r_g , so that our Taylor series expansions for *g* are applicable to first-order. But it gives us some feel for what α has to be, in order to preserve parabolicity, and furthermore, reinforces the idea that α is really *dependent* on r_s / r_g.

For a neutron star, where $r_s / r_g \approx 1/3$, (*) tells us that $\alpha \approx ½$, which is the classically accepted value. In this case, even though r_s / r_g is *not* small, the Taylor series expansions still work fairly well, as you can see in the pictures that follow …

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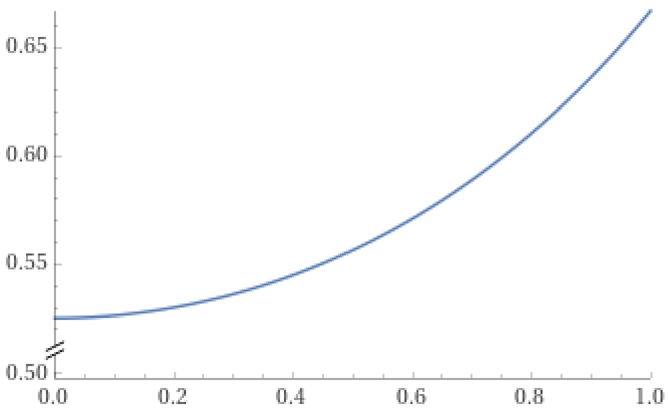
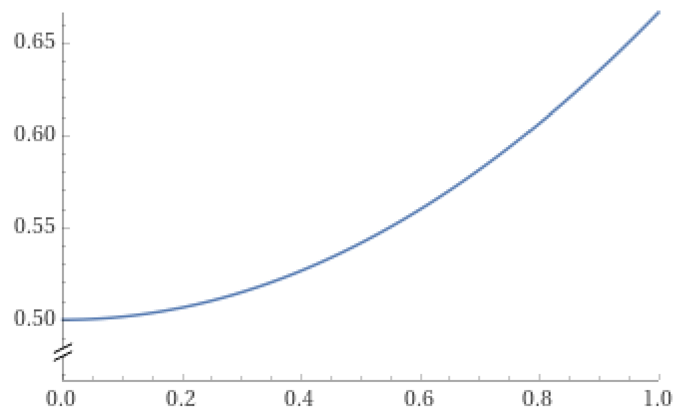
In the pictures below, $r_s = 1/3$, $r_g = 1$, $\alpha = 1/2$, and $0 \le r \le r_g$. The picture on the *left* is the *interior* metric $g = g_{t,t}(r, \alpha)$ from page 417 of this research note. The diagram on the *right* is from the function $f = f\,(r, \alpha)$ on page 416, and reproduced here. In this case r_s / r_g is not small, but we will use it anyway, for purposes of illustration …

$$f\,(r, \alpha) \approx \{1 - (\alpha + 1)r_s / r_g\} \cdot \left(1 + \beta(r_s / r_g^3) \cdot r^2\right)$$

$$\beta = \alpha \,/ \left(1 - (\alpha + 1)r_s / r_g\right)$$





You can see from these pictures, that they agree *exactly* when $r = r_g$, and very nearly agree at the origin O, when $r = 0$. The reason they don't quite agree at O, is because f above is made from Taylor series expansions in g, but *only* to first-order. For a value of $r_s / r_g \approx 1/3$, there is, therefore, bound to be some discrepancy, which we would not see if r_s / r_g was small. Still, for an object like a neutron star, the comparisons are pretty decent, in my opinion …

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Suppose we have an object, such as a neutron star, which now collapses to its Schwarzschild radius r_s, so that $r_s = r_g$. From (*) on page 417, and reproduced here,

$$\alpha = (1 - r_s / r_g) \,/\, (1 + r_s / r_g)\,, \qquad (*)$$

we know that as r_s / r_g *increases*, α *decreases*, and might surmise that in the limit, as r_s / r_g ---› 1, it is the case that α ---› 0. In such a case, our *interior* Schwarzschild time component, $g = g_{t,t}(r, \alpha)$, and shown below, becomes *zero*.

$$\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}}\right)^2$$

Thus, in the collapsed neutron star [$\mathcal{N}_c$], time stands still. It no longer exists …

SEEING THE CONNECTIONS, PART IV

We wish to continue the discussion in Part III of this series, by developing a *second* order expansion for $g = g_{t,t}(r, \alpha)$, as shown below.

$$\left((\alpha+1)\sqrt{1-\frac{r_s}{r_g}}-\alpha\sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2$$

When the dust settles on the calculations, this expansion is [labelling as (†)] …

$$f(r, \alpha) \approx 1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2 + \alpha(r_s / r_g^3) \cdot r^2 - \alpha(\alpha+1)(r_s / 2r_g^2)^2 \cdot r^2 .$$

Now it should be pointed out that there really is *no true* Taylor series expansion of second order, that gives us agreement between f and g, at $r = r_g$, so the *last* term in (†) is actually a *correction* factor that fixes this problem. Thus f and g do agree at $r = 0$ [up to second order], and also agree when $r = r_g$, when we implement the fix.

But what we want to do here, as in Part III, is ask ourselves what happens to α when r_s / r_g ---› 1. And in order to do this, let's rewrite (†) as

$$f(r, \alpha) \approx \{1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2\} \cdot \left(1 + \beta(r_s / r_g^3) \cdot r^2\right) ,$$

where now,

$$\beta = \left(\alpha - \alpha(\alpha+1)(r_s / 4r_g)\right) / \{1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2\} . \quad (*)$$

Now again, our expression from page 416 still holds; that is to say

$$(2\beta - 1)\rho = 3p ,$$

so that our 'equation of state' is attained, *without* respect to sign conventions, if $\beta = 0$ or 1. And when $\beta = 0$, it is the case that $\alpha = 0$, so nothing new here. It is, just as before. And if $\beta = 1$, with the variable $\gamma = r_s / r_g$, then (*) becomes

$$1 - (\alpha+1)\gamma + \alpha(\alpha+1)(\gamma/2)^2 = \alpha - \alpha(\alpha+1)(\gamma/4) . \quad (\sim)$$

Now if, in (~) above, we set $\alpha = 0$, then it is easy to see that $\gamma = 1$. On the other hand, if we now set $\gamma = 1$, then it is again, easy to see that (~) becomes

$$\alpha^2 - 3\alpha = 0 .$$

Thus, either $\alpha = 0$ or $\alpha = 3$. To decide which α is correct, imagine doing the whole exercise all over again, say out to *third* order. We'd eventually wind up with a *cubic* polynomial, instead of the *quadratic* just above, and one of those solutions to the cubic would, in fact, be $\alpha = 0$. But the other two solutions would be *non-zero* values for α, so that if we were to opt for one of the *non-zero*

roots, which one do we pick ? If the two *non-zero* roots are *equal*, it wouldn't matter, but what if they're not ? And similarly for a *fourth* order expansion, and so on. Each expansion will always have $\alpha = 0$ as a root [something that is not too hard to show], but there will be other choices too, especially as the *order* of the expansion increases.

The only logical conclusion here, is to reason that α must be *zero*. Thus, as $r_s / r_g \longrightarrow 1$, it must, in general, be the case that $\alpha \longrightarrow 0$ for the *interior* Schwarzschild metric $g = g_{t,t}(r, \alpha)$, as shown on page 420. And conversely, as $\alpha \longrightarrow 0$, in general, it is the case that $r_s / r_g \longrightarrow 1$.

So in that neutron star [$\mathcal{N}$], which collapses to its Schwarzschild radius [p 419], it is indeed highly likely that $g_{t,t}(r, \alpha) = 0$ for *all* r, such that $0 \le r \le r_s$.

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Now if, in (~), and reproduced below, it is the case that $\gamma = 0$, then necessarily $\alpha = 1$. On the other hand,

$$1 - (\alpha + 1)\gamma + \alpha(\alpha + 1)(\gamma / 2)^2 = \alpha - \alpha(\alpha + 1)(\gamma / 4) . \qquad (\sim)$$

if $\alpha = 1$, then (~) reduces to

$$\gamma^2 - 3\gamma = 0 ,$$

and a similar argument to the one above tells us that the logical choice for γ is *zero*.

And, of course, this agrees with our *first order* analysis in Part II, as $\gamma = r_s / r_g \longrightarrow 0$ or $\alpha \longrightarrow 1$, since there we found that

$$\alpha = (1 - r_s / r_g) / (1 + r_s / r_g) .$$

PUTTING IT ALL TOGETHER

The full metric may now be written as …

$$c^2 d\tau^2 = -\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}}\right)^2 c^2 dt^2 + \left(1 - \frac{r^2 r_s}{r_g^3}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

and from what we've learned previously, $\alpha = 1$ implies $\gamma = 0$, so that the expression above becomes

$$c^2 d\tau^2 = -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) .$$

On the other hand, when $\gamma = 1$, it is the case that $\alpha = 0$, and the full metric is now [$0 \le r \le r_s$] …

$$c^2 d\tau^2 = \left(1 - (r / r_s)^2\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) .$$

SOME MORE PLOTS

In the pictures below, $r_s = 2/5$, $r_g = 1$, $\alpha = 1/2$, and $0 \le r \le r_g$. The picture on the *left* is the *interior* metric $g = g_{t,t}(r, \alpha)$ from page 420, and reproduced below.

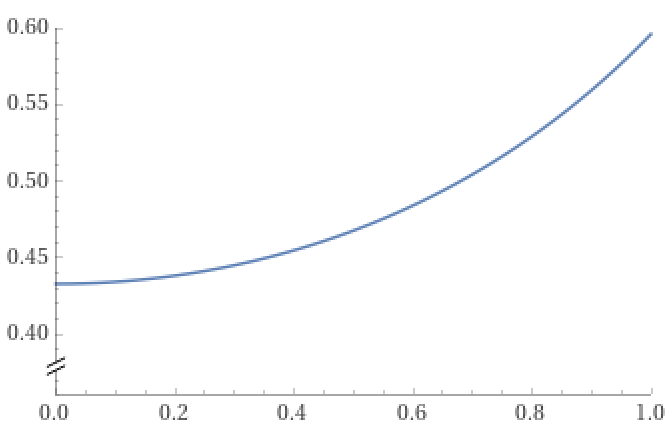
$$\left((\alpha+1)\sqrt{1-\frac{r_s}{r_g}}-\alpha\sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2$$

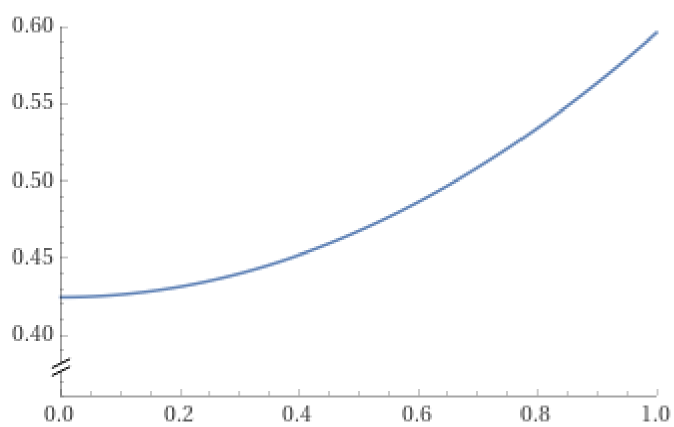
$$f(r, \alpha) \approx 1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2 + \alpha(r_s / r_g^3)\cdot r^2 - \alpha(\alpha+1)(r_s / 2r_g^2)^2 \cdot r^2$$

The diagram on the *right* is from the function $f = f(r, \alpha)$ on page 420, and reproduced above. It is the *second* order expansion of $g_{t,t}(r, \alpha)$, where the *rightmost* term is the correction factor, so f and g agree when $r = r_g$.

In this case $\gamma = r_s / r_g \approx 2/5$ is not small, but it is the value calculated from (~) on page 420, and shown below, when $\alpha = 1/2$, now out to *second* order. Recall the *first* order estimate for r_s / r_g was 1/3 when $\alpha = 1/2$ [see page 416].

$$1 - (\alpha+1)\gamma + \alpha(\alpha+1)(\gamma/2)^2 = \alpha - \alpha(\alpha+1)(\gamma/4)\,. \qquad (\sim)$$





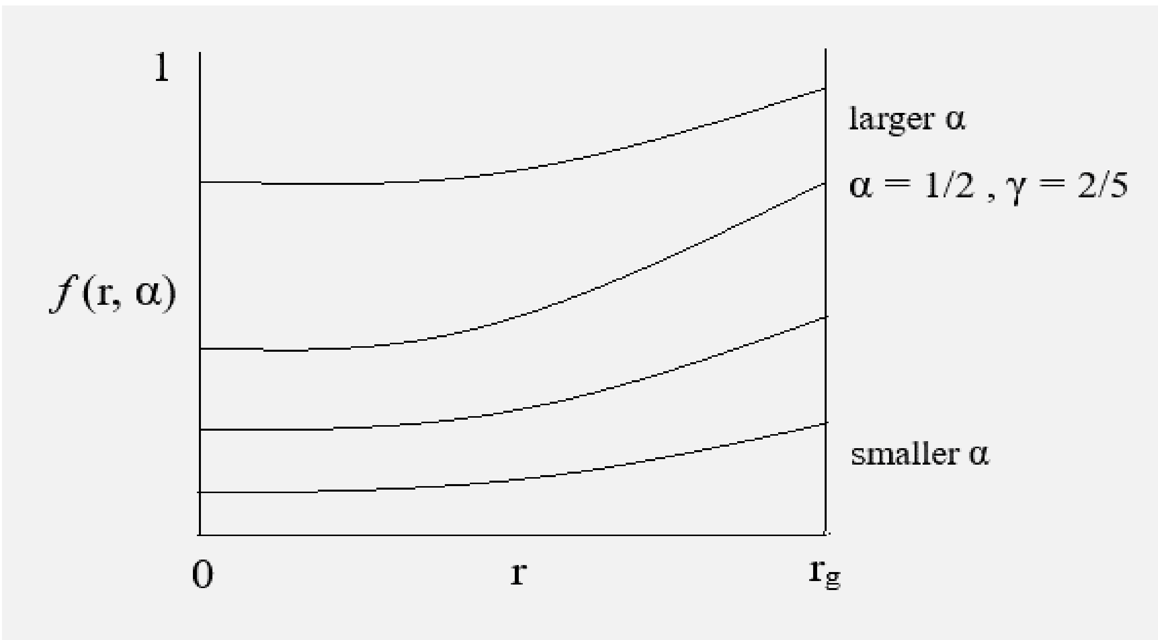
You can see from these pictures, that they agree *exactly* when $r = r_g$, and very nearly agree at the origin O, when $r = 0$. And they track each other quite well for all values of r between 0 and r_g.

SEEING THE CONNECTIONS, PART V

In this note, we are going to study the parabolic curves [f] a little more closely, and develop a formula for the *difference* [Δ] between $f\,(\mathrm{r}, \alpha)$ at $\mathrm{r} = \mathrm{r_g}$ versus $\mathrm{r} = 0$, for *any* given α, in the *second* order expansion of $g_{t,t}(\mathrm{r}, \alpha)$. Recall again, from previous notes, that this expansion is …

$$f\,(\mathrm{r}, \alpha) \;\approx\; 1 \;-\; (\alpha + 1)\mathrm{r_s} / \mathrm{r_g} \;+\; \alpha(\alpha + 1)(\mathrm{r_s} / 2\mathrm{r_g})^2 \;+\; \alpha(\mathrm{r_s} / \mathrm{r_g}^3) \cdot \mathrm{r}^2 \;-\; \alpha(\alpha + 1)(\mathrm{r_s} / 2\mathrm{r_g}^2)^2 \cdot \mathrm{r}^2 \,.$$

In the picture below we see a family of parabolic curves for f, where α ranges from 0 to 1. In these curves, we expect Δ to be small as α begins to increase away from 0, and at some point, Δ should peak in value, and then become smaller again as α approaches 1.



Indeed, from previous research, $\alpha = 0$ represents the *time* component of the *interior* Schwarzschild metric, in a collapsed star [$\mathrm{r_g} = \mathrm{r_s}$], which we believe is *zero*, while $\alpha = 1$ represents the *time* component in a vacuum, which is *one*. So somewhere in between $\alpha = 0$ and $\alpha = 1$ is the peak value for Δ, that we are looking for.

If we define $\gamma = \mathrm{r_s} / \mathrm{r_g}$, then Δ computes to

$$\Delta \;=\; -\gamma \;+\; (\alpha + 1)\gamma \;-\; \alpha(\alpha + 1)(\gamma / 2)^2 \,,$$

and from our expression (~) on page 422 and reproduced here,

$$1 \;-\; (\alpha + 1)\gamma \;+\; \alpha(\alpha + 1)(\gamma / 2)^2 \;=\; \alpha \;-\; \alpha(\alpha + 1)(\gamma / 4) \qquad (\sim)$$

we can substitute what $\alpha(\alpha + 1)(\gamma / 2)^2$ computes to in (~), back into Δ, so that we obtain the following expression for Δ …

$$\Delta \;=\; 1 \;-\; \alpha \;+\; \alpha(\alpha+1)(\gamma/4) \;-\; \gamma\,.$$

Notice here that for $\Delta = 0$ and $\alpha = 0$, it is the case that $\gamma = 1$, which is the case of the collapsed star. On the other hand, if $\Delta = 0$ and $\gamma = 0$, then $\alpha = 1$, which is our vacuum, as we said above.

Now if we rewrite (~) above as a parabolic equation in γ, we obtain

$$\alpha(\alpha+1)\gamma^2 \;+\; (\alpha^2 - 3\alpha - 4)\gamma \;+\; 4(1-\alpha) \;=\; 0\,,$$

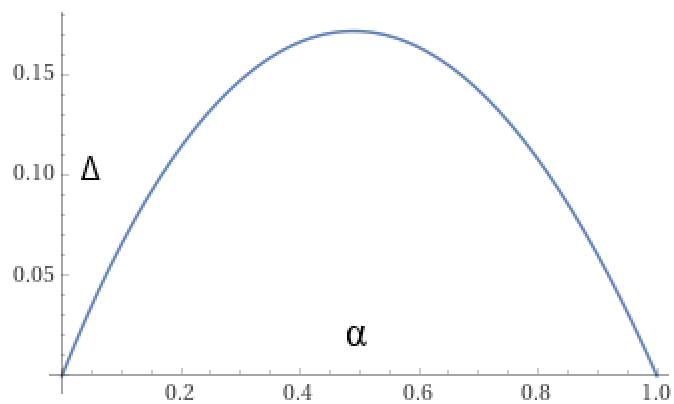
so that the *proper* solution to this quadratic is …

$$\gamma \;=\; \frac{-(\alpha^2 - 3\,\alpha - 4) - \sqrt{(\alpha^2 - 3\,\alpha - 4)^2 + 16\,\alpha\,(\alpha+1)\,(\alpha-1)}}{2\,\alpha\,(\alpha+1)}$$

If we now insert this into Δ above, we finally arrive at our destination, which is …

$$\Delta \;=\; 1 - \alpha + \frac{-(\alpha^2 - 3\,\alpha - 4) - \sqrt{(\alpha^2 - 3\,\alpha - 4)^2 + 16\,\alpha\,(\alpha+1)\,(\alpha-1)}}{2\,\alpha\,(\alpha+1)}\left(\alpha \times \frac{\alpha+1}{4} - 1\right)$$

And here is a plot of Δ, where α ranges between 0 and 1 …



You can see from the picture that Δ peaks at *precisely* $\alpha = ½$, where $\gamma = r_s / r_g \approx 2/5$, which is neutron star territory [see page 422 for some graphs, where Δ is indeed ≈ 0.17]. On either side of $\alpha = ½$, the curve is *perfectly* symmetric, suggesting that the parabolas on page 423 behave in *exactly* the same way *below* $\alpha = ½$, as they do *above* $\alpha = ½$.

This only reinforces our belief that the *time* component of the *interior* Schwarzschild metric, namely $g = g_{t,t}(r, \alpha)$, is *zero* for *all* r, such that $0 \le r \le r_s$, once a star collapses to its Schwarzschild radius …

In looking at our expression for Δ, and reproduced below,

$$\Delta = 1-\alpha+\frac{-(\alpha^2-3\alpha-4)-\sqrt{(\alpha^2-3\alpha-4)^2+16\alpha(\alpha+1)(\alpha-1)}}{2\alpha(\alpha+1)}\left(\alpha\times\frac{\alpha+1}{4}-1\right)$$

it is easy to show that as α ---› 1, Δ ---› 0. But it takes a little more work to show that as α ---› 0, it is *also* the case that Δ ---› 0, because of the denominator term 2α(α + 1).

To show that this is so, let us start by expanding the polynomials inside the square root. Keeping only *first* order terms in α, this becomes, for *small* α …

$$\sqrt{8\alpha + 16}\,,$$

which is approximately α + 4, when expanded as a Taylor series, to *first* order. Now the other term in the numerator is $-(\alpha^2 - 3\alpha - 4)$, and to *first* order, this is 3α + 4. Subtracting α + 4 from 3α + 4 leaves us with 2α, so that Δ becomes, as α ---› 0 …

$$\Delta = 1 - \alpha + (2\alpha / 2\alpha(\alpha+1)) \cdot (\alpha(\alpha+1)/4 - 1)\,.$$

And this reduces to

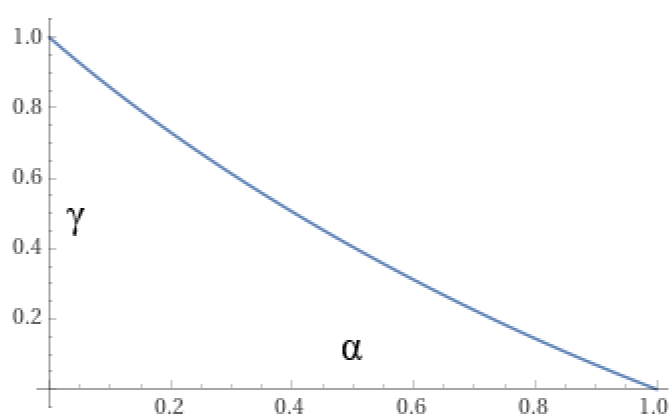
$$\Delta = 1 - \alpha + \alpha/4 - 1/(\alpha+1)\,.$$

If we now let α ---› 0, it is easy to see that Δ ---› 0.

SOME MORE PLOTS

Here is a plot of $\gamma = r_s / r_g$ versus α, using the *second* order expression from the previous page, and reproduced here …

$$\gamma = \frac{-(\alpha^2-3\alpha-4)-\sqrt{(\alpha^2-3\alpha-4)^2+16\alpha(\alpha+1)(\alpha-1)}}{2\alpha(\alpha+1)}$$



SEEING THE CONNECTIONS, PART VI

In this note, we want to go back and study the *first* order model, and show that if Δ_f is the *difference* between $f = f\,(r, \alpha)$ at $r = r_g$ versus $r = 0$, and Δ_g is the *difference* between $g = g_{t,t}(r, \alpha)$ at $r = r_g$ versus $r = 0$, then $\Delta_f \geq \Delta_g$ holds for *all* α between 0 and 1. Thus, if Δ_f ---› 0, so must Δ_g.

Recall again that *f* is the *first* order expansion of *g*, where

$$f\,(r, \alpha) \;\approx\; \{1 \;-\; (\alpha + 1)r_s / r_g\}\cdot \big(1 \;+\; \beta(r_s / r_g^3)\cdot r^2\big)$$

$$\beta \;=\; \alpha / \big(1 \;-\; (\alpha + 1)r_s / r_g\big)\;,$$

and *g* is the *interior* time component of the Schwarzschild metric, as shown below …

$$\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}}\right)^2$$

Since *f* and *g* agree at $r = r_g$, all we really need to do is calculate $\Delta = g(0) - f\,(0)$, and show that Δ is always *greater* than or *equal* to zero, for *any* choice of α.

Now with $\gamma = r_s / r_g$, Δ computes to

$$\Delta \;=\; ((\alpha + 1)\sqrt{1 - \gamma} \;-\; \alpha)^2 \;\;+\;\; (\alpha + 1)\gamma \;\;-\;\; 1\;.$$

Remember too, that in the *first* order model [p 416]

$$\alpha \;=\; (1 \;-\; \gamma) / (1 \;+\; \gamma)\;,$$

and hence, $\gamma = (1 - \alpha) / (1 + \alpha)$, so that we can reduce Δ above to the following inequality test, supposing it to be valid …

$$1 \;-\; \sqrt{1 \;-\; \gamma} \;\;+\;\; \tfrac{1}{2}\cdot(\alpha \;-\; 1) / (\alpha \;+\; 1) \;\;\geq\;\; 0\;.$$

Or more simply, the expression just above becomes

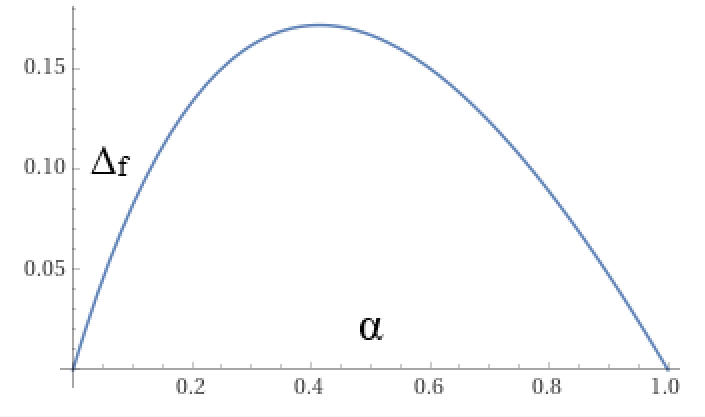
$$(1 \;-\; \gamma / 2)^2 \;\;\geq\;\; 1 \;-\; \gamma\;,$$

which is, indeed, a true statement. Thus, $\Delta_f \geq \Delta_g$ holds for *all* α between 0 and 1, as we said above.

Now Δ_f is relatively easy to calculate and computes to …

$$\Delta_f \;=\; \alpha(1 - \alpha) / (1 + \alpha)\;,$$

and here a plot follows on the next page …

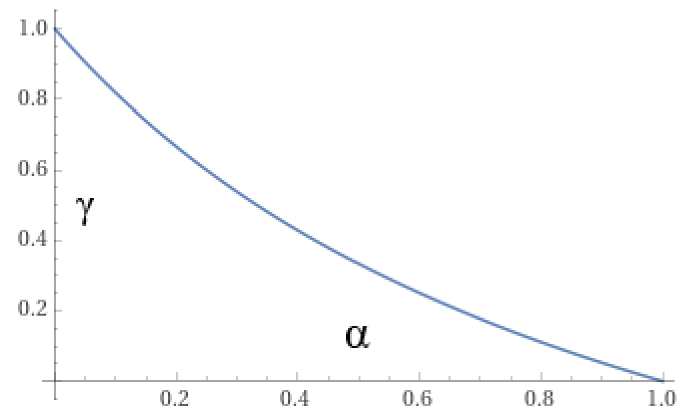


Notice in this picture, as opposed to its counterpart in the *second* order model [p 424], that the parabola is skewed. Nonetheless, the *peak* value of Δ_f is again about ≈ 0.17, which occurs when α is a little greater than 2/5.

Also notice that as α ---› 0 or 1, Δ_f ---› 0 as well. Thus, from our work above, it must *also* be the case that Δ_g ---› 0, which means that at $\alpha = 0$, $g = g_{t,t}(r, \alpha) = 0$ for *all* $0 \le r \le r_s$ (the case of a star collapsing to its Schwarzschild radius, so $\gamma = 1$). When α ---› 1, we see the same effect, and here we are in a vacuum, so that $g = 1$ because $\gamma = 0$.

Were we to repeat the calculations on the previous page, for the *second* order model, we would arrive at a similar conclusion; however, the calculations seem intractable to me. But we do have some pictures to justify our conclusions here; for on page 419 we have plots for a neutron star that confirm our *first* order computations above, and again on page 422 for the *second* order. In both cases $g(0)$ exceeds $f(0)$, even if it's by the slightest amount. Thus, again $\Delta_f \ge \Delta_g$ holds for *all* α between 0 and 1.

Below is a plot of γ versus α, for the *first* order model, which is similar to the *second* order plot on page 425. However, this curve has a little more bend in it …



SOME MORE THOUGHTS, SOME MORE PLOTS

The *first* and *second* order models, discussed above in previous research notes, are largely predictive tools, based on 'equations of state', that tell us what is most likely true when α ---› 0 and hence $\gamma = r_s / r_g$ ---› 1. Indeed, this modelling more generally tells us what is most likely true, as α decreases *away* from 1 (the vacuum) so that γ *increases* away from 0, in the time component *g* of the *interior* Schwarzschild metric, as shown below …

$$\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}}\right)^2$$

Were we to simply solve the differential equations associated with the field equations of general relativity, in this case, we would *not* arrive at *g* above, where α is variable and ranges between 0 and 1; rather, we would conclude that α is *precisely* ½ in *g* and *nothing* else, no matter what γ was. Thus, α becomes the *average* value between 0 and 1 (a curiosity in its own right), and to this day is the classically accepted value for *g*, according to the solutions for the differential equations.

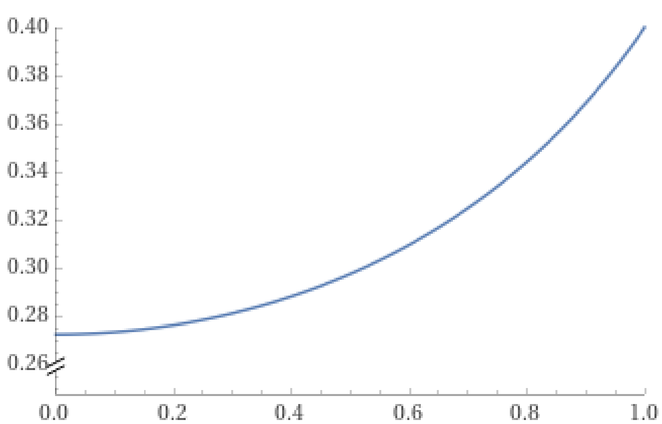
But by doing a lim r ---› 0 analysis of things [p 383 *ff.*], we see that *g* can be generalized, and to me, at least, this is the better approach, as it leads to a very *real* relationship between α and γ that, otherwise, we might never have seen [pp 425 and 427].

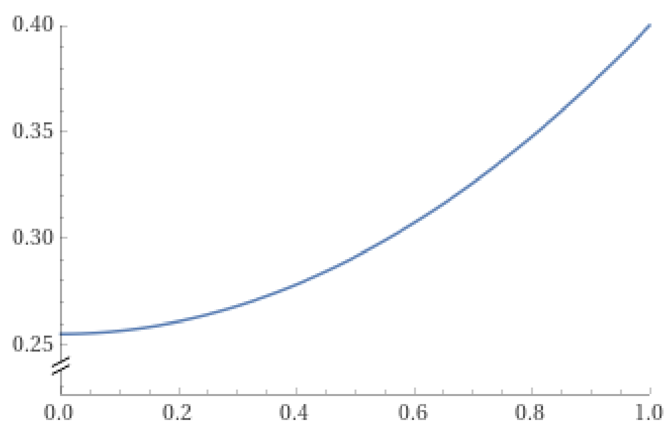
In the pictures below, $r_s \approx 0.6$, $r_g = 1$, $\alpha = 0.3$, and $0 \le r \le r_g$, where α and γ are taken from the *second* order model [p 425]. The picture on the *left* is the *interior* metric $g = g_{t,t}(r, \alpha)$, as shown above.

The diagram on the *right* is from the function $f = f(r, \alpha)$ on page 420, and reproduced below. It is the *second* order expansion of $g_{t,t}(r, \alpha)$, where the *rightmost* term is the correction factor, so *f* and *g* agree when $r = r_g$.

$$f(r, \alpha) \approx 1 - (\alpha + 1)r_s / r_g + \alpha(\alpha + 1)(r_s / 2r_g)^2 + \alpha(r_s / r_g^3) \cdot r^2 - \alpha(\alpha + 1)(r_s / 2r_g^2)^2 \cdot r^2$$

In this case $\gamma = r_s / r_g \approx 0.6$ is *not* small, but it is the value taken from the α-γ chart on page 425, when $\alpha = 0.3$. Indeed, such a γ may not even be *physically* meaningful.





You can see from these pictures, that they agree *exactly* when r = r_g, and nearly agree at the origin O, when r = 0. And they track each other quite well for all values of r between 0 and r_g. Indeed, for a *second* order approximation, *f* tracks *g* surprisingly well, given how large γ really is …

Finally, it should be mentioned that whether we are dealing with the *first* or *second* order model, the *peak* value occurs when $\gamma = r_s / r_g$ is about 2/5, in *both* cases [pp 423-7]. Could it be, at least in the case of a neutron star [$\mathcal{N}$], that $\mathcal{N}$ is most likely to collapse to its Schwarzschild radius, and thus form a black hole, when γ is 2/5 ?

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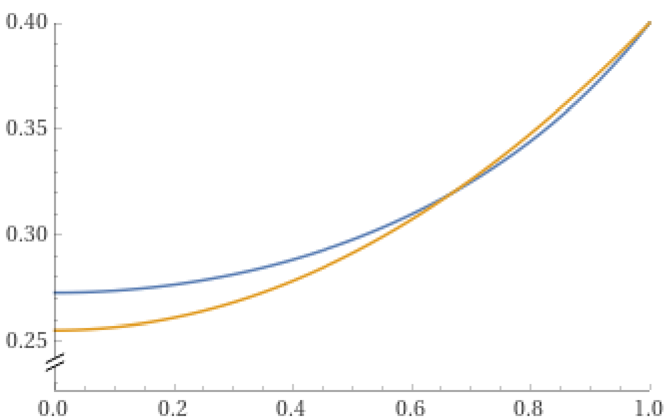
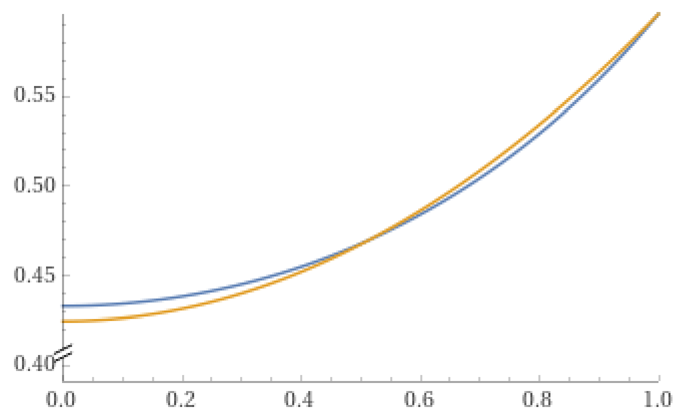
The picture on the *left* is the *overlay* for $r_s \approx 0.6$, $r_g = 1$, $\alpha = 0.3$, and $0 \le r \le r_g$, using the *second* order model, from the last page. The diagram on the *right* is the *overlay* for $r_s \approx 2/5$, $r_g = 1$, $\alpha = 1/2$, and $0 \le r \le r_g$, from page 422, again using the *second* order model. Blue represents the *interior* metric $g = g_{t,t}(r, \alpha)$, as shown on page 428, and gold represents $f\,(r, \alpha)$, also shown on page 428.



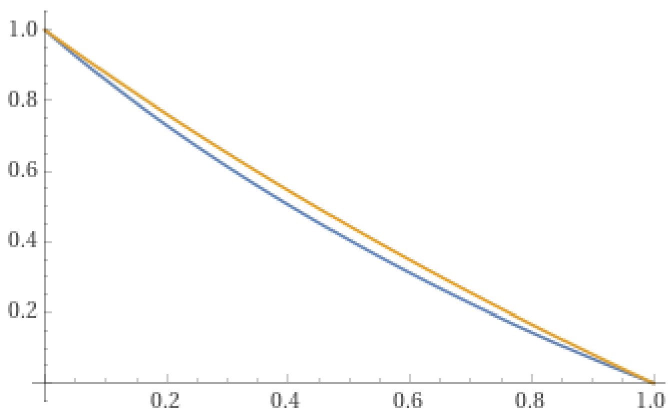


An Interestng Estimator For $\gamma = r_s / r_g$ In The Second Order Model

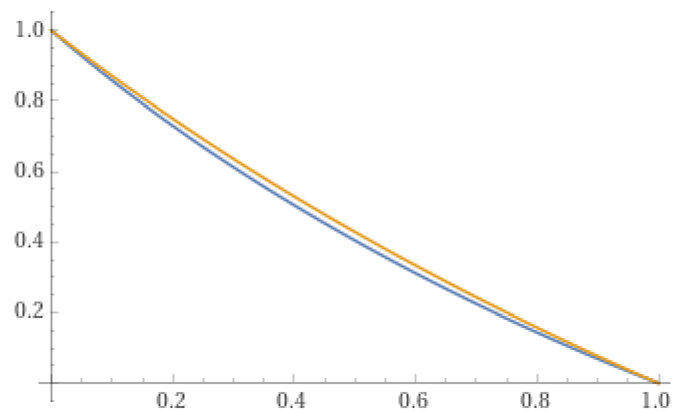
In the *second* order model, we know from previous research, that γ is defined to be …

$$\gamma \;=\; \frac{-\left(\alpha^2 - 3\,\alpha - 4\right) - \sqrt{\left(\alpha^2 - 3\,\alpha - 4\right)^2 + 16\,\alpha\,(\alpha+1)\,(\alpha-1)}}{2\,\alpha\,(\alpha+1)}$$

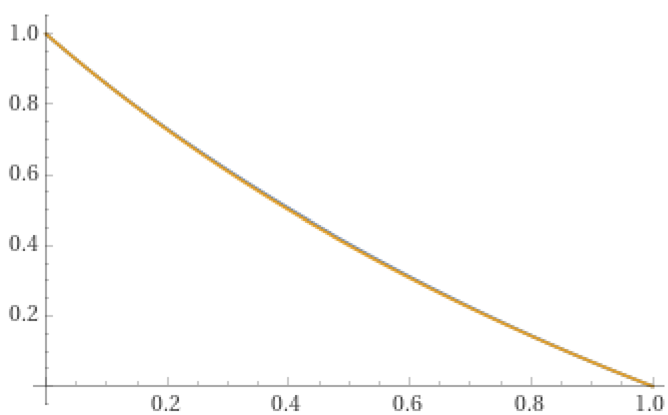
and the question comes up as to whether there might be a simple *estimator* for this expression. It turns out there is, and here are a few plots that demonstrate this idea. The blue curve is γ above, and the gold curve is the estimator $\gamma_{est} \;=\; (1-\alpha)\,/\,(1+\alpha\,/\,n)$, where $n = 2, 3, 4$.



$n \;=\; 4$



$n \;=\; 3$



$n \;=\; 2$

Notice that when $n = 2$, the blue and gold curves coincide, so that we have a near perfect estimator for γ above when $\gamma_{est} \;=\; (1-\alpha)\,/\,(1+\alpha\,/\,2)$.

SEEING THE CONNECTIONS, PART VII

In this note, we now want to go back and study the *second* order model, and show that if Δ_f is the *difference* between $f = f$ (r, α) at r = r_g versus r = 0, and Δ_g is the *difference* between $g = g_{t,t}(r, \alpha)$ at r = r_g versus r = 0, then $\Delta_f \geq \Delta_g$ holds for *all* α between 0 and 1. Thus, if Δ_f ---› 0, so must Δ_g.

Recall again that *f* is the *second* order expansion of *g*, where

$$f\,(r, \alpha) \;\approx\; 1 \;-\; (\alpha + 1)r_s / r_g \;+\; \alpha(\alpha + 1)(r_s / 2r_g)^2 \;+\; \alpha(r_s / r_g^3) \cdot r^2 \;-\; \alpha(\alpha + 1)(r_s / 2r_g^2)^2 \cdot r^2\,,$$

and *g* is the *interior* time component of the Schwarzschild metric, as shown below …

$$\left((\alpha+1)\sqrt{1-\frac{r_s}{r_g}}-\alpha\sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2$$

Since *f* and *g* agree at r = r_g , all we really need to do is calculate Δ = *g*(0) – *f* (0), and show that Δ is always *greater* than or *equal* to zero, for *any* choice of α.

Now Δ computes to [where γ is now γ_{est} = (1 – α) / (1 + α / 2) from page 430]

$$\Delta \;=\; ((\alpha+1)\sqrt{1-\gamma}\;-\;\alpha)^2 \;-\; \alpha(\alpha + 1)(\gamma / 2)^2 \;+\; (\alpha + 1)\gamma \;-\; 1\,,$$

and from our expression (~) on page 422, and reproduced here …

$$1 \;-\; (\alpha + 1)\gamma \;+\; \alpha(\alpha + 1)(\gamma / 2)^2 \;=\; \alpha \;-\; \alpha(\alpha + 1)(\gamma / 4)\,, \qquad (\sim)$$

the *inequality* test for Δ becomes …

$$\Delta \;=\; ((\alpha+1)\sqrt{1-\gamma}\;-\;\alpha)^2 \;+\; \alpha(\alpha + 1)(\gamma / 4) \;-\; \alpha \;\geq\; 0\,.$$

Using the fact that γ = (1 – α) / (1 + α / 2), it is not hard to show that α = (1 – γ) / (1 + γ / 2), so that after some mild algebra, the inequality test for Δ reduces to [0 ≤ γ ≤ 1] …

$$\gamma^2 \;-\; 24\gamma \;+\; 32 \;\geq\; 8(4 - \gamma)\sqrt{1-\gamma} \qquad (*)$$

To see that (*) is true, we'll do a plot, which is shown below. In this plot, the blue curve is the *left* side of (*) and the gold curve is the *right* side of (*). You can see from the picture that the gold curve is always *bounded* by the blue curve, whence (*) holds.

Thus, for the *second* order model, it is *always* the case that $\Delta_f \geq \Delta_g$ when γ = (1 – α) / (1 + α / 2) is the estimator from page 430.

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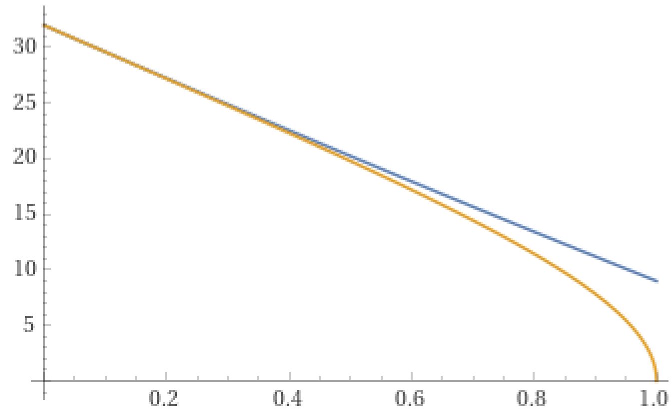

Now as α ---› 0 or 1, Δ_f ---› 0 [see pages 423-5]. Thus, from our work above, it must *also* be the case that Δ_g ---› 0, which means that at $\alpha = 0$, $g = g_{t,t}(r, \alpha) = 0$ for *all* $0 \le r \le r_s$ (the case of a star collapsing to its Schwarzschild radius, so $\gamma = 1$). When α ---› 1, we see the same effect, and here we are in a vacuum, so that $g = 1$ because $\gamma = 0$ (note that because *f* and *g* are parabolic, and hence *increasing* functions, relative to the *first* and *second* order models, *both* flatten out to straight lines, as α ---› 0 or 1, since again, $\Delta_f \ge \Delta_g$).

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If we bring back our expression (*) from the previous page, as shown below [$0 \le \gamma \le 1$],

$$\gamma^2 - 24\gamma + 32 \ge 8(4-\gamma)\sqrt{1-\gamma} \qquad (*)$$

and *square* each side, and then collect terms, (*) reduces to …

$$\gamma^2(\gamma + 8)^2 \ge 0 .$$

And of course, this is a true statement. Furthermore, in the range $0 \le \gamma \le 1$, this can be reduced even more, by writing

$$\gamma(\gamma + 8) \ge 0 .$$

SEEING THE CONNECTIONS, PART VIII

We wish to continue the discussion in Part IV of this series, by developing a *third* order expansion for $g = g_{t,t}(r, \alpha)$, as shown below.

$$\left((\alpha+1)\sqrt{1-\frac{r_s}{r_g}}-\alpha\sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2$$

When the dust settles on the calculations, this expansion is [labelling as (†)] …

$$f(r, \alpha) \approx 1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2 + \alpha(\alpha+1)(r_s / 2r_g)^3 + \alpha(r_s / r_g^3)\cdot r^2$$

$$- \alpha(\alpha+1)(r_s / 2r_g^2)^2 \cdot r^2 - \alpha(\alpha+1)(r_s / 2r_g^2)^3 \cdot r^3 .$$

Now it should be pointed out that there really is *no true* Taylor series expansion of *third* order, that gives us agreement between *f* and *g*, at $r = r_g$, so the *last* two terms in (†) are actually *correction* factors that fix this problem. Thus *f* and *g* do agree at $r = 0$ [up to third order], and *also* agree when $r = r_g$, when we implement the fix.

But what we want to do here, as in Part IV, is ask ourselves what happens to α when r_s / r_g ---› 1. And in order to do this, let's rewrite (†) as

$$f(r, \alpha) \approx \{1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2 + \alpha(\alpha+1)(r_s / 2r_g)^3\}\cdot$$

$$\left(1 + \beta(r_s / r_g^3)\cdot r^2 + \text{some term in } r^3\right),$$

where now,

$$\beta = \left(\alpha - \alpha(\alpha+1)(r_s / 4r_g)\right) / \{1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2 + \alpha(\alpha+1)(r_s / 2r_g)^3\} . \qquad (*)$$

Now notice here, that the 'some term in r^3' isn't calculated with any precision, and that is because when we do a lim r ---› 0 analysis on pages 396-7, any terms *higher* than r^2 are ignored anyway, so that when calculating β, only the terms in r^2 matter. What is new, though, is the *last* term in (*) itself.

Now again, our expression from page 416 still holds; that is to say

$$(2\beta - 1)\rho = 3p ,$$

so that our 'equation of state' is attained, *without* respect to sign conventions, if $\beta = 0$ or 1. And when $\beta = 0$, it is the case that $\alpha = 0$, so nothing new here. It is, just as before. And if $\beta = 1$, with the variable $\gamma = r_s / r_g$, then (*) becomes

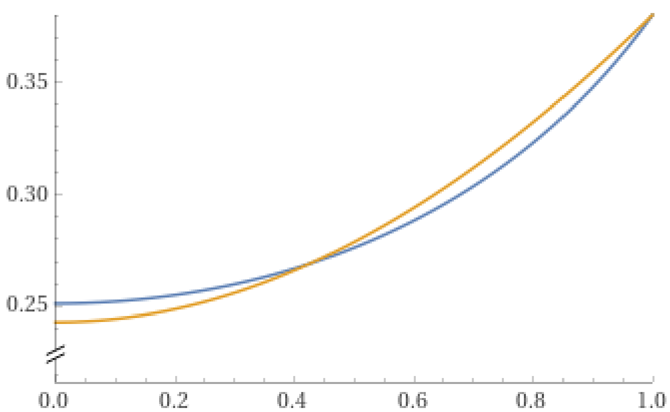
$$1 - (\alpha+1)\gamma + \alpha(\alpha+1)(\gamma/2)^2 + \alpha(\alpha+1)(\gamma/2)^3 = \alpha - \alpha(\alpha+1)(\gamma/4) . \qquad (\sim)$$

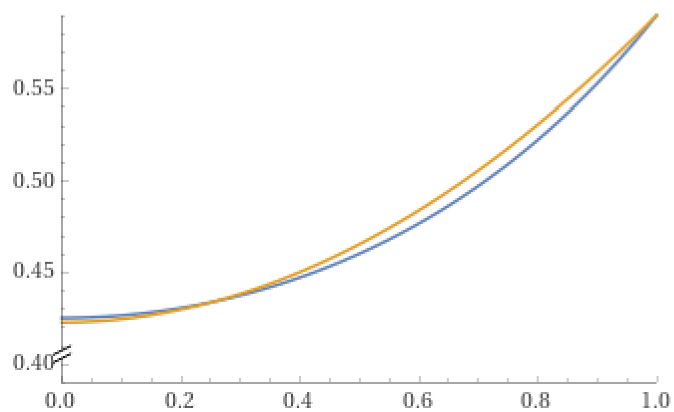
We can write (~) as a *cubic* polynomial in terms of γ, and this is …

$$\alpha\gamma^3 \;+\; 2\alpha\gamma^2 \;+\; 2(\alpha - 4)\gamma \;+\; 8(1-\alpha)\,/\,(1+\alpha) \;=\; 0\,. \qquad (\sim)$$

When $\alpha = ½$, $\gamma \approx 0.41$ in (~), and this compares favorably with the *second* order estimate, which is ≈ 0.40. When $\alpha = 0.3$, $\gamma \approx 0.62$ in (~), and this *also* compares favorably with the *second* order estimate, which is ≈ 0.61 [the comparisons are actually better than this, but I'm rounding to two digits here].

The picture below on the *left* is the *overlay* for $r_s \approx 0.62$, $r_g = 1$, $\alpha = 0.3$, and $0 \;\le\; r \;\le\; r_g$, using the *third* order model, from the last page. The diagram below on the *right* is the *overlay* for $r_s \approx 0.41$, $r_g = 1$, $\alpha = 1/2$, and $0 \;\le\; r \;\le\; r_g$, again using the *third* order model. Blue represents the *interior* metric $g = g_{t,t}(r, \alpha)$, as shown on page 433, and gold represents $f\,(r, \alpha)$, also shown on page 433.





You can see from these pictures, that they agree *exactly* when $r = r_g$, and very nearly agree at the origin O, when $r = 0$. And they track each other quite well, for all values of r between 0 and r_g. And this is encouraging, especially in the case where $\gamma = r_s\,/\,r_g \approx 0.62$, since this is not at all a *small* value. Indeed, one has to wonder if 'high γ values' like this even have any physical meaning …

Finally, notice in (~), and reproduced below,

$$\alpha\gamma^3 \;+\; 2\alpha\gamma^2 \;+\; 2(\alpha - 4)\gamma \;+\; 8(1-\alpha)\,/\,(1+\alpha) \;=\; 0 \qquad (\sim)$$

that when $\gamma = 0$, it is the case that $\alpha = 1$, which is our vacuum. On the other hand, when $\alpha = 0$, it is the case that $\gamma = 1$, which is the case of a star collapsing to its Schwarzschild radius.

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TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR SECOND AND THIRD ORDER MODELS

α	γ (*second* order)	γ (*third* order)
0	1	1
0.2	0.730	0.741
0.4	0.504	0.513
0.6	0.311	0.314
0.8	0.144	0.144
1	0	0

You can see from the table above that there is very little difference in γ, when comparing these two models. The difference appears to be *no more* than about 0.01 for *high* values of γ, and for low values of γ, they almost match.

An Interesting Estimator For $\gamma = r_s / r_g$ In The Third Order Model

For the *third* order model, we are left with solving a *cubic* equation, as shown below, and this can be a difficult task.

$$1 - (\alpha + 1)\gamma + \alpha(\alpha + 1)(\gamma/2)^2 + \alpha(\alpha + 1)(\gamma/2)^3 = \alpha - \alpha(\alpha + 1)(\gamma/4) . \quad (\sim)$$

However, suppose we were to rewrite (~) above, as an *infinite* expansion on the left side, thusly,

$$1 - (\alpha + 1)\gamma + \alpha(\alpha + 1)(\gamma/2)^2 + \alpha(\alpha + 1)(\gamma/2)^3 + \alpha(\alpha + 1)(\gamma/2)^4 + \alpha(\alpha + 1)(\gamma/2)^5 + \ldots$$

$$= \alpha - \alpha(\alpha + 1)(\gamma/4) . \quad (\dagger)$$

Then because $0 \le \gamma \le 1$, we would expect the solutions for γ in (~) and (†) to be *very* close. Now gathering terms by collapsing the series in (†), on the left side, leads to

$$1 - (\alpha + 1)\gamma + \alpha(\alpha + 1)(\gamma/2)^2 [1 + \gamma/2 + (\gamma/2)^2 + \ldots] = \alpha - \alpha(\alpha + 1)(\gamma/4) ,$$

or more simply,

$$1 - (\alpha + 1)\gamma + \alpha(\alpha + 1)(\gamma/2)^2 [1 / (1 - \gamma/2)] = \alpha - \alpha(\alpha + 1)(\gamma/4) .$$

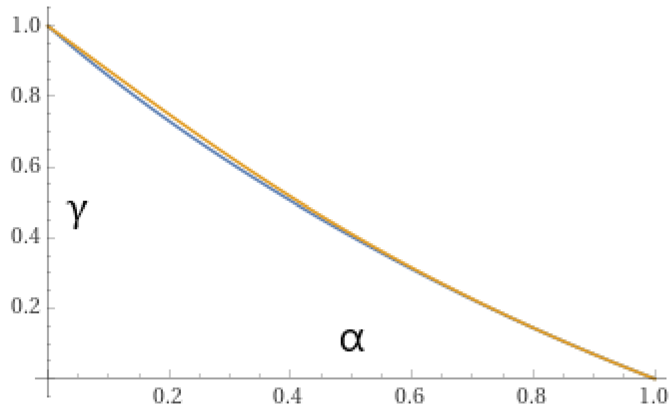
Now the expression just above is actually a *parabolic* equation, which evaluates to

$$(\alpha + 1)(\alpha + 4)\gamma^2 + 2(\alpha + 2)(\alpha - 3)\gamma + 8(1 - \alpha) = 0 ,$$

and the *proper* solution to this quadratic is [labelling as (*)]

$$\gamma = \frac{1}{(\alpha + 1)(\alpha + 4)}\left(-(\alpha + 2)(\alpha - 3) - \sqrt{(\alpha + 2)^2 (\alpha - 3)^2 + 8(\alpha + 4)(\alpha + 1)(\alpha - 1)}\right)$$

And this becomes our *estimator* for the cubic (~) on the last page. Now we don't actually have an α-γ plot for the cubic, but we do for the *second* order model, so what you see in the picture below is a comparison between the *second* order α-γ, and the cubic *estimato*r α-γ (*), just above.



In the picture above, blue represents the *second* order α-γ plot, and gold is the cubic *estimato*r α-γ plot. You can see how closely they compare to one another. Were we to plot the real *cubic* chart, which follows from (~) on the last page, we would find that it was sandwiched in between the blue and the gold !

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR SECOND AND THIRD ORDER MODELS, AS WELL AS THE CUBIC ESTIMATOR

α	γ (*second* order)	γ (*third* order)	γ (*cubic* estimator)
0	1	1	1
0.2	0.730	0.741	0.749
0.4	0.504	0.513	0.516
0.6	0.311	0.314	0.315
0.8	0.144	0.144	0.144
1	0	0	0

*****

SEEING THE CONNECTIONS, PART IX

For the *second* and *third* order models, we can also take the α-γ equations, and write them as *parabolic* expressions in α. Recall these equations are, respectively,

$$1 - (\alpha+1)\gamma + \alpha(\alpha+1)(\gamma/2)^2 = \alpha - \alpha(\alpha+1)(\gamma/4)$$

$$1 - (\alpha+1)\gamma + \alpha(\alpha+1)(\gamma/2)^2 + \alpha(\alpha+1)(\gamma/2)^3 = \alpha - \alpha(\alpha+1)(\gamma/4)$$

And the parabolic forms are, for the *second* and *third* order models, respectively,

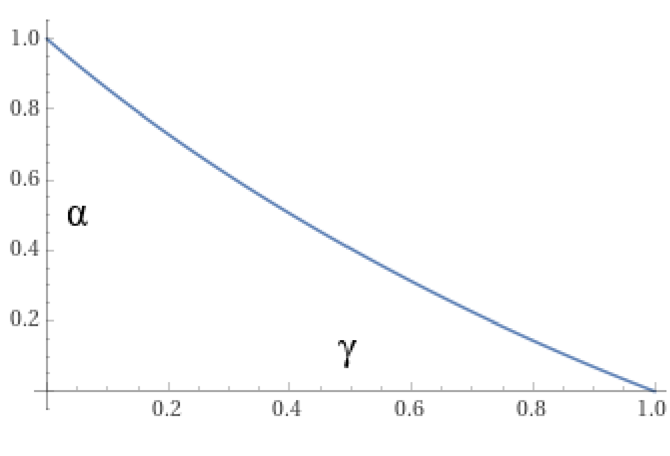
$$\big((\gamma/2)^2 + (\gamma/4)\big)\alpha^2 + \big((\gamma/2)^2 - 3\gamma/4 - 1\big)\alpha + \big(1 - \gamma\big) = 0$$

$$\big((\gamma/2)^3 + (\gamma/2)^2 + (\gamma/4)\big)\alpha^2 + \big((\gamma/2)^3 + (\gamma/2)^2 - 3\gamma/4 - 1\big)\alpha + \big(1 - \gamma\big) = 0$$

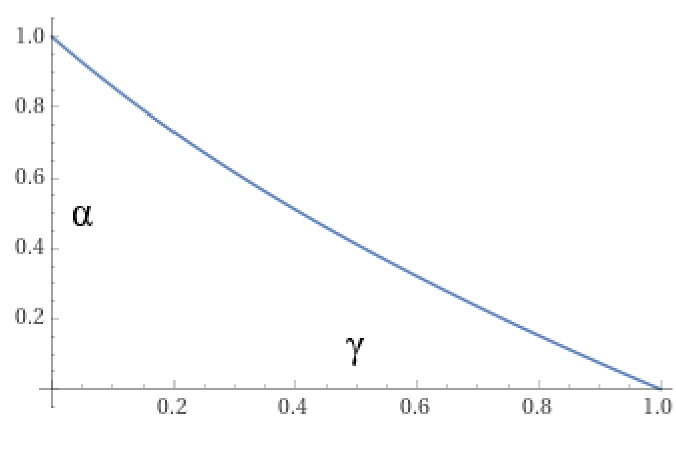
Now here, we can calculate α precisely in these quadratics; and the *proper* solutions are shown below $[0 \le \alpha, \gamma \le 1]$, for *second* order and *third* order, as well as the *nearly* identical plots …

$$\alpha = \frac{1}{2\left(\left(\frac{\gamma}{2}\right)^2 + \frac{\gamma}{4}\right)}\left(\left(3\times\frac{\gamma}{4} + 1 - \left(\frac{\gamma}{2}\right)^2\right) - \sqrt{\left(3\times\frac{\gamma}{4} + 1 - \left(\frac{\gamma}{2}\right)^2\right)^2 + 4\left(\left(\frac{\gamma}{2}\right)^2 + \frac{\gamma}{4}\right)(\gamma - 1)}\right)$$

$$\alpha = \left(\left(3\times\frac{\gamma}{4} + 1 - \left(\frac{\gamma}{2}\right)^2 - \left(\frac{\gamma}{2}\right)^3\right) - \sqrt{\left(3\times\frac{\gamma}{4} + 1 - \left(\frac{\gamma}{2}\right)^2 - \left(\frac{\gamma}{2}\right)^3\right)^2 + 4\left(\left(\frac{\gamma}{2}\right)^3 + \left(\frac{\gamma}{2}\right)^2 + \frac{\gamma}{4}\right)(\gamma - 1)}\right) \Big/ \left(2\left(\left(\frac{\gamma}{2}\right)^3 + \left(\frac{\gamma}{2}\right)^2 + \frac{\gamma}{4}\right)\right)$$



second order



third order

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR SECOND AND THIRD ORDER MODELS

γ	α (*second* order)	α (*third* order)
0	1	1
0.1	0.858	0.858
0.2	0.730	0.731
0.3	0.613	0.616
0.4	0.504	0.510
0.5	0.404	0.412
0.6	0.311	0.321
0.7	0.225	0.234
0.8	0.144	0.152
0.9	0.069	0.074
1	0	0

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR SECOND AND THIRD ORDER MODELS

α	γ (*second* order)	γ (*third* order)
0	1	1
0.1	0.858	0.867
0.2	0.730	0.741
0.3	0.613	0.623
0.4	0.504	0.513
0.5	0.404	0.410
0.6	0.311	0.314
0.7	0.225	0.226
0.8	0.144	0.144
0.9	0.069	0.069
1	0	0

***** *****

SEEING THE CONNECTIONS, PART X

We wish to continue the discussion in Part VIII of this series, by developing a *fourth* order expansion for $g = g_{t,t}(r, \alpha)$, as shown below.

$$\left((\alpha+1)\sqrt{1-\frac{r_s}{r_g}}-\alpha\sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2$$

When the dust settles on the calculations, this expansion is [labelling as (†)] …

$$f(r, \alpha) \approx 1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2 + \alpha(\alpha+1)(r_s / 2r_g)^3 + 5\alpha((\alpha+1)/4)(r_s / 2r_g)^4$$

$$+ \alpha(r_s / r_g^3)\cdot r^2 - \alpha(\alpha+1)(r_s / 2r_g^2)^2\cdot r^2 - \alpha(\alpha+1)(r_s / 2r_g^2)^3\cdot r^3 - 5\alpha((\alpha+1)/4)(r_s / 2r_g^2)^4\cdot r^4$$

Now it should be pointed out that there really is *no true* Taylor series expansion of *fourth* order, that gives us agreement between *f* and *g*, at $r = r_g$, so the *last* three terms in (†) are actually *correction* factors that fix this problem. Thus *f* and *g* do agree at $r = 0$ [up to fourth order], and *also* agree when $r = r_g$, when we implement the fix.

Also note that the Taylor series expansion of $(1 - x)^{1/2}$ is, out to fourth order …

$$1 - x/2 - x^2/8 - x^3/16 - 5x^4/128 ,$$

which explains the existence of the number 5 in *two* of the terms associated with $f(r, \alpha)$.

But what we want to do here, as in Part VIII, is ask ourselves what happens to α when $r_s / r_g \longrightarrow 1$. And in order to do this, let's rewrite (†) as

$$f(r, \alpha) \approx \{1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2 + \alpha(\alpha+1)(r_s / 2r_g)^3 + 5\alpha((\alpha+1)/4)(r_s / 2r_g)^4\}\cdot$$

$$\left(1 + \beta(r_s / r_g^3)\cdot r^2 + \text{some term in } r^3 + \text{some term in } r^4\right) ,$$

where now,

$$\beta = \left(\alpha - \alpha(\alpha+1)(r_s / 4r_g)\right) / \{1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2 + \alpha(\alpha+1)(r_s / 2r_g)^3 +$$

$$5\alpha((\alpha+1)/4)(r_s / 2r_g)^4\} . \quad (*)$$

Now notice here, that the 'some term in r^3 *and* in r^4' isn't calculated with any precision, and that is because when we do a lim $r \longrightarrow 0$ analysis on pages 396-7, any terms *higher* than r^2 are ignored anyway, so that when calculating β, only the terms in r^2 matter. What is new, though, is the *last* term in (*) itself.

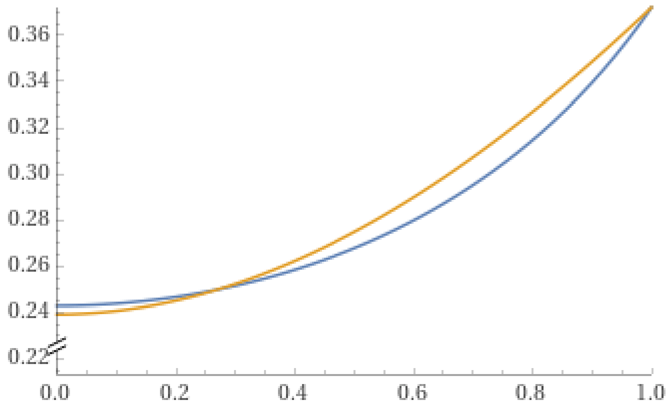
Now again, our expression from page 416 still holds; that is to say …

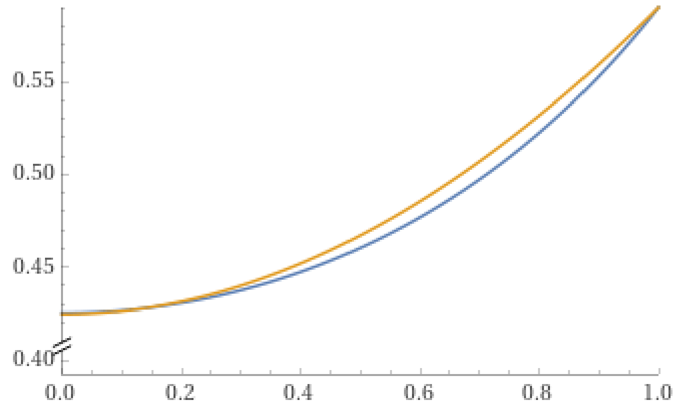
$$(2\beta - 1)\rho = 3p \,,$$

so that our 'equation of state' is attained, *without* respect to sign conventions, if $\beta = 0$ or 1. And when $\beta = 0$, it is the case that $\alpha = 0$, so nothing new here. It is, just as before. And if $\beta = 1$, with the variable $\gamma = r_s / r_g$, then (*) becomes [labelling as (~)] …

$$1 - (\alpha+1)\gamma + \alpha(\alpha+1)(\gamma/2)^2 + \alpha(\alpha+1)(\gamma/2)^3 + 5\alpha((\alpha+1)/4)(\gamma/2)^4 = \alpha - \alpha(\alpha+1)(\gamma/4)\,.$$

When $\alpha = ½$, $\gamma \approx 0.411$ in (~), and this compares favorably with the *third* order estimate, which is ≈ 0.410. When $\alpha = 0.3$, $\gamma \approx 0.628$ in (~), and this *also* compares favorably with the *third* order estimate, which is ≈ 0.623.





The picture above on the *left* is the *overlay* for $r_s \approx 0.63$, $r_g = 1$, $\alpha = 0.3$, and $0 \le r \le r_g$, using the *fourth* order model, from the last page. The diagram above on the *right* is the *overlay* for $r_s \approx 0.41$, $r_g = 1$, $\alpha = 1/2$, and $0 \le r \le r_g$, again using the *fourth* order model. Blue represents the *interior* metric $g = g_{t,t}(r, \alpha)$, as shown on page 439, and gold represents $f\,(r, \alpha)$, also shown on page 439.

You can see from these pictures, that they agree *exactly* when $r = r_g$, and very nearly agree at the origin O, when $r = 0$. And they track each other quite well, for all values of r between 0 and r_g. And this is encouraging, especially in the case where $\gamma = r_s / r_g \approx 0.63$, since this is not at all a *small* value. Indeed, one has to wonder if 'high γ values' like this even have any physical meaning [but see also the corresponding *third* order plots on page 434, which to me, at least, have better balance].

Finally, notice in (~), and reproduced below,

$$1 - (\alpha+1)\gamma + \alpha(\alpha+1)(\gamma/2)^2 + \alpha(\alpha+1)(\gamma/2)^3 + 5\alpha((\alpha+1)/4)(\gamma/2)^4 = \alpha - \alpha(\alpha+1)(\gamma/4)$$

that when $\gamma = 0$, it is the case that $\alpha = 1$, which is our vacuum. On the other hand, when $\alpha = 0$, it is the case that $\gamma = 1$, which is the case of a star collapsing to its Schwarzschild radius.

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TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR 2nd, 3rd AND 4th ORDER MODELS

γ	α (*second* order)	α (*third* order)	α (*fourth* order)
0	1	1	1
0.1	0.858	0.858	0.858
0.2	0.730	0.731	0.731
0.3	0.613	0.616	0.616
0.4	0.504	0.510	0.511
0.5	0.404	0.412	0.415
0.6	0.311	0.321	0.324
0.7	0.225	0.234	0.239
0.8	0.144	0.152	0.157
0.9	0.069	0.074	0.078
1	0	0	0

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR 2nd, 3rd AND 4th ORDER MODELS

α	γ (*second* order)	γ (*third* order)	γ (*fourth* order)	γ (*cubic* est)
0	1	1	1	1
0.1	0.858	0.867	0.872	0.874
0.2	0.730	0.741	0.747	0.749
0.3	0.613	0.623	0.628	0.629
0.4	0.504	0.513	0.516	0.516
0.5	0.404	0.410	0.411	0.411
0.6	0.311	0.314	0.315	0.315
0.7	0.225	0.226	0.226	0.226
0.8	0.144	0.144	0.144	0.144
0.9	0.069	0.069	0.069	0.069
1	0	0	0	0

Notice in the tables above, that there is very *little* difference in the *third* and *fourth* order models, which means at *fourth* order, we are approaching *convergence*. And indeed, in the second table, just above, we see that even the *fourth* order estimates for γ are bounded above by the *cubic* estimator developed on pages 435-6. And that is because this *cubic* estimator was fashioned from an *infinite* series expansion, and thus, might be an *upper* bound for *all* order models; that is to say,

$$n = 1, 2, 3, 4, \ldots$$

EVALUATING A FOURTH ORDER EXPANSION FOR $g = g_{t,t}(r, \alpha)$

In this note, we are going to show how a *fourth* order expansion for $g = g_{t,t}(r, \alpha)$, as shown below, might be developed.

$$\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}}\right)^2$$

As we said on page 439, the Taylor series expansion of $(1 - x)^{1/2}$ is, out to fourth order …

$$1 - x/2 - x^2/8 - x^3/16 - 5x^4/128\,,$$

so this allows us to write *g*, *approximately*, as follows [we'll refer to the expression below as *f*] …

$$\big((\alpha + 1)(1 - r_s / 2r_g - (r_s / r_g)^2 / 8 - (r_s / r_g)^3 / 16 - 5(r_s / r_g)^4 / 128) - \alpha(1 - (r_s / 2r_g^3)r^2)\big)^2$$

Or, upon rewriting things a bit, we have

$$\big(\{1 - (\alpha + 1)(r_s / 2r_g) - (\alpha + 1)((r_s / r_g)^2 / 8 + (r_s / r_g)^3 / 16 + 5(r_s / r_g)^4 / 128)\}$$

$$+ \{\alpha(r_s / 2r_g^3)r^2\}\big)^2$$

Now define

$$a = 1 - (\alpha + 1)(r_s / 2r_g) \;;\; b = (\alpha + 1)((r_s / r_g)^2 / 8 + (r_s / r_g)^3 / 16 + 5(r_s / r_g)^4 / 128)$$

and let $A = a - b$, and let $B = \alpha(r_s / 2r_g^3)r^2$. Then our expansion, just above, becomes

$$(A + B)^2 = A^2 + 2AB + B^2\,.$$

Now after gathering *relevant* terms up to $(r_s / r_g)^4$, A^2 computes to

$$1 - (\alpha + 1)r_s / r_g + \alpha(\alpha + 1)(r_s / 2r_g)^2 + \alpha(\alpha + 1)(r_s / 2r_g)^3 + 5\alpha((\alpha + 1)/4)(r_s / 2r_g)^4\,,$$

and retaining only the *first* term in 2AB gives us $\alpha(r_s / r_g^3)r^2$. The term B^2 is omitted altogether, so that finally, after adding back the correction factors, so that *f* and *g* agree at $r = r_g$, we have

$$\boxed{\begin{aligned} f(r, \alpha) \approx\ & 1 - (\alpha + 1)r_s / r_g + \alpha(\alpha + 1)(r_s / 2r_g)^2 + \alpha(\alpha + 1)(r_s / 2r_g)^3 + 5\alpha((\alpha + 1)/4)(r_s / 2r_g)^4 \\ & + \alpha(r_s / r_g^3)\cdot r^2 - \alpha(\alpha + 1)(r_s / 2r_g^2)^2\cdot r^2 - \alpha(\alpha + 1)(r_s / 2r_g^2)^3\cdot r^3 - 5\alpha((\alpha + 1)/4)(r_s / 2r_g^2)^4\cdot r^4 \end{aligned}}$$

AN UPPER BOUND FOR THE INFINITE EXPANSION

In this note, we are going to develop an *infinite* order expansion for $g = g_{t,t}(r, \alpha)$ as shown below, in the *first* term, as per the methodology in the last note [the *second* term is always to *first* order].

$$\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}} \right)^2$$

When the dust settles on the calculations, a pattern 'seems to emerge', so that the *first* term in *g* above is [providing the pattern below holds or 'nearly holds' beyond the penultimate term] …

$$1 - (\alpha + 1)r_s / r_g + \alpha(\alpha + 1)(r_s / 2r_g)^2 + \alpha(\alpha + 1)(r_s / 2r_g)^3 + 5\alpha((\alpha + 1)/4)(r_s / 2r_g)^4$$

$$+ 7\alpha((\alpha + 1)/4)(r_s / 2r_g)^5 + 9\alpha((\alpha + 1)/4)(r_s / 2r_g)^6 + \ldots$$

and this can be written as

$$1 - (\alpha + 1)r_s / r_g + \alpha(\alpha + 1)(r_s / 2r_g)^2 + \alpha(\alpha + 1)(r_s / 2r_g)^3 +$$

$$(1/4)\alpha(\alpha + 1)(r_s / 2r_g)^4 (5 + 7(r_s / 2r_g) + 9(r_s / 2r_g)^2 + \ldots)$$

Now the *infinite* series, just above, actually has a *closed* form, so that we may write the expansion thusly,

$$1 - (\alpha + 1)r_s / r_g + \alpha(\alpha + 1)(r_s / 2r_g)^2 + \alpha(\alpha + 1)(r_s / 2r_g)^3 + (1/4)\alpha(\alpha + 1)(r_s / 2r_g)^4 \cdot \psi(\gamma) ,$$

where $\gamma = r_s / r_g$ and ψ computes to

$$\psi = 5 / (1 - \gamma/2) + \gamma / (1 - \gamma/2)^2 .$$

And hence, we can complete the form for $f(r, \alpha)$, in the case of an *infinite* expansion, and this is …

$$f(r, \alpha) \approx 1 - (\alpha + 1)r_s / r_g + \alpha(\alpha + 1)(r_s / 2r_g)^2 + \alpha(\alpha + 1)(r_s / 2r_g)^3 + (1/4)\alpha(\alpha + 1)(r_s / 2r_g)^4 \cdot \psi$$

$$+ \alpha(r_s / r_g^3) \cdot r^2 - \alpha(\alpha + 1)(r_s / 2r_g^2)^2 \cdot r^2 - \alpha(\alpha + 1)(r_s / 2r_g^2)^3 \cdot r^3 - (1/4)\alpha(\alpha + 1)(r_s / 2r_g^2)^4 \cdot \psi \cdot r^4$$

And so, at this juncture, as per our past research notes, we can form the α-γ equation, which is, labelling as (~),

$$1 - (\alpha + 1)\gamma + \alpha(\alpha + 1)(\gamma/2)^2 + \alpha(\alpha + 1)(\gamma/2)^3 + (1/4)\alpha(\alpha + 1)(\gamma/2)^4 \cdot \psi = \alpha - \alpha(\alpha + 1)(\gamma/4).$$

And because (~) is fashioned out of an *infinite* series, solutions in γ *will* form an *upper* bound for all *finite* order expansions that precede it. Thus, (~) is the *ultimate* definition of *convergence*.

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR 2nd, 3rd, 4th AND ∞ ORDER MODELS

α	γ (*second* order)	γ (*third* order)	γ (*fourth* order)	γ (∞ order)
0	1	1	1	1
0.1	0.858	0.867	0.872	0.879
0.2	0.730	0.741	0.747	0.753
0.3	0.613	0.623	0.628	0.632
0.4	0.504	0.513	0.516	0.517
0.5	0.404	0.410	0.411	0.412
0.6	0.311	0.314	0.315	0.315
0.7	0.225	0.226	0.226	0.226
0.8	0.144	0.144	0.144	0.144
0.9	0.069	0.069	0.069	0.069
1	0	0	0	0

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR 2nd, 3rd, 4th AND ∞ ORDER MODELS

γ	α (*second* order)	α (*third* order)	α (*fourth* order)	α (∞ order)
0	1	1	1	1
0.1	0.858	0.858	0.858	0.858
0.2	0.730	0.731	0.731	0.731
0.3	0.613	0.616	0.616	0.616
0.4	0.504	0.510	0.511	0.512
0.5	0.404	0.412	0.415	0.416
0.6	0.311	0.321	0.324	0.327
0.7	0.225	0.234	0.239	0.243
0.8	0.144	0.152	0.157	0.163
0.9	0.069	0.074	0.078	0.083
1	0	0	0	0

You can see from these tables, that the *infinite* order expansion forms an upper bound for *all* other order expansions, and that there really is not that much difference between the *infinite* order and the *finite* orders. Indeed, the difference between the *infinite* order and *fourth* order is less than 0.01 everywhere.

But the important takeaway here is that *any finite* order expansion, as per our methodology [p 442], will *always* be bounded by the *infinite* order expansion, as per the tables above.

AN UPPER BOUND FOR THE INFINITE EXPANSION, PART II

In this note, we are going to develop another *infinite* order expansion for $g = g_{t,t}(r, \alpha)$ as shown below, in the *first* term, as per the methodology on page 442 [the *second* term is always to *first* order].

$$\left((\alpha+1)\sqrt{1-\frac{r_s}{r_g}}-\alpha\sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2$$

While I believe the expansion in Part I has merit, the one we are going to write down here has been verified by me out to the *sixth* order, and so I believe it is the correct form.   And here it is [$f(r, \alpha)$] …

$$1 - (\alpha+1)r_s / r_g + \sum_{n=2}^{\infty} \alpha(\alpha+1)\, c_n \cdot (r_s / 2r_g)^n + \alpha(r_s / r_g{}^3) \cdot r^2$$

$$- \sum_{n=2}^{\infty} \alpha(\alpha+1)\, c_n \cdot (r_s / 2r_g{}^2)^n \cdot r^n$$

And here, for $n > 1$,

$$c_n = 2(2n-3)!! / n!\ .$$

Some verification is in order, so let's bring back our expansion for the *first* term in *g*, from Part I, which is [labelling as (*)]

$$1 - (\alpha+1)r_s / r_g + \alpha(\alpha+1)(r_s / 2r_g)^2 + \alpha(\alpha+1)(r_s / 2r_g)^3 + 5\alpha((\alpha+1)/4)(r_s / 2r_g)^4$$

$$+ 7\alpha((\alpha+1)/4)(r_s / 2r_g)^5 + 9\alpha((\alpha+1)/4)(r_s / 2r_g)^6 + \ldots$$

When $n = 2$ or 3, c_n computes to 1, and this agrees with the coefficient in the *third* and *fourth* terms, in (*) above, which are correct. When $n = 4$, c_n computes to 5/4, and this is also correct for the *fifth* term. When $n = 5$, c_n computes to 7/4, and this is also correct for the *sixth* term. Now when $n = 6$, c_n computes to 21/8, which is correct, and this differs from the value of 9/4 above.

Thus, we have enough evidence now, to conclude that the formula for c_n is, indeed, valid. And so, at this juncture, as per our past research notes, we can form the α-γ equation, which is, labelling as (~),

$$1 - (\alpha+1)\gamma + \sum_{n=2}^{\infty} \alpha(\alpha+1)\, c_n \cdot (\gamma/2)^n = \alpha - \alpha(\alpha+1)(\gamma/4).$$

Now I am not aware of a *closed* form for this series in (~), which means it won't be possible to form an upper bound on γ, as we did in Part I. But, we do have some tables below which include values from (~) above, out to an *eighth* order expansion. And here they are …

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR 2nd, 3rd, 4th AND 8th ORDER MODELS

α	γ (*second* order)	γ (*third* order)	γ (*fourth* order)	γ (8th order)
0	1	1	1	1
0.1	0.858	0.867	0.872	0.879
0.2	0.730	0.741	0.747	0.754
0.3	0.613	0.623	0.628	0.632
0.4	0.504	0.513	0.516	0.517
0.5	0.404	0.410	0.411	0.412
0.6	0.311	0.314	0.315	0.315
0.7	0.225	0.226	0.226	0.226
0.8	0.144	0.144	0.144	0.144
0.9	0.069	0.069	0.069	0.069
1	0	0	0	0

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR 2nd, 3rd, 4th AND 8th ORDER MODELS

γ	α (*second* order)	α (*third* order)	α (*fourth* order)	α (8th order)
0	1	1	1	1
0.1	0.858	0.858	0.858	0.858
0.2	0.730	0.731	0.731	0.731
0.3	0.613	0.616	0.616	0.616
0.4	0.504	0.510	0.511	0.512
0.5	0.404	0.412	0.415	0.416
0.6	0.311	0.321	0.324	0.326
0.7	0.225	0.234	0.239	0.242
0.8	0.144	0.152	0.157	0.161
0.9	0.069	0.074	0.078	0.081
1	0	0	0	0

Interestingly, if you compare values in the *eighth* order column, with their counterparts on page 444, for the *infinite* order expansion in Part I, they are virtually identical …

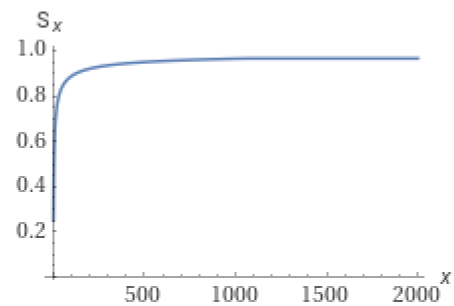
AN UPPER BOUND FOR THE INFINITE EXPANSION, PART III

In our α-γ equation below [labelling as (~)], the only outstanding issue is the *convergence* of the *infinite* summation when $\gamma = 1$. If we can show that we have convergence here, then the model has validity for *all* values of $0 \le \alpha, \gamma \le 1$.

$$1 - (\alpha + 1)\gamma + \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (\gamma/2)^n = \alpha - \alpha(\alpha + 1)(\gamma/4).$$

Now if we set $\gamma = 1$ in this *infinite* sum, as shown below, it actually converges to 1. And the picture below from the Wolfram site shows this *asymptotic* convergence, when *n* runs from 2 to 2000 in the summation.

$$\sum_{n=2}^{\infty} c_n \cdot (\gamma/2)^n$$



$\gamma = 1$

Now out to 2000, the value of the infinite sum is ≈ 0.97477, so the convergence is very slow. But at $n = \infty$, we finally arrive at our destination, and in (~) above, α will evaluate to 0 now, when $\gamma = 1$.

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR 2nd, 3rd, 4th AND 100th ORDER MODELS

γ	α (*second* order)	α (*third* order)	α (*fourth* order)	α (100th order)
0	1	1	1	1
0.1	0.858	0.858	0.858	0.858
0.2	0.730	0.731	0.731	0.731
0.3	0.613	0.616	0.616	0.616
0.4	0.504	0.510	0.511	0.512
0.5	0.404	0.412	0.415	0.416
0.6	0.311	0.321	0.324	0.327
0.7	0.225	0.234	0.239	0.244
0.8	0.144	0.152	0.157	0.165
0.9	0.069	0.074	0.078	0.087
1	0	0	0	0

You can see in the table above, out to the 100th order, that the values in the *rightmost* column agree very favorably with their counterparts in the *second* table on page 446. But since we don't have a *closed* form for the *infinite* summation, discussed on the last page, we can't compare with the *first* table on page 446.

However, in the *first* table on page 446, because we now have established *convergence* for this model, the 8th order estimates [p 446] should suffice as a reasonable *upper* bound, just as the 100th order estimates do for the table on the last page.

OTHER CONSIDERATIONS

We'd like to offer a more *mathematical* proof that the series below converges, when $\gamma = 1$ …

$$\sum_{n=2}^{\infty} c_n \cdot (\gamma / 2)^n$$

Now recall here, for $n > 1$, that

$$c_n = 2(2n-3)!! / n!\ ,$$

and so for *large* values of *n*, we'll defer to Stirling's formula, which is …

$$n! \sim \sqrt{2\pi n} \cdot (n / e)^n \qquad \text{… for any suitably } \textit{large } n$$

$$n!! \sim \sqrt{2n} \cdot (n / e)^{n/2} \qquad \text{… for any suitably } \textit{large odd } n$$

Thus, $c_n / 2$ becomes …

$$\sqrt{2(2n-3)} \cdot ((2n-3) / e)^{(2n-3)/2} \ / \ \sqrt{2\pi n} \cdot (n / e)^n\ ,$$

And since the *square root* terms in the numerator and denominator won't contribute anything of substance to the *infinite* summation for $n \geq N$, for some suitably large N (since they're *both* of order $\sqrt{n}$) we can, therefore, focus on the expression

$$((2n-3) / e)^{(2n-3)/2} \ / \ (n / e)^n\ .$$

Now after some mild algebra, the expression above reduces to [labelling as (*)] …

$$2^n n^n (1 - (3/2n))^n e^{3/2} \ / \ n^n (2n-3)^{3/2}\ ,$$

and since for *large n*,

$$(1 - (3/2n))^n \approx e^{-3/2}\ ,$$

(*) reduces to the following …

$$2^n n^n \;/\; n^n (2n - 3)^{3/2} \,,$$

or more simply,

$$2^n \,/\, (2n - 3)^{3/2} \,.$$

Thus, our summation, as shown below, for $\gamma = 1$,

$$\sum_{n = N}^{\infty} c_n \cdot (\gamma / 2)^n$$

reduces to a consideration of the series

$$\sum_{n = N}^{\infty} 1 \,/\, (2n - 3)^{3/2}$$

and this most certainly *converges.*

AN UPPER BOUND FOR THE INFINITE EXPANSION, PART IV

In this note, we are going to develop still another *infinite* order expansion for $g = g_{t,t}(r, \alpha)$ as shown below, in the *correction* factors, as per the methodology on page 442 [the *second* term below is always to *first* order].

$$\left((\alpha + 1)\sqrt{1 - \frac{r_s}{r_g}} - \alpha\sqrt{1 - \frac{r^2 r_s}{r_g^3}}\right)^2$$

From Part II [pp 445-6], our expansion [$f(r, \alpha)$] was, which we'll call A …

$$1 - (\alpha + 1)r_s / r_g + \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (r_s / 2r_g)^n + \alpha(r_s / r_g^3) \cdot r^2$$

$$- \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (r_s / 2r_g^2)^n \cdot r^n$$

so that only the *first* term, in the *second* infinite series above, participated in the formation of the α-γ equation, because terms in r^n, where $n > 2$, are always discarded.

And here, for $n > 1$,

$$c_n = 2(2n - 3)!! / n!\,.$$

Now we'd like to rewrite f above thusly, which we can, because we are dealing with the *correction* factors, and we'll call this expansion B …

$$1 - (\alpha + 1)r_s / r_g + \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (r_s / 2r_g)^n + \alpha(r_s / r_g^3) \cdot r^2$$

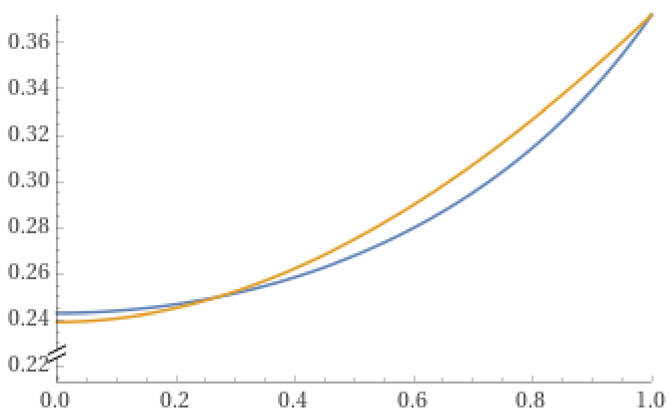
$$- \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (1 / 2^n)(r_s / r_g)^{(n-1)}\, (r_s / r_g^3) \cdot r^2$$

In doing so, *every* term in the *second* infinite series, just above, participates in the formation of the α-γ equation, which now becomes [labelling as (~)] …

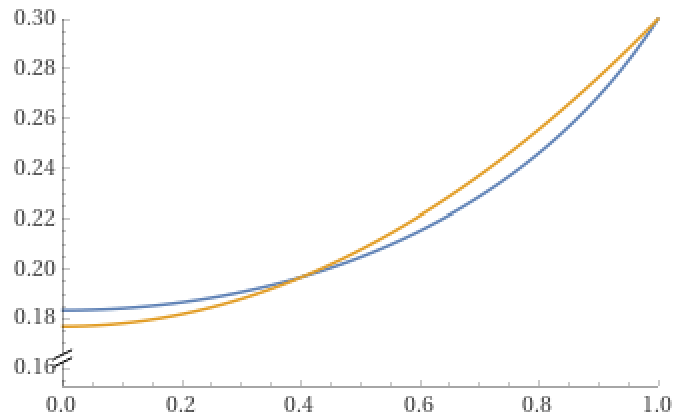
$$1 - (\alpha + 1)\gamma + \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (\gamma / 2)^n = \alpha - \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (1 / 2^n)(\gamma)^{(n-1)}$$

Notice too, that in B, f and g agree at $r = r_g$ and compute to $1 - r_s / r_g$, as they should.

To show that B may be the superior choice, we'll offer two plots; the first from A out to *fourth* order, and the second from B, also out to *fourth* order.



Model A , $\gamma \approx 0.63$



Model B , $\gamma \approx 0.70$

The picture above on the *left* is the *overlay* for $r_s \approx 0.63$, $r_g = 1$, $\alpha = 0.3$, and $0 \le r \le r_g$, out to *fourth* order using model [A], from the last page. The diagram above on the *right* is the *overlay* for $r_s \approx 0.70$, $r_g = 1$, $\alpha = .264$, and $0 \le r \le r_g$, again out to *fourth* order using model [B]. Blue represents the *interior* metric $g = g_{t,t}(r, \alpha)$, as shown on page 450, and gold represents $f\,(r, \alpha)$, also shown on page 450, for both models.

You can see from the pictures above that f and g track each other quite well for both models, but from my perspective, anyway, *even* with a *higher* γ value, model B does the better job.

.

TABLE OF VALUES α VERSUS $\gamma = r_s / r_g$ FOR 4th AND 100th ORDER A & B MODELS

γ	α (4th order) A	α (100th order) A	α (4th order) B	α (100th order) B
0	1	1	1	1
0.1	0.858	0.858	0.860	0.860
0.2	0.731	0.731	0.738	0.738
0.3	0.616	0.616	0.629	0.630
0.4	0.511	0.512	0.530	0.532
0.5	0.415	0.416	0.438	0.443
0.6	0.324	0.327	0.350	0.360
0.7	0.239	0.244	0.264	0.282
0.8	0.157	0.165	0.178	0.206
0.9	0.078	0.087	0.090	0.127
1	0	0	0	0

Notice in the table above that the B values are somewhat higher, and this is to be expected, given the construction of the α-γ equation [~] for B on page 450. Out to the 100^{th} order, however, we can conclude with confidence, that the B values in the table form an *ultimate* upper bound, and indeed, any lower *finite* series will be contained by them. Additionally, since *every* term in the *second* infinite series in (~), on the *right-hand* side, participates in the formation of the α-γ equation for B, this is as far as we can go with an r^2 model.

And finally, notice in the A model, that the α-γ equation is a *special* case of (~) in B, in so much as the *second* infinite series in (~), on the *right-hand* side, terminates at $n = 2$.

OTHER CONSIDERATIONS

The *true* Taylor series expansion of *g* on page 450, where the *second* term in *g* is to *first* order only, computes to [labelling as (*)] …

$$1 - (\alpha + 1)r_s / r_g + \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (r_s / 2r_g)^n + \alpha(r_s / r_g^3) \cdot r^2$$

$$- \alpha(\alpha + 1)\{(1/2)(r_s / r_g) + (1/8)(r_s / r_g)^2 + (1/16)(r_s / r_g)^3 + (5/128)(r_s / r_g)^4 + \ldots\}(r_s / r_g^3) \cdot r^2$$

Note that on the *second* line above, the coefficients are simply the Taylor series expansion of the expression $(1 - x)^{1/2}$, beginning at the term x; that is to say …

$$1 - x/2 - x^2/8 - x^3/16 - 5x^4/128 - \ldots$$

Thus, when forming the function *f* as we did for model B [p 450], so that *f* and *g* agree at $r = r_g$, it behooves us to emulate the second line in (*) above, so that the agreement holds. And this is precisely what we did …

A PHILOSOPHICAL DISCUSSION OF WHETHER f AND g SHOULD AGREE AT $r = r_g$

In developing expressions for f that serve as expansions for g, we have imposed the requirement that they agree when $r = r_g$. But why would we do this, given that we could take the *true* Taylor series expansion for f on page 452, and use it to build any α-γ equation we wanted.

From my perspective, there are several reasons. First, we want f to mimic g as closely as possible, so that the corresponding α-γ equation is a reflection of this emulation. But secondly, we want to develop models that are consistent with one another, as we 'roll back' from the *infinite* order, down to the *finite* order, and finally *first* order.

If we take the *true* Taylor series expansion above, and *remove both* infinite series, we arrive at a *first* order equation for f [pp 416-17], which is …

$$f(r, \alpha) \approx 1 - (\alpha + 1)r_s / r_g + \alpha(r_s / r_g^3) \cdot r^2 .$$

Now notice here, by *default*, that at $r = r_g$, both f and g agree and compute to $1 - r_s / r_g$. It isn't something we have to work for actually … it is simply there. And so the natural question to ask is if it should be there, going forward, as we build out higher order expansions for f.

And in my opinion, the answer is yes; for it is hard to ignore this agreement at *first* order, as though, perhaps, it was nothing more than chance. Indeed, the *first* order analysis may be giving us a hint or clue as to how things work, in the *higher* orders, as we endeavour to unlock the mysterious connection between α and γ. And so for the time being, anyway, we'll retain this requirement …

NOTE TO THE READER

This particular release contains a number of cosmetic enhancements [pp 258-305]. A few annotations were also made in the previous research note [pp 450-452]. And the original essay [see pp 1-52] was, for the most part, moved over to the Times New Roman font.

***** *****

A PROOF THAT $S = \sum_{n=2}^{\infty} c_n \cdot (\gamma/2)^n$ MAY COMPUTE TO 1 WHEN $\gamma = 1$

On page 447 we surmised that S was indeed 1, when $\gamma = 1$, and here we'd like to offer up a rather *heuristic* proof that this may be true. Recalling our α-γ equation from page 450, and reproduced below [labelling as (~)] …

$$1 - (\alpha + 1)\gamma + \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (\gamma/2)^n = \alpha - \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (1/2^n)(\gamma)^{(n-1)}$$

we can, after collecting terms, rewrite this as [labelling as (*)] …

$$\sum_{n=2}^{\infty} c_n \cdot (\gamma/2)^{(n-1)} = 2\{\alpha - 1 + (\alpha + 1)\gamma\} / \alpha(\alpha + 1)(\gamma + 1)$$

Now on the *right-hand* side in (*) above, divide top and bottom by $\alpha + 1$ to obtain [again, labelling as (*)] …

$$\sum_{n=2}^{\infty} c_n \cdot (\gamma/2)^{(n-1)} = 2\{-(1 - \alpha) / (1 + \alpha) + \gamma\} / \alpha(1 + \gamma)$$

And since we know that as $\gamma \dashrightarrow 1$, it is the case that $\alpha \dashrightarrow 0$, write for *small* α …

$$(1 - \alpha) / (1 + \alpha) = (1 - \alpha)^2 / (1 + \alpha)(1 - \alpha) \approx 1 - 2\alpha\,.$$

Now substituting this result into the previous equation (*), just above, we obtain with $\gamma \dashrightarrow 1$ …

$$\sum_{n=2}^{\infty} c_n \cdot (1/2)^{(n-1)} = 2\{-1 + 2\alpha + 1\} / 2\alpha = 4\alpha / 2\alpha = 2\,,$$

and so our sum S must be 1, where again, $c_n = 2(2n - 3)!! / n!$.

OTHER CONSIDERATIONS

When considering (*) above, it should be noted that as $\gamma \dashrightarrow 1$, and hence $\alpha \dashrightarrow 0$, α will have some dependency on γ, so that $\alpha = \alpha(\gamma)$. However, it is unlikely we will ever know what this dependency computes to. Even so, we can still write the *right-hand* side of (*) as

$$\lim_{\gamma \dashrightarrow 1} 2\{-1 + 2\alpha(\gamma) + \gamma\} / \alpha(\gamma)(1 + \gamma)$$

and in the limit, as $\gamma \dashrightarrow 1$, the expression above should resolve as

$$\lim_{\gamma \dashrightarrow 1} 4\alpha(\gamma) \,/\, 2\alpha(\gamma)\ ,$$

which computes to 2.

Now if we bring back our α-γ equation from that last page, but *replace* the *infinite* expansion on the *right* side by a *finite* one; that is to say …

$$1 \ - \ (\alpha + 1)\gamma \ + \sum_{n=2}^{\infty} \alpha(\alpha + 1)\, c_n \cdot (\gamma/2)^n \ = \ \alpha \ - \sum_{n=2}^{N} \alpha(\alpha + 1)\, c_n \cdot (1/2^n)(\gamma)^{(n-1)}$$

where now *N* is some integer greater than 1; and attempt to compute S as we did above, we could and probably will *not* get the right result, because now α has a dependency *not only* on γ, but *also* *N*, as does S.

And this will matter as we let $\gamma \dashrightarrow 1$, so that $\alpha \dashrightarrow 0$. Such a dependency on *N* is easily removed by letting $N \dashrightarrow \infty$ in the summation.

In fact, if we were to compute S from the expression just above, using the same techniques as on the last page, we would find that S computed to

$$2 \ - \sum_{n=2}^{N} c_n \cdot (1/2^n)$$

Clearly this is not the correct result, but it becomes the right result, in the limit, as $N \dashrightarrow \infty$. For then

$$S \ = \ \lim_{N \dashrightarrow \infty} \ 2 \ - \sum_{n=2}^{N} c_n \cdot (1/2^n) \ = \ 2 \ - \ S\ ,$$

so that again, S is equal to 1.

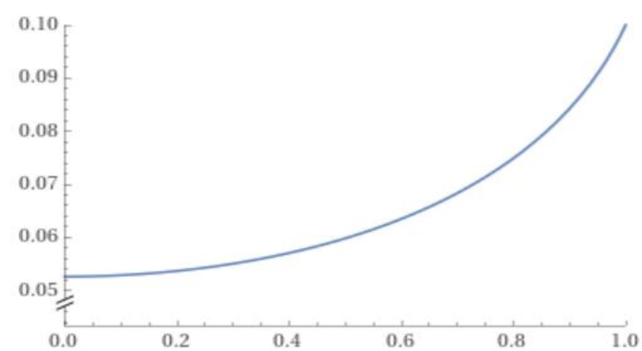
In turn, this means that when choosing a model for the α-γ equation, say model A or model B, discussed on pages 450-2, it would be wise to opt for B if we want consistency with the derivation of S (provided, of course, our methodology here makes sense). Indeed, in model A, the value of *N* is 2 in the α-γ equation above, whereas in model B, $N = \infty$.

But finally, though, our approach is *speculative*, and so our comments here must be tempered by this very fact.

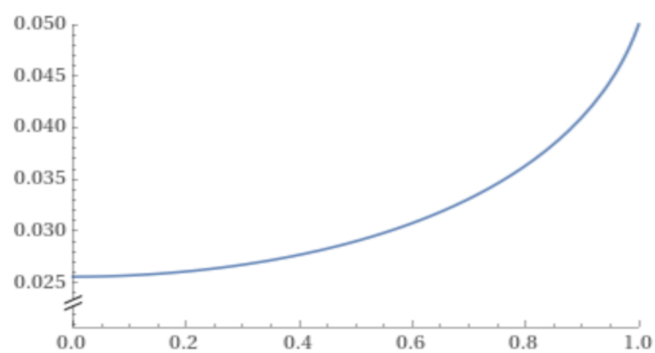
TABLE OF VALUES α VERSUS HIGH $\gamma = r_s / r_g$ FOR THE 100th ORDER B MODEL

γ	α (100th order) B
0.90	0.127
0.91	0.118
0.92	0.110
0.93	0.101
0.94	0.091
0.95	0.082
0.96	0.071
0.97	0.060
0.98	0.047
0.99	0.030
1	0

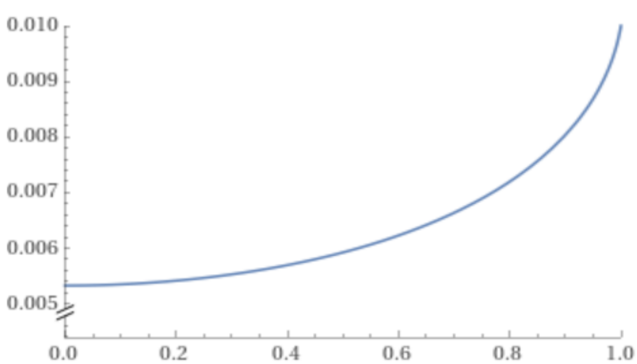
The pictures below are from model [B], first outlined on pages 450-2. They show the *interior* time component $g = g_{t,t}(r, \alpha)$, also found on page 450, for *high* values of γ in the table above. And here, $r_g = 1$, and $0 \le r \le r_g$. Note that *all* three curves are parabolic, as they should be …



$\gamma = 0.90$



$\gamma = 0.95$



$\gamma = 0.99$

AN INEQUALITY CONCERNING γ AND $S = \sum_{n=2}^{\infty} c_n \cdot (\gamma/2)^n$ FOR THE B MODEL

Let us bring back our α-γ equation for the B model, which is …

$$1 - (\alpha+1)\gamma + \sum_{n=2}^{\infty} \alpha(\alpha+1)\, c_n \cdot (\gamma/2)^n = \alpha - \sum_{n=2}^{\infty} \alpha(\alpha+1)\, c_n \cdot (1/2^n)(\gamma)^{(n-1)}$$

and rewrite it as …

$$1 - (\alpha+1)\gamma + \alpha(\alpha+1)S = \alpha - \alpha(\alpha+1)S/\gamma .$$

Note that the equation, just above, is a quadratic in α, which can be written as …

$$(\gamma+1)S \cdot \alpha^2 + (\gamma+1)(S-\gamma) \cdot \alpha + \gamma(1-\gamma) = 0 .$$

And since *both* α and γ are between 0 and 1, solutions in α for this quadratic will be *real-valued,* so that the *square root* term in the solution must be *real* itself. That is to say, for the general quadratic

$$Ax^2 + Bx + C = 0 ,$$

$B^2 - 4AC \geq 0$ applies here.

Thus, we may write

$$[(\gamma+1)(S-\gamma)]^2 \geq 4\gamma S(1-\gamma)(1+\gamma) ,$$

or more simply,

$$(S-\gamma)^2 \geq 4\gamma S(1-\gamma)/(1+\gamma) ,$$

which is what we wanted to show. And again, this inequality applies in the range $0 \leq \gamma \leq 1$.

And note also that when $\gamma = 0$ or 1, $S - \gamma$ is *exactly* 0, so that the inequality above now becomes an equality. This statement presumes, of course, that when $\gamma = 1$, S is equal to 1 as well.

*****

From our inequality, just above, we may deduce that

$$-S^2 + (2\gamma(3-\gamma)/(1+\gamma)) \cdot S - \gamma^2 \leq 0 ,$$

and the inequality above will hold true, provided S is *less* than or *equal* to S_{root}, where S_{root} is the *proper* solution of the quadratic

$$-S^2 + (2\gamma(3-\gamma) / (1+\gamma)) \cdot S - \gamma^2 = 0 .$$

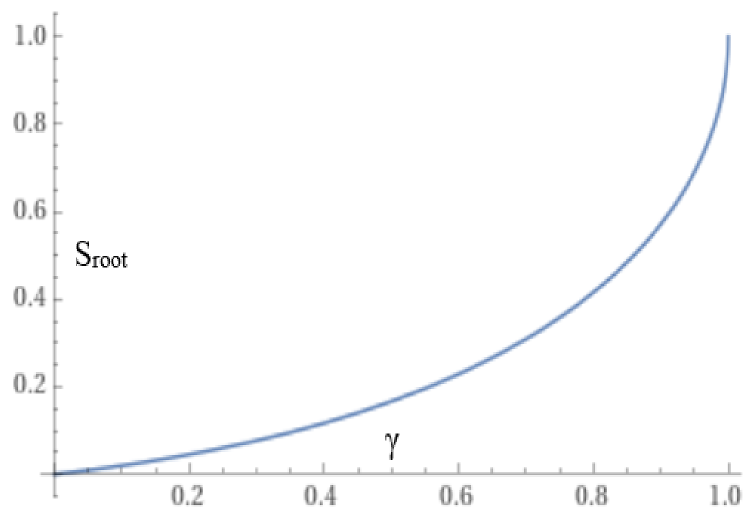
That is to say [labelling as (*)] …

$$S \le \frac{1}{2}\left(2\gamma \times \frac{3-\gamma}{1+\gamma} - \sqrt{\left(2\gamma \times \frac{3-\gamma}{1+\gamma}\right)^2 - 4\gamma^2}\right)$$

where S_{root} is the *right-hand* side of the inequality just above. The inequality, itself, is valid for all $0 \le \gamma \le 1$.

Note that if $\gamma = 0$, then $S = 0$, as is the *right* side of (*), and if $\gamma = 1$, then $S = 1$, as is the *right* side of (*). And if, for example, $\gamma = 0.999$, then S computes to ~ 0.93695, out to 4000 iterations, while the *right* side of (*) evaluates to 0.95530.

A plot of S_{root} versus γ is shown below …

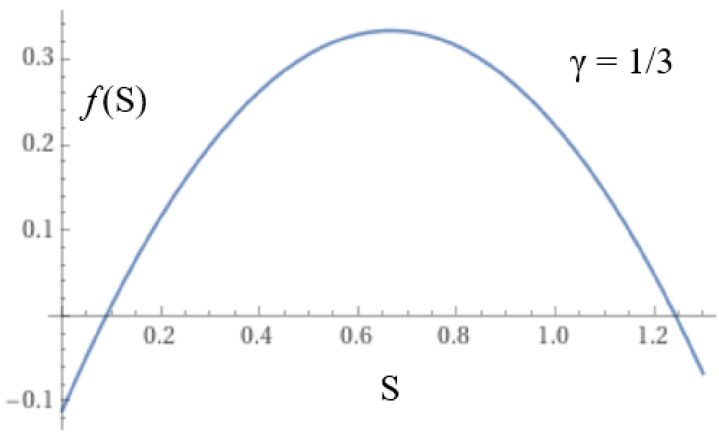


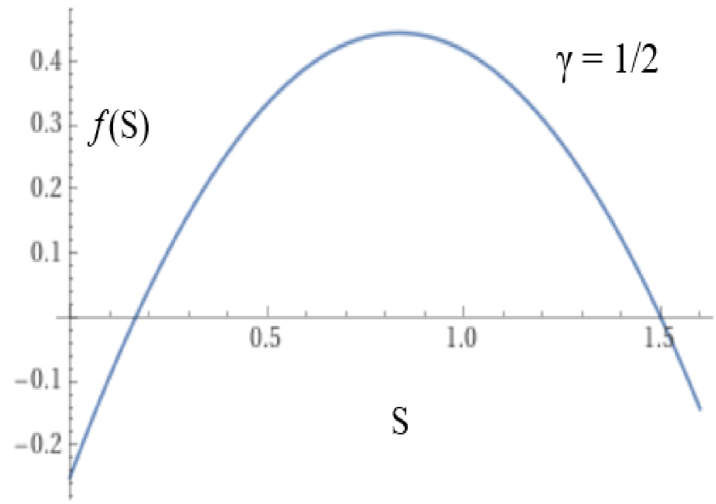
To demonstrate that our inequality [shown below] only holds when S is *less* than or *equal* to S_{root},

$$-S^2 + (2\gamma(3-\gamma)/(1+\gamma)) \cdot S - \gamma^2 \leq 0$$

we'll offer two plots of the function below [where S is now a *context* dependent variable] …

$$f(S) = -S^2 + (2\gamma(3-\gamma)/(1+\gamma)) \cdot S - \gamma^2$$





In the *left* picture we see what $f(S)$ looks like when $\gamma = 1/3$. Two roots appear, with one *less* than the other, which will *always* be the case for *any* $0 < \gamma < 1$ [if $\gamma = 0$ or 1 then the roots are both 0 or both 1]. It is the *smaller* root that we are interested in, which we call S_{root}, and that is because S can never be greater than the *larger* root, as we'll now show.

From our original inequality on page 457 and shown below, where S is our summation,

$$(S-\gamma)^2 \geq 4\gamma S(1-\gamma)/(1+\gamma)$$

we see that *either* $S > \gamma$ or $S < \gamma$ for *all* $0 < \gamma < 1$, with equality being reached when $\gamma = 0$ or 1. Now if for some γ *between* 0 and 1 it was the case that $S = \gamma$ (because the sign of $S - \gamma$ had flipped in some neighborhood of γ), the *right-hand* side of the inequality, just above, would be *zero*, which is not possible. A simple numerical test now shows that $S \leq \gamma$, for all $0 \leq \gamma \leq 1$, is the correct expression.

.
.
.

$$\frac{1}{2}\left(2\gamma\times\frac{3-\gamma}{1+\gamma}+\sqrt{\left(2\gamma\times\frac{3-\gamma}{1+\gamma}\right)^2-4\gamma^2}\right)$$

The *second* and *larger* root for *f* (S) is shown above, and it is easy to see from this expression, that the root [call it S_{root_2}] is always *greater* than or *equal* to γ, for all $0 \leq \gamma \leq 1$ [simply discard the *square root* term, and analyze what's left]. Thus, from our previous result, we have …

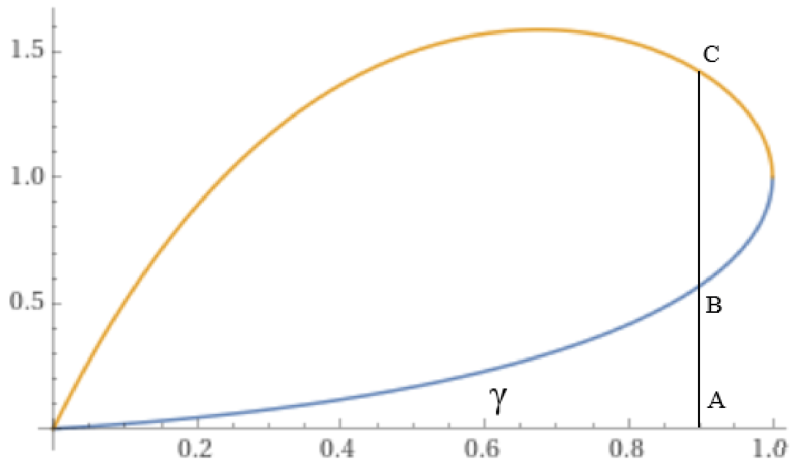
$$S \leq \gamma \leq S_{root\_2} ,$$

which shows us that S *must* be bounded above by the *larger* root of *f* (S). Indeed, it is actually true that $S \leq min(1, S_{root\_2})$, since we already know S can never be greater than 1.

So when looking at the pictures on the previous page, we are only interested in the range

$$0 \leq S \leq S_{root\_2} ,$$

and here the parabolas split, *negative* up to S_{root}, and then *positive* up to S_{root_2}. Thus, $f(S) \leq 0$ can only hold if $S \leq S_{root}$, which is what we wanted to show [p 458].



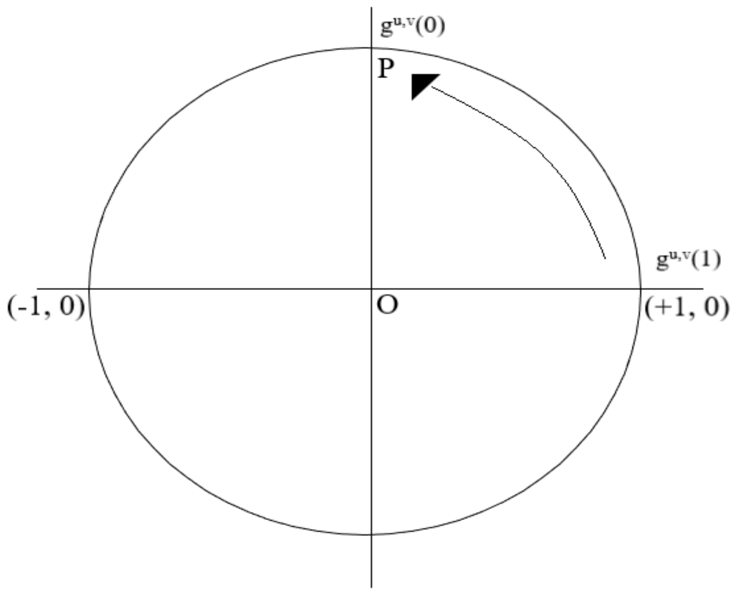
In the picture above is a plot of the two roots of *f* (S); the *larger* is gold and the *smaller* is blue. Along the AB segment, $f(S) \leq 0$ holds, and along the BC segment $f(S) \geq 0$ holds. The *x*-axis is γ.

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$

On pages 321-3, we came up with a form for the field equations in the *coupled* case, for the *two* dimensional plane, which is …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta))] \quad (\S)$$

Here, we wish to put forth another argument that the correct parametrization for $g^{u,v}$ is $\cos(\theta)$, as shown above, and we'll start with the following diagram.

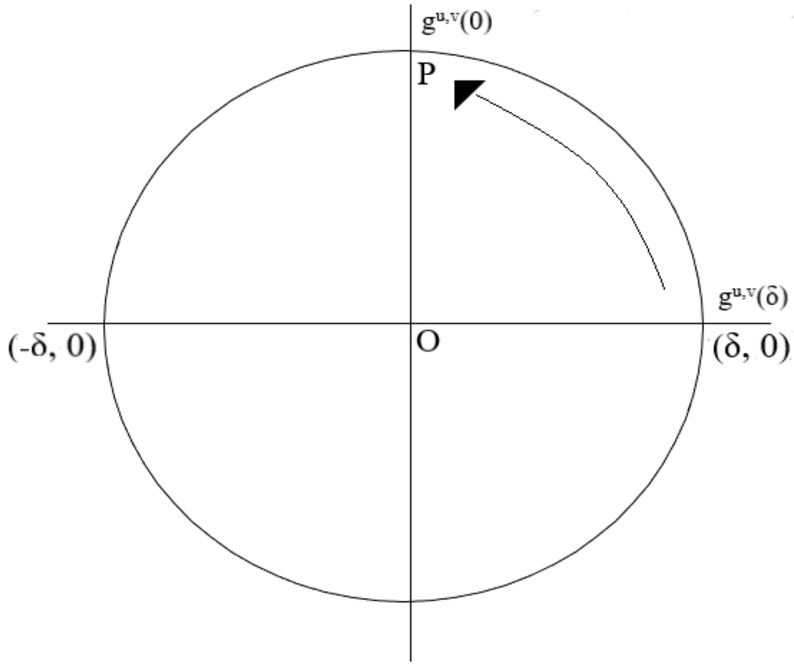


The diagram above shows the location of the dark energy singularities associated with (§), and these are at the origin O, and at (±1, 0). Additionally, we know that (§) is *exact* when θ is equal to 0 or π, and believe that $arg[g^{u,v}]$ *diminishes* as we move away from the singularity at (+1, 0), and head northward to P. Indeed, we believe $arg[g^{u,v}] = 0$ at P.

If we make the assumption that $arg[g^{u,v}]$ reaches a *minimum* value at P [pp 323-5], *irrespective* of where the singularity is on the *x*-axis, and call this minimum value $0 \le \delta < 1$; then it must *also* be true that *if* the singularities at (±1, 0) are now relocated to (±δ, 0) [where $\delta > 0$], $arg[g^{u,v}]$ must now be *less than* δ at P [because $arg[g^{u,v}] = \delta$ at (±δ, 0), is *exact*]. That is to say, at P we have

$$0 \le arg[g^{u,v}] < \delta \quad \text{(a contradiction, from our assumptions, unless we let } \delta \dashrightarrow 0)$$

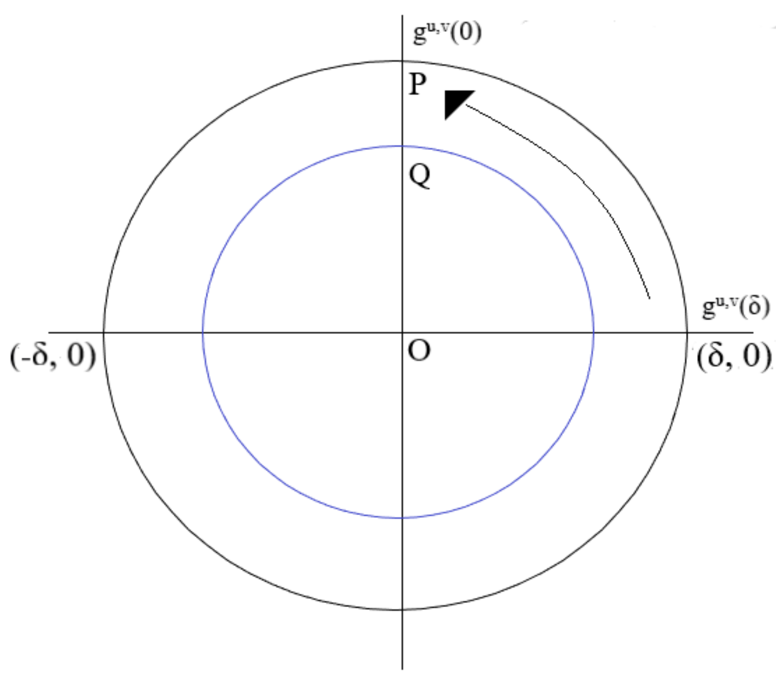
By letting δ ---› 0, we see this forces $arg[g^{u,v}] = 0$ at P, so that whatever the parametrization of $g^{u,v}$ is, it must be such that $g^{u,v}(1)$ holds at (+1, 0) and $g^{u,v}(0)$ holds at P, as in the diagram above. Other arguments [p 321 *ff.*] now suggest that $\cos(\theta)$ is the correct parametrization for $g^{u,v}$, which becomes $g^{u,v}(\delta\cos(\theta))$, if the singularities are at (± δ, 0). And recall too, that $g^{u,v}(\pm\delta)$ are equal to one another, because of symmetry.



THE SINGULARITY INVARIANCE POSTULATE [$\delta \geq 0$]

This postulate simply states that *arg* [$g^{u,v}$] reaches a *minimum* value of *zero* at P, *irrespective* of where the dark energy singularities lie on the *x*-axis. In the picture above, the singularities are at the origin O, and at ($\pm\delta$, 0). In the picture below, *arg* [$g^{u,v}$] is still *zero* at any other point Q on the *y*-axis, because the radius of the circle is arbitrary.

I believe this principle will be important in terms of understanding how the gravitational tensor $g^{u,v}$ interacts with dark energy, itself.



Finally, the field equations in *two* dimensions, with dark energy singularities at the origin O and at $(\pm\delta, 0)$, become, for the *coupled* case …

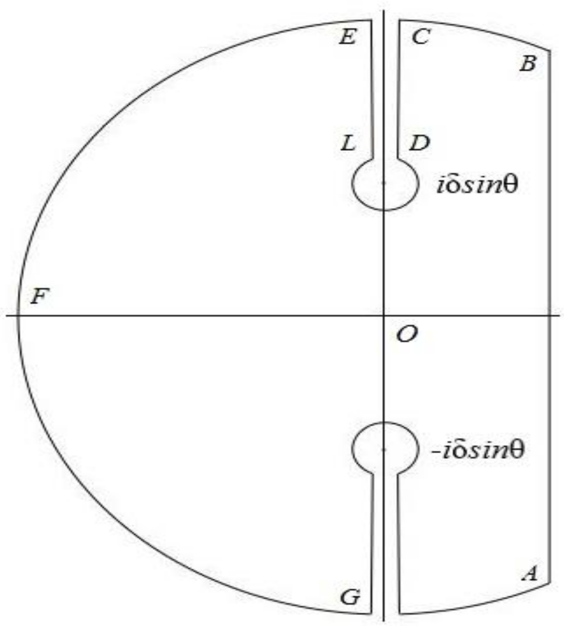
$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] \qquad (\S)$$

Note that the simplest way to calculate the *second* dark energy component above, is by examining the dark energy contour integral in the *uncoupled* case, and reasoning that it [the component] must be the *same*, as in the *coupled* case.

For in the *uncoupled* case, our dark energy *contour* integral becomes, for the physical singularity at $(\delta, 0)$ …

$$\kappa\sigma e^{r\delta\cos(\theta)}\int_\gamma e^{sr}/\sqrt{(s - i\delta sin(\theta))(s + i\delta sin(\theta))}\, ds \qquad (\ddagger)$$

where γ is the contour shown below. And when evaluated along the branch lines, and combined with the singularity at $(-\delta, 0)$, we recover the *second* dark energy component in (§) above.



Also note that as δ ---› 0, (§) reduces to

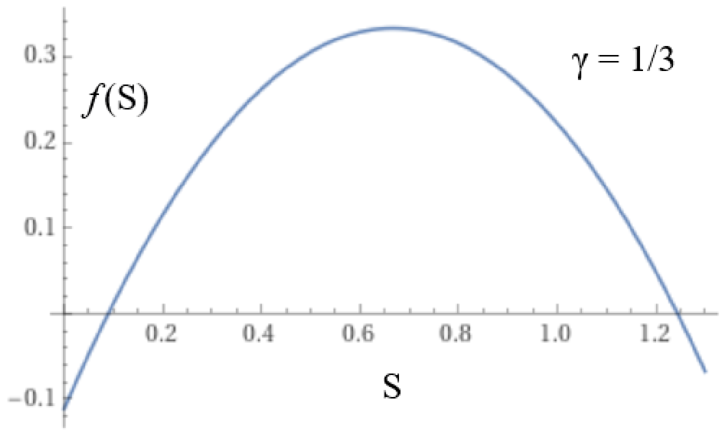
$$G^{u,v} \approx 3\sigma \cdot g^{u,v}(0)\,,$$

and this is what we would expect, because it is the ‘equivalent’ of three dark energy singularities now present at O. And finally, generalizing to *three* dimensions would work the same way [p 323].

AN INEQUALITY CONCERNING γ AND $S = \sum_{n=2}^{\infty} c_n \cdot (\gamma/2)^n$ FOR THE B MODEL, PART II

Some brief remarks are in order concerning the methodology used in Part I [pp 457-60]. On page 459, we define the function

$$f(S) = -S^2 + (2\gamma(3-\gamma)/(1+\gamma)) \cdot S - \gamma^2$$



and offer two plots, one of which is shown above for $\gamma = 1/3$. But we *also* know that S is defined to be the summation, as shown in the *title* to this note, so how do we reconcile this fact with the argument to $f(S)$? After all, for a given choice of γ, S will compute to a certain value.

In my opinion, the best way to see this, within the context of $f(S)$, is to allow S to become a *fluid* variable, which can take on *any* value as shown in the picture above, in a *hypothetical* sense. That is to say, we know what S *really* is, from the summation; but let's *pretend*, hypothetically, that it can take on *any* non-negative value, along the *x-axis*, relative to $f(S)$, and then form a constraint on what its range must actually be.

If we then follow the arguments through to completion on pages 459-60, it should now be a little easier to see how the inequality we developed for S, comes about.

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part II

Let us recall our equivalency theorem from pages 332-3, and reproduced here, where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, in the *coupled* case …

$$(1) = g^{u,v}(\alpha) \cdot J_0(r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true} \qquad (*)$$

Here, the following expression is (1),

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\alpha + iy) \; - \; g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) \; + \; g^{u,v}(\alpha - iy)]\}\, dy \,/ \sqrt{y^2 - \varepsilon^2}$$

and the expression below is (2), where the dark energy singularities are at O and at $(\pm 1, 0)$ …

$$G^{u,v} \approx \sigma[g^{u,v}(0) \; + \; 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] \, .$$

Now from Part I [pp 461-3], we now know that (1) is *equal* to $g^{u,v}(\alpha) \cdot J_0(r\varepsilon)$ in *two* cases; and these are when $\alpha = 1$, $\varepsilon = 0$, and when $\alpha = 0$, $\varepsilon = 1$. Note that because the dark energy component can be calculated from the dark energy contour integral, in the *uncoupled* case, all that really matters in the statement

$$(1) = g^{u,v}(\alpha) \cdot J_0(r\varepsilon) \qquad (\sim)$$

is the parametrization of $g^{u,v}$, just above, on the right-hand side.

Now if we look at ($\sim$) a little more closely, in the *two* cases where it is true, we see that *arg*[$g^{u,v}$] is equal to α, on the right-hand side of ($\sim$), which just happens to be the *average* value of the *two* arguments associated with $g^{u,v}$ in (1). That is to say, when $\alpha = 0$ or 1, it is the case that

$$\alpha \; = \; \{(\alpha - iy) \; + \; (\alpha + iy)\} / 2 \; = \; \textit{arg}[g^{u,v}] \text{ in } (\sim) \text{ [right-hand side]}$$

I suspect this is true in general, for *any* choice of α and ε, where θ is between 0 and π, say. If so, then from (*), (2) is *also* true for all such θ.

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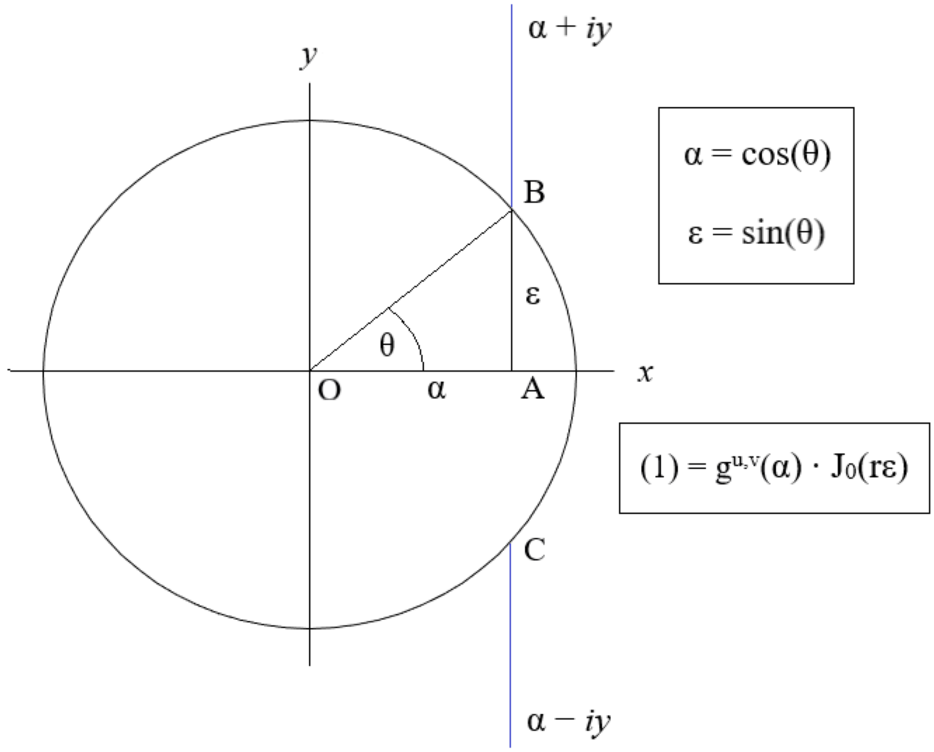


In the picture above is a *unit* circle, and here the dark energy singularities are at O and at $(\pm 1, 0)$. Superimposed on this picture is the α-line (blue) in the *complex* plane, where the integration in (1) takes place, implicitly. It starts at B and travels northward to $\alpha + i\infty$, and similarly, goes from C to $\alpha - i\infty$ in the opposite direction, when looking at the arguments associated with $g^{u,v}$ in (1).

We surmise that this integration, for *all* θ between 0 and π, is actually equal to

$$g^{u,v}(\alpha) \cdot J_0(r\varepsilon) ,$$

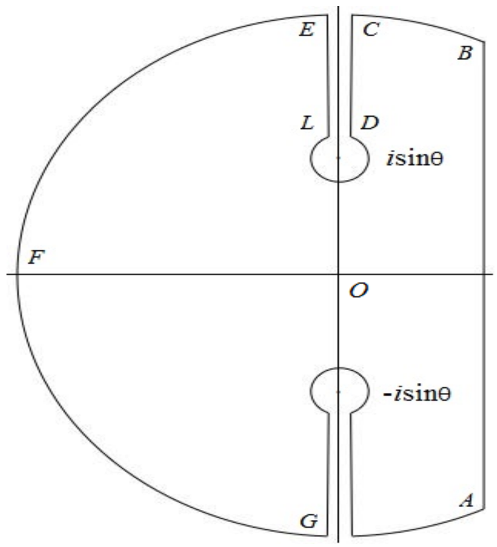
where α is the segment OA, and ε is the segment AB, which is equal to AC. And the variable r in J_0 above, is the radius of a circle in an (r, θ) layout, which shows up in (1) in the cos( ) and sin( ) terms.

*** ***

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part III

In considering our dark energy contour integral below (*), in the *coupled* case, where the singularity is at (+1, 0), we note that as $\theta \dashrightarrow 0$, the *branching* points in the contour γ below merge into a simple *pole*, so that it is easy to calculate the residue.

$$\kappa\sigma e^{r\cos(\theta)} \int_\gamma e^{sr} g^{u,v}(s + \cos(\theta)) / \sqrt{(s - i\sin(\theta))(s + i\sin(\theta))}\, ds \qquad (*)$$



Indeed, our expression (*) becomes, where γ is now a *simple* contour that omits the branch lines above,

$$\lim_{\theta \dashrightarrow 0} \kappa\sigma e^{r} \int_\gamma \{e^{sr} g^{u,v}(s + \cos(\theta)) / s\}\, ds$$

and this resolves as …

$$\lim_{\theta \dashrightarrow 0} \sigma e^{r} \cdot g^{u,v}(\cos(\theta))$$

Thus, *arg*[$g^{u,v}$] in the *coupled* equations, *already* presents itself as $\cos(\theta)$, *even* as the angle $\theta \dashrightarrow 0$, thereby only bolstering our view that (2) [p 465] is the correct form we are looking for …

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part IV

Let us consider (1) and (2) again [pp 465-6], with $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, in the *coupled* case, where (1) is …

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\alpha + iy) - g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) + g^{u,v}(\alpha - iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

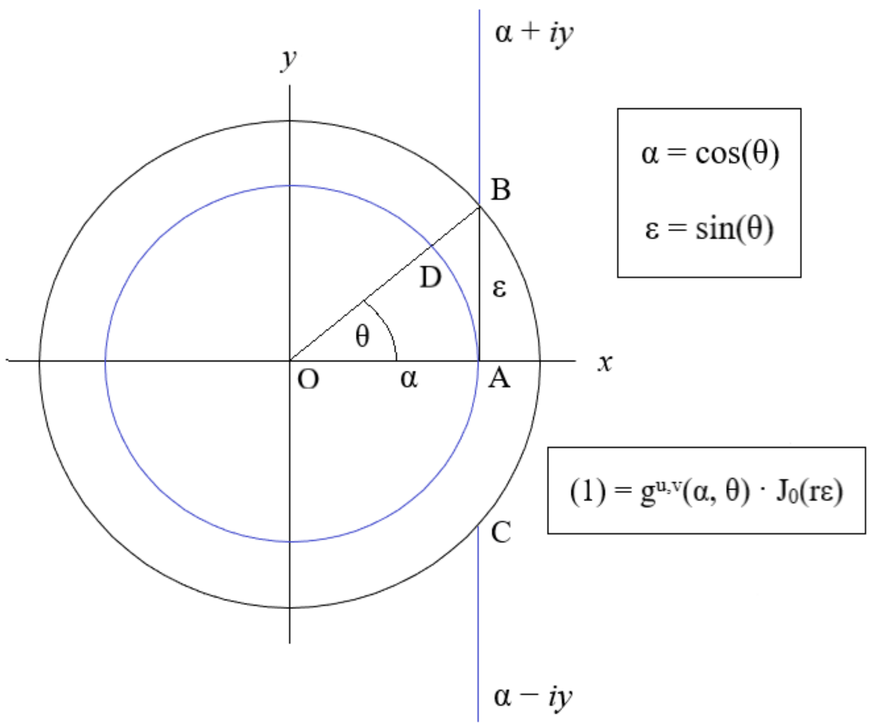
and (2) is …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)]\,.$$

Recalling our equivalency, it is probably *more* correct to say

$$(1) = g^{u,v}(\alpha, \theta) \cdot J_0(r\varepsilon) \text{ if and } only \text{ if (2) is true}, \qquad (*)$$

because for the *second* dark energy component in (2), $g^{u,v}(\alpha)$ will depend on θ, since α itself, is a *radial* measure [pp 323-5, 369-70].



Thus in the picture above, when calculating $g^{u,v}(\alpha)$, it must be done along the radial line $\boldsymbol{\ell}_\theta$, where α is now OD, which is equal to OA. Here, the dark energy singularities are at O and at $(\pm 1, 0)$, and the black circle above has a radius of 1. And in Cartesian coordinates, $g^{u,v}(\alpha, \theta) = g^{u,v}(\alpha^2, \alpha\varepsilon)$.

Now let's consider the *three* dimensional coupled equations, as shown below, in a *mathematical* coordinate system,

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) + 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) + 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] \quad (*)$$

with *physical* singularities at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) .$$

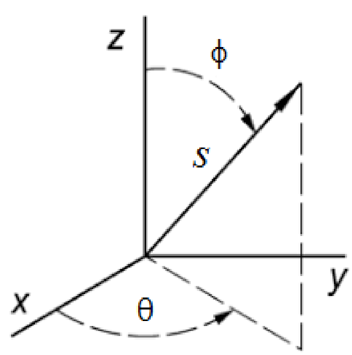
And here, we know from previous research, that $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$.

In this case, where we are on a *unit* sphere centered at O, the arguments $\cos(\alpha)$, $\sin(\beta)$ and $\cos(\phi)$ above, associated with $g^{u,v}$, are *radial* measures on the line $\ell_{\theta\phi}$, so that if we want to convert these arguments to Cartesian coordinates, we repeat the exercise on the previous page.

Since we know, in a *mathematical* coordinate system, that

$$x = r\sin(\phi)\cos(\theta) \ ; \ y = r\sin(\phi)\sin(\theta) \ ; \ z = r\cos(\phi)$$

all we need to do is let r become $\cos(\alpha)$, $\sin(\beta)$ and $\cos(\phi)$, for the corresponding terms associated with $g^{u,v}$ in (*), and calculate x, y, and z, accordingly. And, of course, use *absolute* values for r, if need be.



Mathematical Coordinate System

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part V

Let us consider (1) and (2) again [pp 465-6], where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, in the *coupled* case, with singularities at the origin O and $(\pm 1, 0)$. Then (1) is …

$$2\kappa \int_{\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\alpha + iy) - g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) + g^{u,v}(\alpha - iy)]\}\, dy \,/ \sqrt{y^2 - \varepsilon^2}$$

and (2) is …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] \,.$$

Recalling our equivalency,

$$(1) = g^{u,v}(\alpha, \theta) \cdot J_0(r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true },\qquad (\sim)$$

let us set $\varepsilon = 0$, and allow the singularities at $(\pm 1, 0)$ to *converge* at the origin O. Then (1) becomes

$$2\kappa \int_{0}^{\infty} \{\cos(yr)[g^{u,v}(iy) - g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) + g^{u,v}(-iy)]\}\, dy \,/\, y \qquad \ldots (*)$$

which is *always* equal to $g^{u,v}(0)$, provided $g^{u,v}$ is *well-behaved* in the *complex* plane, *real-valued* on the *x*-axis, and we can secure *convergence* along our contour γ [p 467]. Indeed, we showed it to be true when $g^{u,v}(s) = \zeta(\beta - s)$, where $\beta > 1$ [pp 330-2].

Thus, from our equivalency $(\sim)$ above, it must *also* be true that

$$G^{u,v} \approx 3\sigma \cdot g^{u,v}(0) \,,$$

since α is now 0 in (*), as is ε. And, of course, this agrees with our *original* calculations on pages 197-9, using the dark energy *contour* integral for a singularity at O. Thus, we have a confirmation of our harmonic expression (1) in this case, as well as our equivalency $(\sim)$.

*****

As another confirmation, suppose $g^{u,v} = c$, where c is some constant. Then (1) computes to

$$4\kappa \boldsymbol{i} c \int_{\varepsilon}^{\infty} \sin(yr)\, dy \,/\, \sqrt{y^2 - \varepsilon^2} \;=\; c \cdot 2/\pi \int_{\varepsilon}^{\infty} \sin(yr)\, dy \,/\, \sqrt{y^2 - \varepsilon^2} \;=\; c \cdot J_0(r\varepsilon) \,,$$

so that from our equivalency $(\sim)$, $g^{u,v}(\alpha)$ in (2) is equal to c, which is, indeed, the case.

But the question of how to solve (2), reproduced below, is still rather puzzling …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] \quad (2)$$

For in (2), there are *two* terms on the *right*-hand side that we need to deal with. First, $g^{u,v}(0)$, which is associated with the *first* dark energy component σ, and then $g^{u,v}(\alpha)$, which is associated with the *second* component $2\sigma\cosh(r\alpha)J_0(r\varepsilon)$.

Now $g^{u,v}(0)$ *doesn't* depend on θ, because it is associated with the singularity at the origin O, but the *second term* $g^{u,v}(\alpha)$, associated with the singularities at (±1, 0), *does* depend on θ, because α is a *radial* measure.

Perhaps the best approach is to look at the equation

$$G^{u,v} \approx 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta)) \quad (\S)$$

associated with the dark energy singularities at (±δ, 0), where δ > 0. The solution to (§), namely $g^{u,v}$, will be a function of (r, θ, σ, δ), but if we let δ ---› 0, it becomes the *equivalent* of the solution to

$$G^{u,v} \approx 2\sigma \cdot g^{u,v}(0) .$$

Thus, if we take the solution to (§), namely, $g^{u,v}(r, \theta, \sigma, \delta)$, and replace σ by σ / 2 in it, as δ ---› 0, we should, in theory, arrive at a solution to

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0) ,$$

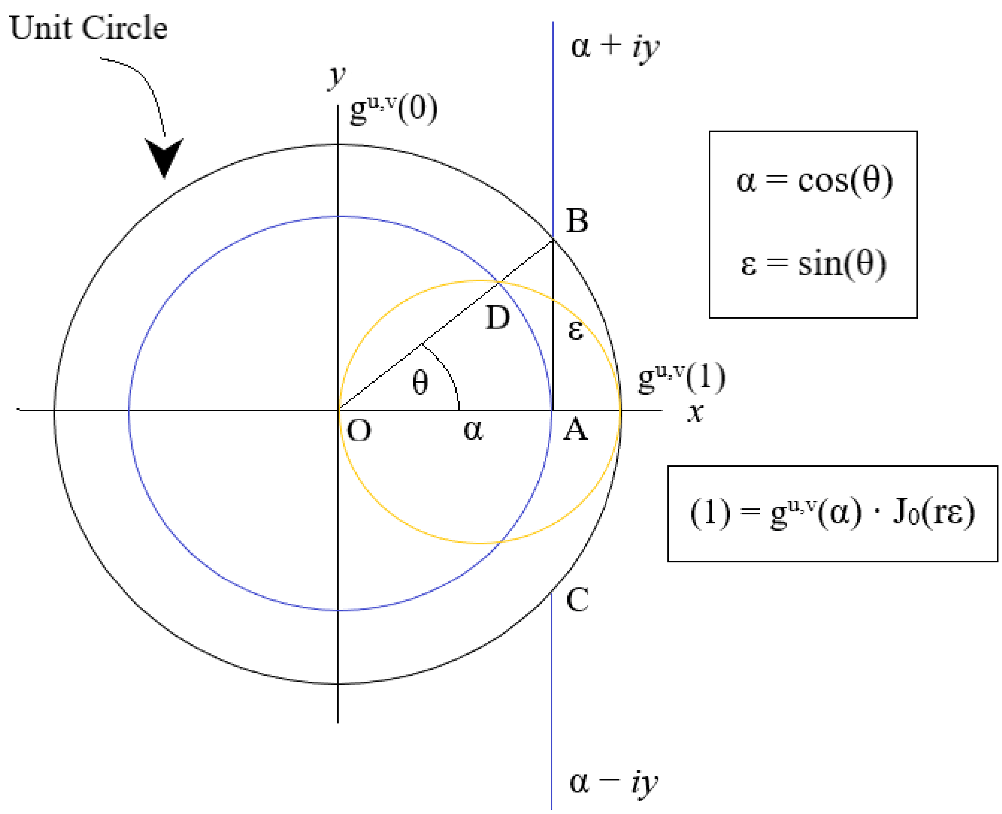
which will *only* depend on (r, σ), because of *radial* symmetry associated with the singularity at O ...

That is to say, the solution $g^{u,v}(r, \sigma)$ to the equation just above, will *not* depend on θ, and perhaps by *adding* the two solutions together; namely $g^{u,v}(r, \theta, \sigma, \delta)$ and $g^{u,v}(r, \sigma)$, we might arrive at a solution which incorporates *all* three singularities; that is to say, the one at O and the two at (±δ, 0).

NOTE TO THE READER

In (2) above, when α = 0, the temptation is to factor $g^{u,v}(0)$ out of the bracketed expression, which we actually did on pages 321-9, purely for *symbolic* reasons. In reality, $g^{u,v}(0)$ and $g^{u,v}(\alpha)$ are two *very* different gravitational tensors – the first being associated with the singularity at O, and thus having no dependency on θ at all, while the second $[g^{u,v}(\alpha)]$ does *very* much depend on θ, because it is associated with the singularities at (±1, 0). Thus, no such factorization is really possible … …

Also note, that when talking about $g^{u,v}(r, \sigma)$ having *no* dependency on θ or ϕ, it is implied that we are referring to the primary directions of *r* and *t*.



In the diagram above, in the *coupled* case, where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, with singularities at the origin O and $(\pm 1, 0)$, we recall that (1) is …

$$2\kappa \int_{\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\alpha + iy) \;-\; g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) \;+\; g^{u,v}(\alpha - iy)]\}\, dy \,/ \sqrt{y^2 - \varepsilon^2}$$

and (2) is …

$$G^{u,v} \;\approx\; \sigma[g^{u,v}(0) \;+\; 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)]\,.$$

Recalling our equivalency,

$$(1) = g^{u,v}(\alpha) \cdot J_0(r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true}\,, \qquad (\sim)$$

we can picture the argument associated with $g^{u,v}(\alpha)$ in two ways: as the segment OD meeting the *blue* circle at D or, as the segment OD meeting the *gold* circle at D, where the gold circle is the *polar* plot of $\alpha = \cos(\theta)$, running from $\theta = -\pi / 2$ to $\pi / 2$.

As θ moves from 0 to $\pi / 2$, the *radius* of the blue circle will shrink to *zero*, while the polar plot in gold will remain the same …

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part VI

In this note, we want to talk briefly about how we *might* solve the *coupled* equations in *two* dimensions, when there are singularities at the origin O, and at $\{(\pm1, 0), (0, \pm1)\}$. Notationally, the form for these equations is …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta)) + 2\cosh(r\sin(\theta))J_0(r\cos(\theta))g^{u,v}(\sin(\theta))] ,$$

but in reality, it only makes sense to deal with it component by component, as we'll see below … ..

Starting with the first component, for the singularity at O, we have

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0) ,$$

and however we arrive at a solution to this equation, we'll call it $g^{u,v}(r, \sigma)$, since it doesn't depend on θ. The *second* component, associated with the singularities at $(\pm1, 0)$, was already dealt with in the last research note, and so again, we'll call the solution to the equation below $g^{u,v}(r, \theta, \sigma)$, since it *does* depend on θ …

$$G^{u,v} \approx 2\sigma \cdot \cosh(r\cos(\theta))J_0(r\sin(\theta))g^{u,v}(\cos(\theta)) . \qquad (*)$$

To find the solution for the *third* component, simply rotate $g^{u,v}(r, \theta, \sigma)$ by 90 degrees, since the singularities here are at $(0, \pm1)$. We'll call this solution $g^{u,v}(r, \theta, \sigma \mid \text{rotate})$. Now we can obtain a *total* solution by *adding* together $g^{u,v}(r, \sigma)$, $g^{u,v}(r, \theta, \sigma)$ and $g^{u,v}(r, \theta, \sigma \mid \text{rotate})$.

In *three* dimensions, with singularities at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) ,$$

the *coupled* equations are, notationally, in a *mathematical* coordinate system …

$$\begin{aligned} G^{u,v} \approx \sigma[g^{u,v}(0) &+ 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) + \\ &2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) + \\ &2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] . \end{aligned}$$

And here, we know from previous research, that $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, so that, roughly speaking, we can use the *same* approach as above. For the singularity at O, we solve the equation

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0) ,$$

and call this solution $g^{u,v}(r, \sigma)$, since it doesn't depend on θ *or* ϕ.

For the *second* component, associated with the singularities at $(\pm 1, 0, 0)$, one would now solve the equation

$$G^{u,v} \approx 2\sigma \cdot \cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) ,$$

and call the solution $g^{u,v}(r, \theta, \phi, \sigma)$. And for the *third* component, the solution is simply a 90 degree *rotation* of $g^{u,v}(r, \theta, \phi, \sigma)$ in the *x-y* plane, since we are now dealing with the singularities at $(0, \pm 1, 0)$. Let's call this solution $g^{u,v}(r, \theta, \phi, \sigma \mid \text{rotate})$.

That only leaves us with the *last* component, associated with the singularities at $(0, 0, \pm 1)$; namely,

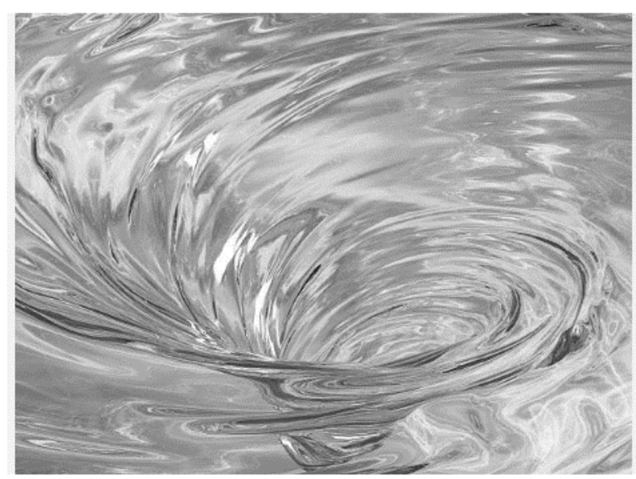
$$G^{u,v} \approx 2\sigma \cdot \cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi)) ,$$

and this is *no* different than solving (*) on the last page, where θ has been replaced by ϕ. We'll call this solution $g^{u,v}(r, \phi, \sigma)$, since it *doesn't* depend on θ – a manifestation of our coordinate system, actually, which is the (r, θ, ϕ) layout.

Now we can obtain a *total* solution, by adding all *four* solutions together; that is to say, $g^{u,v}(r, \sigma)$, along with $g^{u,v}(r, \theta, \phi, \sigma)$, $g^{u,v}(r, \theta, \phi, \sigma \mid \text{rotate})$ and $g^{u,v}(r, \phi, \sigma)$ …

OTHER CONSIDERATIONS

It might just be that at or near a singularity, a 'whorl effect' appears in $g^{u,v}$, in the *coupled* case. For example, we know already that with singularities at the origin O and at $(\pm 1, 0)$, approaching O along the *x*-axis is *not* the same as approaching O along the *y*-axis, when looking at the coupled equations in *two* dimensions. And this could lead to whorls in $g^{u,v}$ as shown in the diagrams below.



An Interesting Decomposition Concerning The Primes

Let n > 3 be *any* integer. Then there is always a *finite* sequence of *increasing* primes $\{p_k\}$, according to our construction below, such that

$$n = p_1 + p_2 + \ldots + p_k \quad ; \quad p_1 \leq p_2 \leq \ldots \leq p_k \quad ; \quad p_1 = 2 \text{ or } 3 \qquad (*)$$

To show that this is so, we'll prove it by going in the *reverse* direction. So let's start by letting p_1 be the *largest* prime at least 2 below n, and let $\Delta = n - p_1$. If Δ = 2 or 3, we are done. If not, find the *largest* prime p_2 at least 2 below Δ, and now let $\Delta = n - p_1 - p_2$. If Δ = 2 or 3, then again, we are done (note that $p_2 \leq p_1$).

Otherwise, continue the process until eventually we have Δ = 2 or 3, and let $p_k = \Delta$, to complete the summation. Finally, by reversing this summation, our theorem is proved.

As an example, if n = 29, it would resolve as 29 = 23 + 3 + 3. On the other hand, 28 = 23 + 3 + 2, and by reversing these summations, we see how they validate our theorem.

What makes this somewhat fascinating to me, at least, is that it holds for *any* n > 3. Thus, even for a number like 10^{1000}, (*) will be true, even though we may never know what the sequence $\{p_k\}$ really is, according to our construction.

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part VII

In the diagram below, on the following page, we want to consider the dark energy over *two* circles [A and B] and compare charts. Here, we are dealing with a *three* dimensional model, with singularities at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) ,$$

where the *coupled* equations are, notationally, in a *mathematical* coordinate system …

$$\begin{aligned} G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) &+ \\ 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) &+ \\ 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] \,. &\quad (*) \end{aligned}$$

And here, we know from previous research, that $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$.

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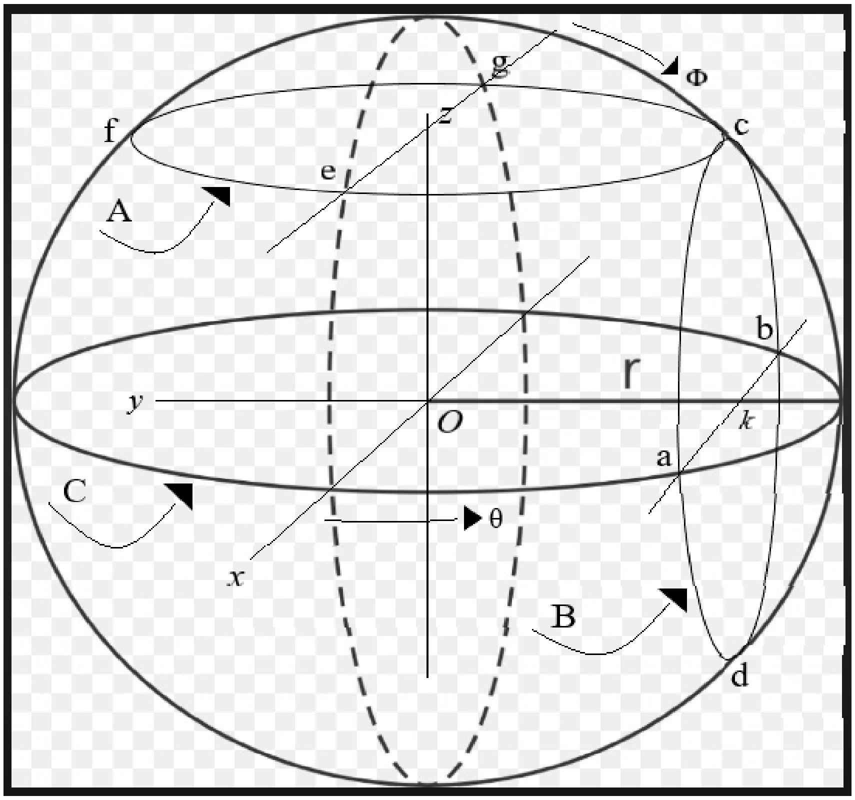
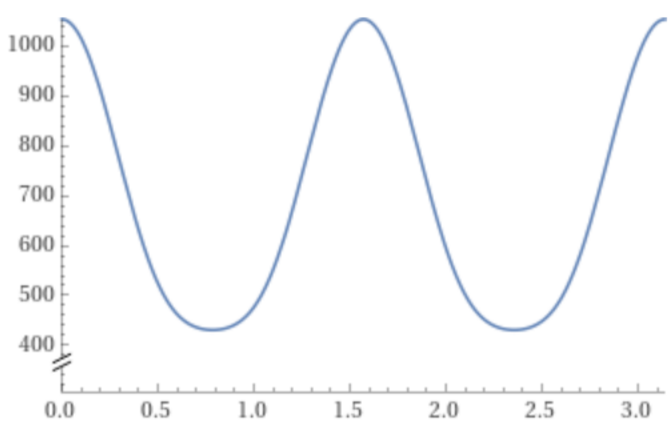
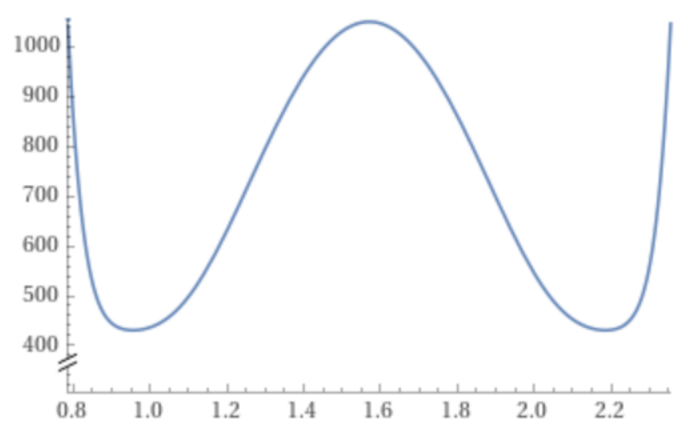


For circle A above, where r = 11, σ = 1, 0 ≤ θ ≤ π, and ϕ = π / 4, the *total* dark energy in (*), on the semi-circle *ecg* is shown in the picture on the *left* below. For circle B, where $k = (\sqrt{2}\,/2) \cdot r$, σ = 1, π / 4 ≤ θ ≤ 3π / 4 and π / 4 ≤ ϕ ≤ π / 2, the *total* dark energy on the semi-circle *acb* is shown on the *right* below. The constant *k* is the center of circle B.



Circle A *ecg* , θ = *x*



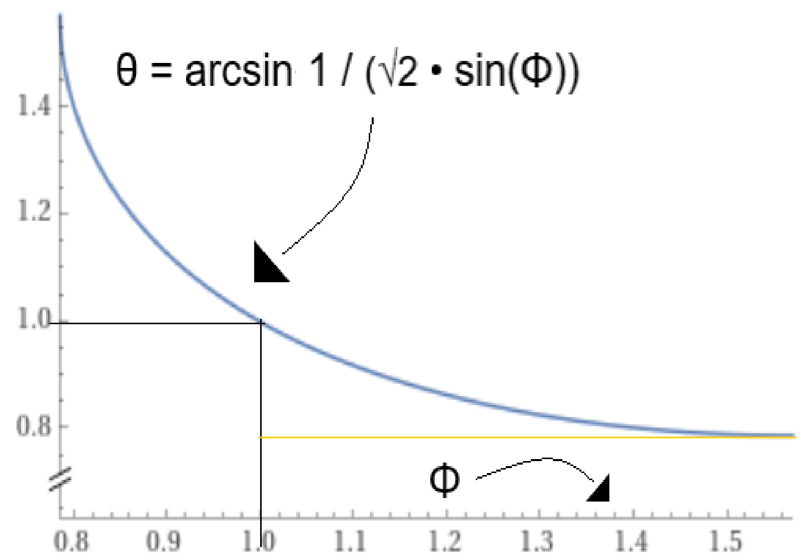
Circle B *acb* , θ = *x*

One would think that the two energy profiles above would be the *same*, given the location of the singularities, and given that circle B is simply a 90 degree rotation of A. However, because of where circle B is situated, the *y* component *k* does not change, so that the equation below *must* hold for this circle [B].

$$y = r \cdot \sin(\phi)\sin(\theta) = k$$

In turn, this creates a *non-linear* relationship between θ and ϕ in circle B, which *skews* or *distorts* θ in the *right* diagram above, versus θ in the *left* diagram above, where no distortion occurs, because in the *left* diagram above, θ is *independent* of ϕ.

Thus, the two diagrams really are the same, up to this distortion effect, and I think this is the correct way to interpret pictures like this, when a dependency between θ and ϕ exists. Below is a plot of θ versus ϕ when $k = (\sqrt{2}/2) \cdot r$, and $\pi / 4 \leq \theta \leq \pi / 2$, $\pi / 4 \leq \phi \leq \pi / 2$.



Finally, when considering solutions to *coupled* forms, like those shown below, it should be kept in mind that $G^{u,v}$ must be constructed in a manner *consistent* with the *right*-hand side of these forms. Here, $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively …

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0)\ ,$$

$$G^{u,v} \approx 2\sigma \cdot \cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))\ .$$

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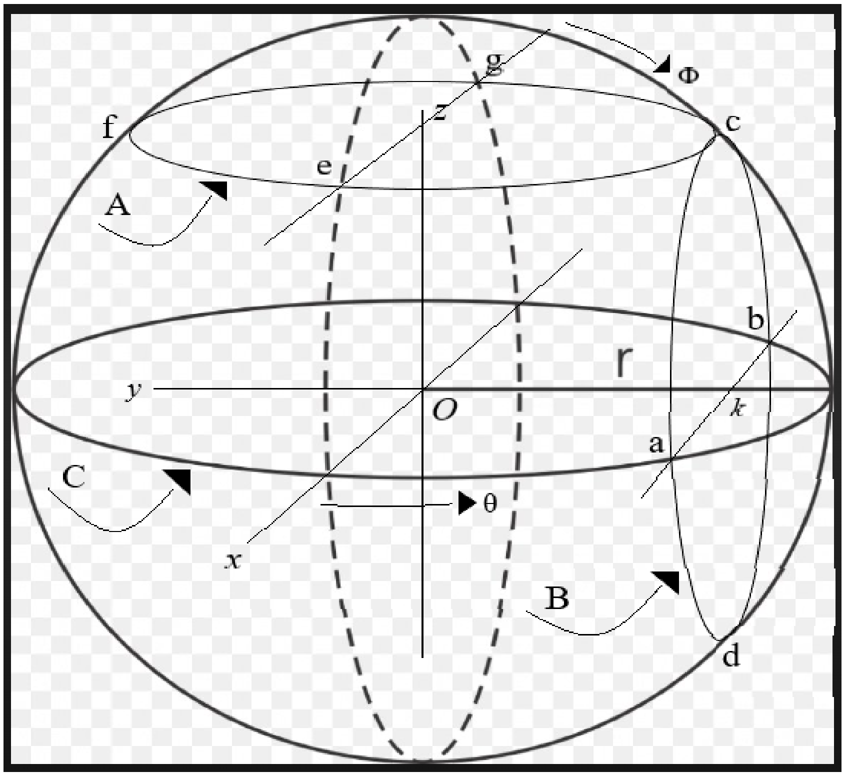
Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part VIII



In the diagram above, the *total* dark energy ξ from (*) on page 475, at points *e* and *g* on the dashed circle, maps *uniquely* to these same points on circle A. Similarly, ξ at points *a* and *b* on circle B maps *uniquely* to these same points on circle C, and at the point *c* on circles A and B, where they touch, ξ is also the same value.

In general, then, for any point *p* on *any* circle B, we can always map (or associate) ξ at *p* with some circle A, uniquely, so that ultimately, the *entire* sphere is covered off in this fashion.

Thus, when calculating the *total* dark energy ξ over the sphere S of radius r, it does not matter whether we choose to integrate over the sum total of *all* circles of type B, moving from the *east* pole of S to the *west* pole of S, or circles of type A, moving from the *north* pole of S to the *south* pole of S. Either way, we will get the same result for ξ.

The only difference is one of simplicity. If we choose to go with circles of type A, moving from *north* to *south* [pp 374-5], the calculations will be much simpler because θ is *independent* of ϕ …

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part IX

Some brief remarks are in order concerning the *coupled* field equations. When we write an equation like

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) + 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) + 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] \,, \quad (*)$$

with singularities at the origin O, and at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) \,;$$

where $G^{u,v} = C^{u,v} - kT^{u,v}$, and $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; we are saying that $G^{u,v}$ is the *remainder* after subtracting out *tangible* matter from the space-time fabric. And here, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, in a *mathematical* coordinate system.

Thus, while dark energy appears in (*), it does so on the right-hand side of (*), and is to be thought of as *intangible* matter, which *still* has an influence on the gravitational tensor $g^{u,v}$. It therefore has no presence in the stress tensor $T^{u,v}$, in my opinion.

So in the simple case where we only have the *vacuum* in our space-time fabric, plus *one* dark energy singularity at the origin O of our coordinate system, $T^{u,v}$ would be zero, and our *coupled* field equations would look like

$$C^{u,v} \approx \sigma \cdot g^{u,v}(0) \,,$$

where $C^{u,v} = R^{u,v} - \frac{1}{2} R g^{u,v}$, and $R^{u,v}$ is the Ricci tensor and R the Ricci scalar. And if there was *also* a static, perfect star centered at O, $T^{u,v}$ in the *comoving* frame, would be comprised of *pressure* and *density* in the star, along the diagonal components, and the coupled field equations would now look like

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0) \,.$$

And the same formulation applies to (*) above, where now the dark energy singularities have been extended to include

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) \,.$$

In my view, this is the correct way to interpret the *left-hand* side of the *coupled* field equations, but there may be other interpretations as well.

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part X

Suppose we are looking for a *total* solution ‘in one go’ to the *coupled* equations below,

$$G^{u,v} \approx \sigma[2\cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) + 2\cosh(r\sin(\beta))J_0(r\cos(\beta))g^{u,v}(\sin(\beta)) + 2\cosh(r\cos(\phi))J_0(r\sin(\phi))g^{u,v}(\cos(\phi))] , \quad (*)$$

with singularities now at

$$(1, 0, 0) \quad (-1, 0, 0) \quad (0, 1, 0) \quad (0, -1, 0) \quad (0, 0, 1) \quad (0, 0, -1) .$$

Because this is being done ‘in one go’, we are abandoning the ‘component by component’ approach discussed in previous research notes in this section [p 461 *ff.*], and we’ll call this *total* solution, which we believe exists, $g^{u,v}(r, \theta, \phi, \sigma \mid \text{total})$.

Now suppose we remove the singularities along the *y* and *z* axes, so that only the singularities along the *x*-axis [$(\pm 1, 0, 0)$] remain. Question: will it be possible to reduce $g^{u,v}(r, \theta, \phi, \sigma \mid \text{total})$, mathematically speaking, so that it now reflects the solution we would have gotten, had we solved the equation

$$G^{u,v} \approx 2\sigma \cdot \cosh(r\cos(\alpha))J_0(r\sin(\alpha))g^{u,v}(\cos(\alpha)) ,$$

associated with $(\pm 1, 0, 0)$?

Either such a reduction is possible or it isn’t. If it is, then the ‘in one go’ approach might work, but if it isn’t, then the ‘component by component’ approach may be our only option, since here such a reduction is obvious [pp 473-4].

*****

We’ll now offer a rather *heuristical* proof that *any* total solution for the coupled equations, with singularities at the origin O, and at S = $\{(\pm\delta_x, 0, 0), (0, \pm\delta_y, 0), (0, 0, \pm\delta_z)\}$, where δ_x, δ_y, and δ_z are all *greater* than *zero*, is always reducible. It should be said that whether there is or isn’t a perfect star centered at O, does not matter here.

Let us assume the solution at O is *separate* and *apart* from the solution for S, so that our *total* solution may be written as

$$\mathcal{T} = g^{u,v}(r, \sigma) + g^{u,v}(r, \theta, \phi, \sigma, \delta_x, \delta_y, \delta_z) .$$

Now let δ_z ---› 0, so that the solution becomes

$$3 \cdot g^{u,v}(r, \sigma) \ + \ g^{u,v}(r, \theta, \phi, \sigma, \delta_x, \delta_y) \ ,$$

which, up to the constant 3, is a subset of $\mathscr{T}$. If we now let δ_y ---› 0, the solution becomes

$$5 \cdot g^{u,v}(r, \sigma) \ + \ g^{u,v}(r, \theta, \phi, \sigma, \delta_x) \ ,$$

which again, is a subset of $\mathscr{T}$. And finally, letting δ_x ---› 0, we obtain $7 \cdot g^{u,v}(r, \sigma)$, and the theorem is now proved.

So no matter how we arrive at a total solution $\mathscr{T}$ for the coupled equations, it must, in the end, be *reducible*, if our logic above holds true.

OTHER CONSIDERATIONS

Let δ_z ---› 0, so that the solution becomes (because the two *z-axis* singularities converge at O),

$$g^{u,v}(r, \sigma) \ + \ g^{u,v}(r, 2\sigma) \ + \ g^{u,v}(r, \theta, \phi, \sigma, \delta_x, \delta_y) \ .$$

If we now let δ_y ---› 0, the solution becomes

$$g^{u,v}(r, \sigma) \ + \ 2g^{u,v}(r, 2\sigma) \ + \ g^{u,v}(r, \theta, \phi, \sigma, \delta_x) \ ,$$

and finally, letting δ_x ---› 0, we obtain

$$g^{u,v}(r, \sigma) \ + \ 3g^{u,v}(r, 2\sigma) \ . \qquad (\sim)$$

Were we to let δ_x, δ_y, and δ_z converge to O *simultaneously*, we could argue the reduction would lead to

$$g^{u,v}(r, \sigma) \ + \ g^{u,v}(r, 6\sigma) \ ,$$

so that from (~)

$$3g^{u,v}(r, 2\sigma) \ = \ g^{u,v}(r, 6\sigma) \ .$$

In turn, this must mean that

$$2g^{u,v}(r, \sigma) \ = \ g^{u,v}(r, 2\sigma) \ ,$$

so that (~) above becomes $7 \cdot g^{u,v}(r, \sigma)$.

*****

We'll now demonstrate this result, using a slightly different approach. So let us begin by considering the *coupled* equation below …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] , \quad (*)$$

with singularities at O, and at $(\pm\delta, 0)$, for some $\delta > 0$. And let us presume, as we did above, that the *total* solution to this equation is of the form

$$\mathcal{T} = g^{u,v}(r, \sigma) + g^{u,v}(r, \theta, \sigma, \delta) .$$

Letting $\delta \dashrightarrow 0$, the solution will become

$$g^{u,v}(r, \sigma) + g^{u,v}(r, 2\sigma) ,$$

but *also*, the form for (*) now becomes

$$G^{u,v} \approx 3\sigma \cdot g^{u,v}(0) ,$$

with a solution $g^{u,v}(r, 3\sigma)$. Thus, it must be the case that

$$g^{u,v}(r, \sigma) + g^{u,v}(r, 2\sigma) = g^{u,v}(r, 3\sigma) ,$$

and so, an *associative* principle concerning σ emerges; namely that if *a* and *b* are *non-zero*, and *real-valued* numbers, then it must be the case that

$$g^{u,v}(r, a\sigma) + g^{u,v}(r, b\sigma) = g^{u,v}(r, (a + b)\sigma) . \quad (\dagger)$$

Extending (*) so that it includes the singularities at $(0, \pm\delta)$, we find that as $\delta \dashrightarrow 0$

$$g^{u,v}(r, \sigma) + 2g^{u,v}(r, 2\sigma) = g^{u,v}(r, 5\sigma) ,$$

and thus, using our associative principle, we have

$$2g^{u,v}(r, 2\sigma) = g^{u,v}(r, 5\sigma) - g^{u,v}(r, \sigma) = g^{u,v}(r, 5\sigma) + g^{u,v}(r, -\sigma) = g^{u,v}(r, 4\sigma) .$$

Replacing σ by $\sigma / 2$ now gives us the desired result; namely that

$$2g^{u,v}(r, \sigma) = g^{u,v}(r, 2\sigma) .$$

We could have discovered this result simply by letting *a* and *b* equal 1 in (†), but it's nice to see too that it can be produced by extending (*), as we did above.

*****

Let's now reconsider (*) above, but this time assume a *total* solution $\mathcal{T}$ where we *don't* split off the solution at O. Again, as $\delta \dashrightarrow 0$, $\mathcal{T}$ becomes $g^{u,v}(r, 3\sigma)$, and we'll call this $\mathcal{T}_0$. Now let's consider the *total* solution $\mathcal{T}'$ to the following equation …

$$G^{u,v} \approx 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta)) ,$$

and let δ ---› 0, so that $\mathcal{T}'$ becomes $g^{u,v}(r, 2\sigma)$, and we'll call this $\mathcal{T}'_0$. We now want to consider the meaning of the statement

$$\mathcal{T}_0 - \mathcal{T}'_0 .$$

It is, in fact, the *total* solution at O, after *removing* two of the three singularities that were there, once we let δ ---› 0 initially, to arrive at $\mathcal{T}_0$. As such, we may write

$$\mathcal{T}_0 - \mathcal{T}'_0 = g^{u,v}(r, \sigma) ,$$

or equivalently,

$$g^{u,v}(r, \sigma) + g^{u,v}(r, 2\sigma) = g^{u,v}(r, 3\sigma) ,$$

which is *exactly* what we concluded on the last page, by first assuming a split in the total solution $\mathcal{T}$ to (*).

Thus, it does *not* matter whether we *do* or *don't* assume a split in $\mathcal{T}$, for either way, we arrive at the *same* conclusion, and so our associative principle concerning σ, holds in general.

NOTE TO THE READER

When talking about $g^{u,v}(r, \sigma)$ having *no* dependency on θ or ϕ, it is implied that we are referring to the primary directions of *r* and *t*, in these notes.

*****

It might just be, for a *coupled* equation like

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] , \quad (*)$$

with singularities at O, and at (±δ, 0), for some δ > 0; that the *total* solution to this equation is of the form (at least for the *radial* and *time* components)

$$\mathcal{T} = g^{u,v}(r) + g^{u,v}(r, \sigma) + g^{u,v}(r, \theta, \sigma, \delta) .$$

Letting δ ---› 0 gives us

$$g^{u,v}(r) + g^{u,v}(r, 3\sigma) ,$$

and letting σ ---› 0 yields $g^{u,v}(r)$, since from our *associative* principle below [p 482], $g^{u,v}(r, \sigma) = 0$ if $\sigma = 0$.

$$2g^{u,v}(r, \sigma) = g^{u,v}(r, 2\sigma)$$

Using $\mathcal{T}$ as the template, we would then calculate $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; given that in a *comoving* frame, $T^{u,v}$ will be comprised of *pressure* and *density* along the diagonal, for a perfect *two-dimensional* star, centered at the origin O.

We might start by looking for a *diagonal* solution to $g^{u,v}$, and thus would calculate $C^{u,v}$ and $T^{u,v}$ along the diagonal, using the whole of $\mathcal{T}$, and solve (*) above, accordingly.

And in this case, because we are dealing with an (r, θ, t) layout, it is reasonable to suppose $\mathcal{T}$ should look the same for all (u, v), where $u = v$, and $u = 1, 2, 3$.

There can be no assurance that this method will work, but at this point, anyway, I believe it's worth considering. And if we decide to take a 'component by component' approach to solving (*) on page 483, assuming our *associative* principle holds true [pp 482-3], we'd start with the expression

$$G^{u,v} \approx 0 ,$$

and solve it by using the *first* term in $\mathcal{T}$ below, as our template for calculating $C^{u,v}$ and $T^{u,v}$, assuming a *perfect* star S*, centered at O …

$$\mathcal{T} = g^{u,v}(r) + g^{u,v}(r, \sigma) + g^{u,v}(r, \theta, \sigma, \delta) .$$

To solve for the singularity at O; namely,

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0) ,$$

we would use the *second* term in $\mathcal{T}$ for doing our calculations, and to solve for the singularities that are at $(\pm\delta, 0)$; namely,

$$G^{u,v} \approx 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta)) ,$$

we would use the *last* term in $\mathcal{T}$ for our calculations. Adding *all* three component solutions together would give us a *total* solution for (*), that allows for σ ---› 0, when S* exists.

The only other option would be to try and solve (*) in its entirety, 'in one go', by using the whole of $\mathcal{T}$ to calculate $C^{u,v}$ and $T^{u,v}$, but I'm not so sure this is a good idea. For more on this, you can also consult the previous research notes [p 461 *ff*.] …

*****

So to see how this works for the *second* component in $\mathcal{T}$, which is $g^{u,v}(r, \sigma)$, associated with the singularity at O; let us assume a perfect star S* centered at O, which is filled with dark energy [σ] everywhere, but *nothing* else. From this, we can calculate an *intangible* stress tensor $T^{u,v}$, comprised of *pressure* and *density* along the *diagonal*, just as we would in the *tangible* case, when dealing with the first term in $\mathcal{T}$, where $\sigma = 0$. So we're imagining dark energy to be a *fluid* here.

The assumptions concerning $g^{u,v}(r, \sigma)$ would be similar to those made for the *tangible* case, except that here we know σ is present and that $g^{u,v}(r, \sigma)$ must *vanish* if there is *no* dark energy; that is to say, $g^{u,v}(r, \sigma) = 0$ if $\sigma = 0$. Therefore, it is reasonable to suppose

$$g^{u,v}(r, \sigma) \; = \; \sigma \cdot g^{u,v}(r) \, ,$$

for such an assumption *also* preserves our *associative* principle mentioned above [see also the notes on pp 482-3]. But there may be other ways to decode $g^{u,v}(r, \sigma)$, as well. For example, we might write, for some function *f* , that preserves *associativity* under addition with respect to σ …

$$g^{u,v}(r, \sigma) \; = \; f(\sigma) \cdot g^{u,v}(r) \, ,$$

or we could even let $f(\sigma) = 1$, and hope that the solution $g^{u,v}(r, \sigma)$ to the equation below preserves *associativity*.

Now we can calculate both $C^{u,v}$ and $T^{u,v}$, using our template from $g^{u,v}(r, \sigma)$ above [which might be $g^{u,v}(r)$, itself]; where here $C^{u,v}$, $T^{u,v}$ can be thought of as *intangible* tensors, and solve the following equation for the *coupled* case …

$$G^{u,v} \; \approx \; \sigma \cdot g^{u,v}(0) \, .$$

We then take this solution $g^{u,v}(r, \sigma)$, valid for *all* $r \geq 0$, and *add* it to the solution $g^{u,v}(r)$ obtained by using the first term in $\mathscr{T}$, when $\sigma = 0$; and so have a *total* solution in the case of a perfect star, centered at O, and filled with *tangible* matter, *plus* one dark energy singularity at O; where the dark energy [σ] itself, permeates the *whole* of space-time. As such, there is no such thing as an *interior* or *exterior* solution for $g^{u,v}(r, \sigma)$, as there is when dealing with *tangible* matter in the star.

Almost like an *overlay*, if you will, where we are considering *two* perfect stars; one filled with a *tangible* fluid, and the other filled with an *intangible* fluid, which we call dark energy, that spreads throughout the *whole* of our space-time fabric.

The case of the third term in $\mathscr{T}$ is much harder to deal with, as it is associated with the singularities at (±δ, 0), where the equation now takes the form

$$G^{u,v} \; \approx \; 2\sigma \cdot \cosh(\delta r\cos(\theta)) J_0(\delta r\sin(\theta)) g^{u,v}(\delta\cos(\theta)) \, .$$

It's still *intangible* matter, but calculating $C^{u,v}$ and $T^{u,v}$ will be much harder. Perhaps we can invoke a *mean-value* theorem here [pp 374-5], to simplify things. I hope to say more in a future note, on this subject …

OTHER CONSIDERATIONS

In dealing with the *two dimensional* perfect star S* of radius r, and filled *only* with dark energy and nothing else, the dark energy at each point in S* is σ, and the *average* value of dark energy over S* is *also* σ [pp 374-5].

Thus, by letting r ---› 0, we see the *average* value of dark energy is *still* σ, and so can define energy density [ρ] in S* to be σ. And since dark energy is an *intangible* fluid, it is reasonable to suppose that the following 'equation of state' holds for S*, where *p* is pressure in the star [p 383 *ff*.] …

$$2p + \rho = 0 .$$

Thus, along the *diagonal*, we can populate the *intangible* stress tensor $T^{u,v}$, in the *comoving* frame, with real values for *pressure* and *density*, and so calculate the intangible $C^{u,v}$ and $T^{u,v}$ in a *relativistic* setting.

As to our template $g^{u,v}(r)$, which is used to calculate $C^{u,v}$ and $T^{u,v}$, it is a function of r in the *radial* and *time* directions [$u = v$, $u = 1$ or 3], and for $u = v = 2$, which is the θ direction, we can let $g_{u,v}(r)$, in its *covariant* form, be equal to r^2 [p 383 *ff*.].

These are just my ideas and opinions on how we go about solving

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0) ,$$

for the *intangible* case, where we are dealing with the singularity at the origin O, and S* is filled *only* with dark energy.

*****

Note that if a component, such as $g^{2,2}$ in the *intangible* case, does *not* depend on σ, then we would not add it to its counterpart in the *tangible* case, when developing a *total* solution. We would only do this if in the intangible case, $g^{u,v}(r, \sigma)$ did depend on σ.

Now looking at (*) below again, we could try to solve it 'in one go' by essentially incorporating the *right-hand* side into the stress tensor. But what will happen when we try to calculate $C^{u,v}$ and $T^{u,v}$, using a template like $g^{u,v}(r, \theta, \sigma, \delta)$? For example, when $\theta = \pi / 2$, am I referring to the *first* component in the bracketed expression below, or am I referring to the *second*, since both are $g^{u,v}(0)$. They are, in fact, two *very* different gravitational tensors, since the first does *not* depend on θ at all, but the second does.

Furthermore, if we were to incorporate only the *first* component $\sigma \cdot g^{u,v}(0)$ into $T^{u,v}$, where now we're dealing with a perfect star [S*] filled with *tangible* matter, it's rather like saying we're going to pair the pressure and density [p and ρ] associated with S*, along the diagonal, with σ. Yet, from our analysis above, of the perfect star filled *only* with dark energy, we have seen that it may have its *own intangible* stress tensor, that follows a certain 'equation of state'. How then, would we reconcile this difference, were we to incorporate in this way ?

For these reasons, and others too, I think the 'component by component' approach is more realistic, and we may have more to say about it in future notes …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] . \quad (*)$$

Now we'll deal with the *third* term in $\mathscr{T}$ below, which is associated with the singularities at (±δ, 0),

$$\mathscr{T} \ = \ g^{u,v}(r) \ + \ g^{u,v}(r, \sigma) \ + \ g^{u,v}(r, \theta, \sigma, \delta)$$

where the equation takes the form

$$G^{u,v} \ \approx \ 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta)) \ . \qquad (\dagger)$$

Our perfect star S*, of *arbitrarily* large radius r, is filled *only* with dark energy from the *quantumlike* fluctuations that come and go in an instant of time, defined by the function

$$\xi \ = \ 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta)) \ .$$

Since ξ satisfies Laplace [pp 374-5], the *average* value of ξ over S* is 2σ, and so we can define the energy density [ρ] in S* to be 2σ. And again, since ξ is *intangible* matter, it is reasonable to suppose that the 'equation of state' for S* is

$$2p \ + \ \rho \ = \ 0 \ ,$$

where p is *average* pressure in the star.

Thus, along the *diagonal*, we can populate the *intangible* stress tensor $T^{u,v}$, in the *comoving* frame, with real values for pressure and density, and so calculate the intangible $C^{u,v}$ and $T^{u,v}$ in a *relativistic* setting.

As to our template $g^{u,v}(r, \theta)$, which is used to calculate $C^{u,v}$ and $T^{u,v}$, it is a function of (r, θ) in the *radial* and *time* directions [$u = v$, $u = 1$ or 3], and for $u = v = 2$, which is the θ direction, we can let $g_{u,v}(r)$, in its *covariant* form, be equal to r^2 [p 383 *ff*.].

The reason we can do this for $g_{2,2}$ is because while there are quantumlike fluctuations along any circle C in S*, centered at O; it is the case that for *any* point q on C, the dark energy at that point belongs to the *radial* fluctuation along some line ℓ_θ which intersects C at q (for pictures of *radial* and *angular* fluctuations, see for example, page 240 and pages 346-7).

Now we can solve (†) [for $u = v$, $u = 1$ or 3], and call this solution $g^{u,v}(r, \theta, \sigma, \delta)$. By adding *all* three solutions together, we arrive at $\mathscr{T}$. And again, the first term in $\mathscr{T}$ is the solution when σ = 0, and we are dealing with a perfect star comprised of *tangible* matter, with pressure p and density ρ.

The second term in $\mathscr{T}$ is the solution when we are dealing with a perfect star of *arbitrarily* large radius r, comprised *only* of dark energy, for the singularity at O. And the third term in $\mathscr{T}$ is like the second, but now the singularities are at (±δ, 0). The hope is that *all* three solutions, added together, will give us what we're looking for …

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Recall that in a *mathematical* coordinate system [labelling as (*)], the *coupled* field equations are ...

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha))g^{u,v}(\delta\cos(\alpha)) + 2\cosh(\delta r\sin(\beta))J_0(\delta r\cos(\beta))g^{u,v}(\delta\sin(\beta)) + 2\cosh(\delta r\cos(\phi))J_0(\delta r\sin(\phi))g^{u,v}(\delta\cos(\phi))] .$$

And here, $\cos(\alpha) = \sin(\phi)\cos(\theta)$ and $\sin(\beta) = \sin(\phi)\sin(\theta)$, and the *physical* singularities associated with $\lambda(s) = \sigma / s$, are at the origin O, and at

$$S = \{(\pm\delta, 0, 0), (0, \pm\delta, 0), (0, 0, \pm\delta)\} .$$

As well, $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; and $C^{u,v} = R^{u,v} - \frac{1}{2}Rg^{u,v}$, where $R^{u,v}$ is the Ricci tensor and R the Ricci scalar.

Now since we are going to be working in a *physical* coordinate system [PCS], we'll want to let θ and ϕ trade places, so we'll keep this in mind, going forward, with respect to (*) above.

In an (r, θ, ϕ, t) layout, using PCS, our *total* solution to (*) is

$$\mathcal{T} = g^{u,v}(r) + g^{u,v}(r, \sigma) + g^{u,v}(r, \theta, \phi, \sigma, \delta) ,$$

where $g^{u,v}(r)$ is the solution when $\sigma = 0$; $g^{u,v}(r, \sigma)$ is the solution associated with the singularity at O, and $g^{u,v}(r, \theta, \phi, \sigma, \delta)$ is the solution that corresponds to the singularities associated with S.

For $g^{u,v}(r)$, this is the case of a perfect star [S*], centered at O, and filled with *tangible* matter, with pressure p and density ρ along the diagonal of $T^{u,v}$, in the *comoving* frame. The *interior* / *exterior* Schwarzschild solution applies here, as it's commonly called … subject to a reconsideration of the *time* component [pp 383-460] for the *interior* case.

For $g^{u,v}(r, \sigma)$, this is the solution associated with the singularity at O, in the case of a perfect star [S*] of *arbitrary* radius r, and filled with *intangible* matter, *only*. By intangible, we mean dark energy [ξ], and the form for the equation we are solving is (because ξ is σ at any point in S*) …

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0) .$$

Just as in our previous notes, the *average* value of σ over S* is σ, itself, so that we can define the energy density [ρ] in S* to be σ. And again, since ξ is *intangible* matter, it is reasonable to suppose that the 'equation of state' for S* is

$$3p + \rho = 0 ,$$

where p is pressure in the star.

Thus, along the *diagonal*, we can populate the *intangible* stress tensor $T^{u,v}$, in the *comoving* frame, with real values for pressure and density; and so, calculate the intangible $C^{u,v}$ and $T^{u,v}$ in a

relativistic setting, using a *generic* function $g^{u,v}(r)$ as the *template* for our calculations, for the *radial* and *time* components.

As to the inner block [r^2, $r^2 \sin^2\theta$] in the *g*-matrix, as shown below in *covariant* form, we leave it as is, so that our solution $g^{u,v}(r, \sigma)$ *only* applies to the first and last entries in *g*.

$$\begin{bmatrix} g_{rr} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -g_{tt} \end{bmatrix}$$

We then *add* to to the Schwarzschild solution $g^{u,v}(r)$ in $\mathcal{T}$, and shown below, the solution $g^{u,v}(r, \sigma)$ for our dark energy singularity at O, in the *radial* and *time* components, only. The inner block is always left alone.

$$\mathcal{T} = g^{u,v}(r) + g^{u,v}(r, \sigma) + g^{u,v}(r, \theta, \phi, \sigma, \delta) .$$

Now we're on to the quantumlike components, the first of which is associated with ($\pm\delta$, 0, 0) in S, and shown below …

$$\xi = 2\sigma \cdot \cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha)) .$$

Our perfect star [S*], of *arbitrary* radius r, is filled *only* with ξ, and because ξ satisfies Laplace [pp 374-5], we can populate the *intangible* $T^{u,v}$, in the *comoving* frame, with pressure *p* and density ρ along the *diagonal*; where $\rho = 2\sigma$, and *average* pressure in the star can be calculated from our 'equation of state'

$$3p + \rho = 0 .$$

We now go ahead and solve the equation below, for the *radial* and *time* components only,

$$G^{u,v} \approx 2\sigma \cdot \cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha))g^{u,v}(\delta\cos(\alpha))$$

by first using a *generic* template $\tau = g^{u,v}(r, \theta, \phi)$ to calculate $C^{u,v}$ and $T^{u,v}$, in a *relativistic* setting. And here, it should be clear that the properties of τ will be driven by ξ, and thus, we should probably assume τ is some kind of Fourier series expansion, at a minimum.

The solution to the equation will be of the form $g^{u,v}(r, \theta, \phi, \sigma, \delta)$, which is then added to the *two* previous solutions that form part of $\mathcal{T}$, as shown below, again in the *radial* and *time* directions …

$$\mathcal{T} = g^{u,v}(r) + g^{u,v}(r, \sigma) + g^{u,v}(r, \theta, \phi, \sigma, \delta) .$$

The inner block [r^2, $r^2 \sin^2\theta$] is always left alone.

The solution to the next quantumlike component, that is associated with $(0, \pm\delta, 0)$ in S, needn't be calculated from the equation below, in my opinion …

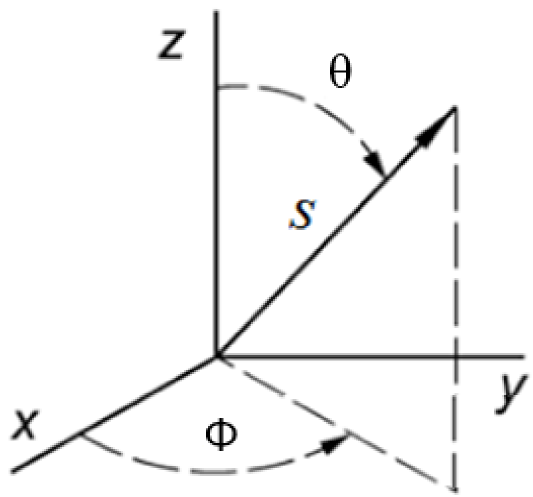
$$G^{u,v} \approx 2\sigma \cdot \cosh(\delta r\sin(\beta))J_0(\delta r\cos(\beta))g^{u,v}(\delta\sin(\beta)) .$$

For because of symmetry, it should be nothing more than a 90 degree rotation of the previous solution above, in the *x*-*y* plane. Again, this solution is added to the other *three* we've already discussed above, in the *radial* and *time* directions only.

That only leaves us with the *last* piece associated with $(0, 0, \pm\delta)$ in S, and in a PCS setting, the equation is to be written as

$$G^{u,v} \approx 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta)) .$$

The solution will be of the form $g^{u,v}(r, \theta, \sigma, \delta)$, and again, the *radial* and *time* components are to be added to the other *four* solutions we've already discussed here. We're still working in three dimensions, but in this case, the dark energy component does not depend on ϕ – a manifestation of our coordinate system, actually … …



Physical Coordinate System

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XI

When solving the equation below, in the *intangible* space, for the *radial* and *time* directions, where $\sigma > 0$

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0) , \qquad (1)$$

so that our perfect star S*, of *arbitrary* radius r, is filled with dark energy *only*; it is perfectly fine to use a template like $g^{u,v}(r)$ to calculate the intangible $C^{u,v}$ and $T^{u,v}$, and thus find a solution to (1); but, the solution $g^{u,v}(r, \sigma)$ itself, must be such that it is *zero* when $\sigma = 0$.

We have demonstrated this idea in different ways in Part X of these research notes, yet here's another approach. Suppose we consider the following equation …

$$G^{u,v} \approx -\sigma \cdot g^{u,v}(0) . \qquad (2)$$

The solution to (2) will be $g^{u,v}(r, -\sigma)$, and we can consider this to be a form of *anti-gravity*, equal and opposite to the solution for (1), which is $g^{u,v}(r, \sigma)$. Thus, *adding* the two solutions should produce *zero* – gravity, in the *intangible* space, would disappear altogether; which is the *same* thing as saying the dark energy singularity at the origin O no longer exists, so that $\sigma = 0$. Thus we may write, in the *intangible* space

$$g^{u,v}(r, \sigma) + g^{u,v}(r, -\sigma) = 0 = g^{u,v}(r, 0) ,$$

which is consistent with our *associative* principle under addition for σ [p 482].

And the same logic applies to other quantumlike components, associated with dark energy singularities, in the *intangible* space. It is perfectly fine to use a template like $\tau = g^{u,v}(r, \theta, \phi)$ to solve the equation below, for the *radial* and *time* directions

$$G^{u,v} \approx 2\sigma \cdot \cosh(\delta r \sin(\beta)) J_0(\delta r \cos(\beta)) g^{u,v}(\delta \sin(\beta)) , \qquad (3)$$

but the solution $g^{u,v}(r, \theta, \phi, \sigma, \delta)$ must always be such that it is *zero* if $\sigma = 0$, in the intangible space.

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And as we said earlier, we could *also* try templates like $\sigma \cdot g^{u,v}(r)$ or $\sigma \cdot g^{u,v}(r, \theta, \phi)$ when solving (1) or (3) above, respectively [p 485], but we won't know what works and what doesn't, until we finally get a solution on paper.

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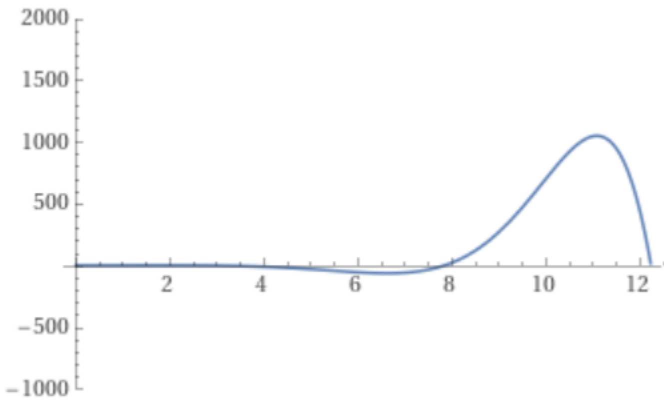
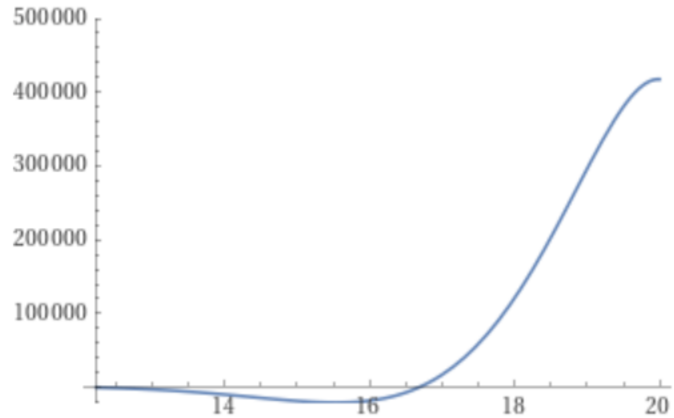
Here are some pictures of the [quantumlike] *radial* fluctuations for a *two*-dimensional star, centered at O, with dark energy singularities at O and at {(±1, 0), (0, ±1)}. It is these fluctuations, in the *intangible* space (dark energy), that are responsible for the *expansion* and *contraction* of the universe. Notice that the fluctuations are both *positive* and *negative*, and that here σ has been normalized to 1 [see also pp 346-7].





These particular pictures show the fluctuations along the line ℓ_θ , where θ is 45 degrees. Note that in the second picture, it is a continuation of the first, using different scaling. In reality, the two graphs, merged together under the *same* scaling, would form a *smooth* picture, because the function that portrays these fluctuations is *continuously* differentiable.

Examining The Equation $G^{u,v} \approx \sigma \cdot g^{u,v}(0)$ Again When $\lambda(s) \approx \sigma / s$

Let us re-examine the equation below, where a dark energy singularity exists at the origin O,

$$G^{u,v} \approx \sigma \cdot g^{u,v}(0) \qquad (*)$$

and consider the case where our perfect star [S*], is centered at O and filled with both *tangible* and *intangible* matter. The latter is dark energy [σ], while the former is some *perfect* fluid, typically associated with *pressure* and *density*.

If we now set σ = 0, then dark energy ceases to exist, and the equation above becomes $G^{u,v} \approx 0$; that is to say,

$$C^{u,v} - kT^{u,v} \approx 0\,,$$

where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; *k* is the Einstein proportionality constant, and $C^{u,v} = R^{u,v} - ½\, R\, g^{u,v}$, where $R^{u,v}$ is the Ricci tensor and R the Ricci scalar. Notice that in this case, the *tangible* $C^{u,v}$, $T^{u,v}$ *do*, in fact, exist.

The solution to the equation above is vintage Schwarzschild, subject to a reconsideration of the *time* component, in the *interior* case [pp 383-460]. The *exterior* solution stays, as is.

Now let's do the opposite, by getting rid of the *tangible* matter altogether and reconsider (*) above, and shown again below …

$$C^{u,v} - kT^{u,v} \approx \sigma \cdot g^{u,v}(0) . \quad (\dagger)$$

In this case, all that remains is *intangible* matter [σ], so that our perfect star [S*] is filled with dark energy [σ] *only*, whose source is the singularity at O.

There are two cases to consider; either *no* such *intangible* stress tensor $T^{u,v}$ exists here, or it does. If it doesn't, we calculate the intangible $C^{u,v}$ using a generic template like $g^{u,v}(r)$, or even $\sigma \cdot g^{u,v}(r)$, and solve the equation below, for the *radial* and *time* directions [see also pp 480-91] …

$$C^{u,v} \approx \sigma \cdot g^{u,v}(0) .$$

If the intangible $T^{u,v}$ *does* exist, we can calculate it using the methods outlined on pages 480-91, and then proceed to solve (†), subject [perhaps] to a reinterpretation of the constant *k*, since this constant was really developed by Einstein when dealing with *tangible* matter.

Either way, we arrive at a total solution (through addition); that is to say

$$\mathcal{G} = g^{u,v}(r) + g^{u,v}(r, \sigma) ,$$

where $g^{u,v}(r)$ is vintage Schwarzschild in the *tangible* case, and $g^{u,v}(r, \sigma)$ is the *intangible* solution, subject to the condition that $g^{u,v}(r, \sigma) = 0$ if $\sigma = 0$.

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If we don't like the idea of splitting S* up into *tangible* and *intangible* matter, as we did above, then (†) can be solved by seeing the *left*-hand side of (†) as purely *tangible*, even though a dark energy singularity exists at O, as reflected in the *right*-hand side of (†). This is probably the easiest interpretation to deal with, and in fact, the constant *k* almost *forces* us to adopt this view of things.

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Loosely speaking, (†) would then be read as 'the remainder, after subtracting out *tangible* matter from the space-time fabric.'

Thus, when calculating $T^{u,v}$ in the *comoving* frame in (†), we would simply diagonalize it with pressure *p* and density ρ, from the perfect star filled with *tangible* matter.

Similar comments apply when there are singularities associated with the [dark energy] *quantumlike* components, but the concepts above still hold. So if, for example, we were looking at the following equation, with singularities at O and at (±δ, 0) …

$$C^{u,v} - kT^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] , \quad (\ddagger)$$

and decided to interpret the *left*-hand side of (‡) as purely *tangible*; then whether we were looking at the *first* or *second* component on the *right*-hand side of (‡), we would always diagonalize $T^{u,v}$ in the *comoving* frame with pressure *p* and density ρ, from the *perfect* star filled with *tangible* matter.

However the issue of templates for $g^{u,v}$ when calculating the *relativistic* $C^{u,v}$, $T^{u,v}$ in (‡), and shown below, remains …

$$C^{u,v} - kT^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] . \quad (‡)$$

For the *first* component $\sigma \cdot g^{u,v}(0)$, on the *right*-hand side, a generic template such as $g^{u,v}(r)$ is sufficient for the *radial* and *time* directions, since $g^{u,v}(0)$ does *not* depend on θ.

For the *second* component; namely

$$2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta)) ,$$

the generic template changes to $g^{u,v}(r, \theta)$, again in the *radial* and *time* directions. It is difficult to see how we could combine the two calculations into a *single* operation, in which case they would have to be done separately, and the solutions then *added* together.

Thus, in the *radial* and *time* directions, we would be solving

$$C^{u,v} - kT^{u,v} \approx \sigma \cdot g^{u,v}(0) , \quad (†)$$

with a solution $g^{u,v}(r, \sigma)$, and we would be solving

$$C^{u,v} - kT^{u,v} \approx 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta)) , \quad (‡)$$

with a solution $g^{u,v}(r, \theta, \sigma, \delta)$, again in the *radial* and *time* directions. In *both* cases, however, the *left*-hand sides of (†) and (‡) would be seen as *purely* tangible, so that we would always diagonalize $T^{u,v}$ in the *comoving* frame with pressure *p* and density ρ, from the *perfect* star filled with *tangible* matter …

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‘there is no such thing as the perfect research paper … unless, of course, all the pages are blank’

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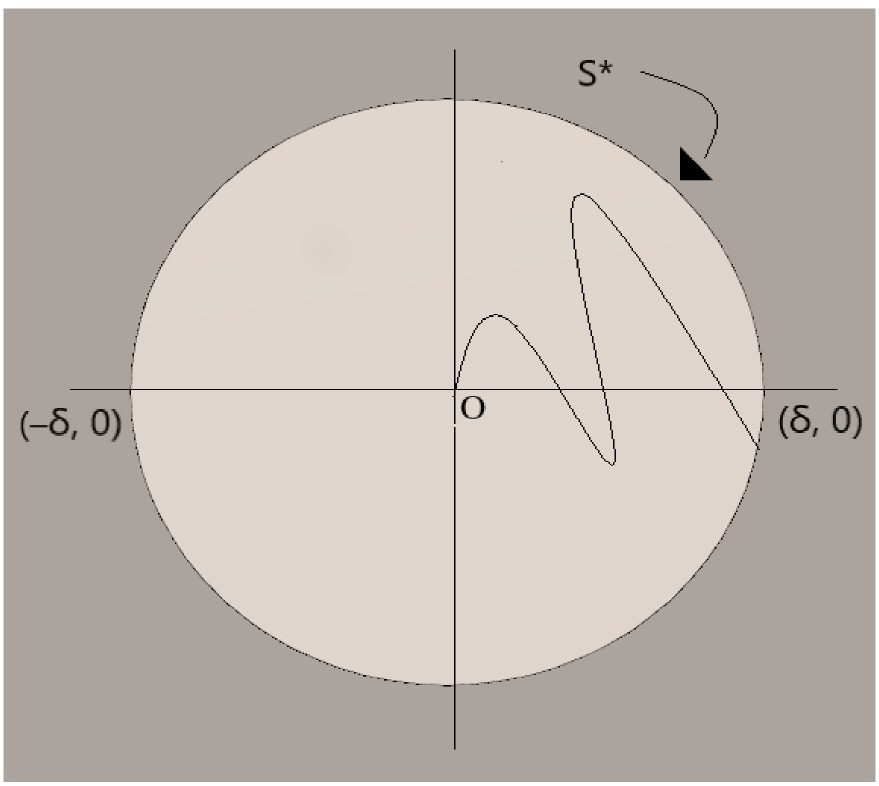
In the picture below, is a perfect, *two*-dimensional star [S*], filled with both *tangible* matter and *intangible* matter (outside the star is dark energy, itself). The *tangible* matter (perfect fluid) is to be associated with pressure p and density ρ.

The *intangible* matter is dark energy, associated with the singularities at O and at $(\pm\delta, 0)$; and is made up of σ from the *first* term on the *right*-hand side of (‡) below, and also the *quantum* fluctuations [ξ], shown as an oscillating line in the picture.

$$C^{u,v} - kT^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] \qquad (‡)$$

$$\xi = 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))$$

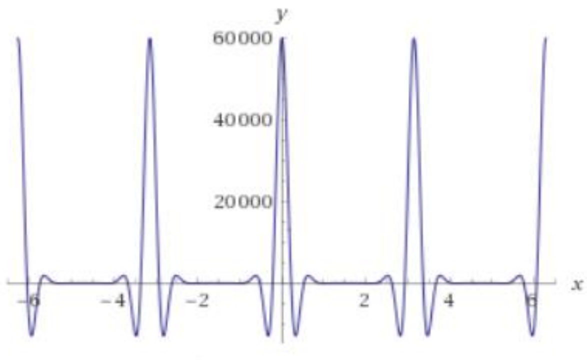
When calculating the *relativistic* stress tensor $T^{u,v}$, we would always *diagonalize* $T^{u,v}$ in the *comoving* frame with pressure p and density ρ, for *either* component on the right-hand side of (‡), as per our previous remarks, *if* we interpret (‡) as ‘the remainder, after subtracting out *tangible* matter from the space-time fabric.’ This is an important notion, when solving (‡) for $g^{u,v}$ in the *radial* and *time* directions …



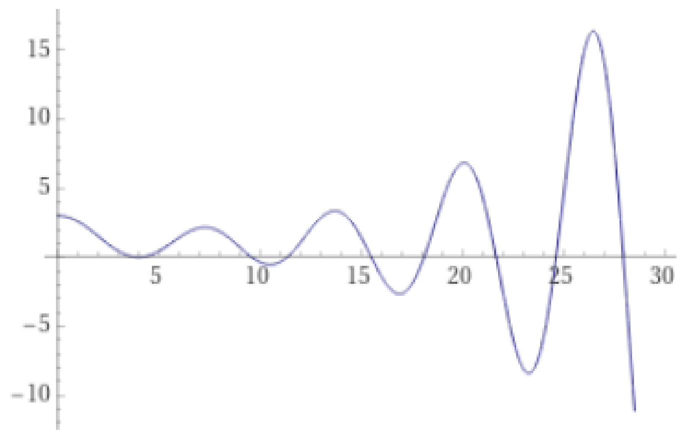
Some comments on page 493 were rewritten in this release, to reflect more accurately, what I was trying to say at the time …

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XII

Up until now, we've spoken mainly about solving the *coupled* equations in the *radial* and *time* directions, but there are *also* quantumlike fluctuations in the *angular* directions of θ and ϕ, that we may wish to incorporate into our modelling down the road [see, for example, pp 272-3].



r = 11 , θ = *x* , 2D star , *angular*



r = *x*-axis , θ = 80° , *radial*

In the pictures above (σ = 1), for a *two*-dimensional star, with singularities at the origin O, and at (±1, 0), the diagram on the *left* shows the *angular* fluctuations along the circle of radius r = 11. The diagram on the *right* shows the *radial* fluctuation emanating from the origin O, at an angle of 80 degrees to the *x*-axis. Notice that the radial fluctuation *grows* in amplitude, as r increases, and that the *angular* fluctuations are really a *cross-sectional* view of the *radial* fluctuations, for a given radius r.

If we now bring back our *coupled* field equations below, that reflect this scenario, where δ = 1; then

$$C^{u,v} - kT^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] \qquad (\ddagger)$$

we see that in an (r, θ, t) layout, there may be justification for including *u* = *v* = 2 (the θ term) in our calculations, when solving (‡). In doing so, it could lead to a better understanding of how $g^{u,v}$ behaves, particularly in the *primary* direction of θ, itself. Indeed, since these are *quantumlike* fluctuations that come and go in an instant of time, perceiving them through $g^{u,v}$ in the *angular* sense, may be just as valuable as perceiving them through $g^{u,v}$ in the *radial* sense.

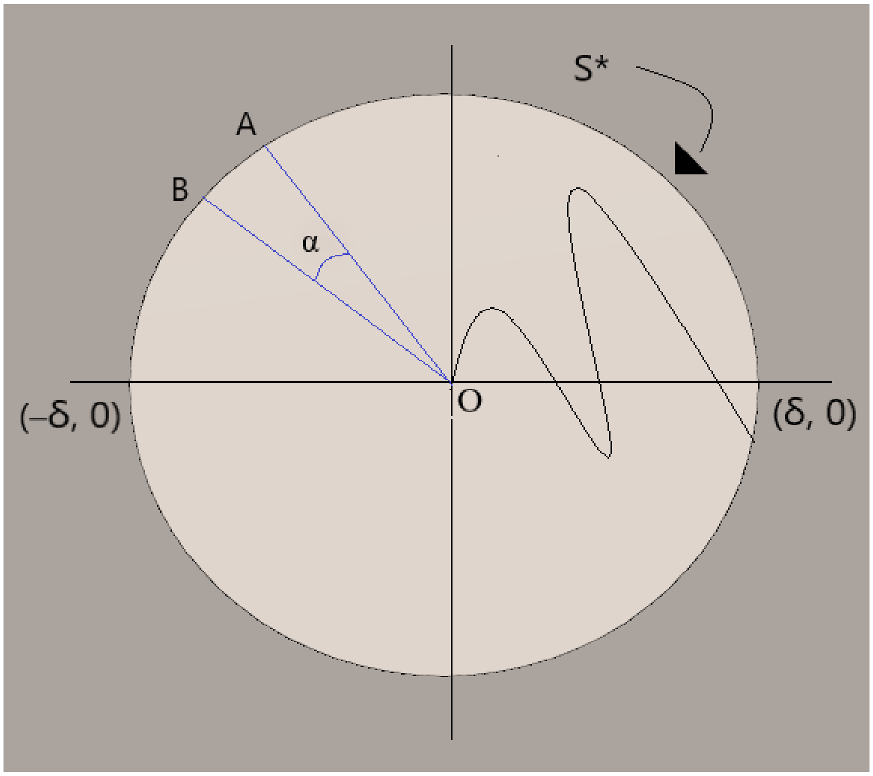
Similar remarks apply for the *three*-dimensional star; but in this case we would include *u* = *v* = 2 or 3 (the θ and ϕ terms) in our calculations, when solving the *coupled* equations, using a *physical* coordinate system in an (r, θ, ϕ, t) layout [see page 488 for the *notational* form for these equations, where there, they are written using *mathematical* coordinates].

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XIII

We'd now like to offer up an interpretation of the ds^2 metric for the solution to our equation shown below, with singularities at the origin O, and at $(\pm\delta, 0)$. We'll assume our *two*-dimensional *perfect* star [S*], is filled with both *tangible* matter (perfect fluid) and *intangible* matter (dark energy).

$$C^{u,v} - kT^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] \qquad (\ddagger)$$

For the *first* component on the right-hand side of $(\ddagger)$; namely $\sigma \cdot g^{u,v}(0)$, the solution $\tau = g^{u,v}(r, \sigma)$ does *not* depend on θ, so that only the *radial* and *time* components in ds^2 will be affected by τ.



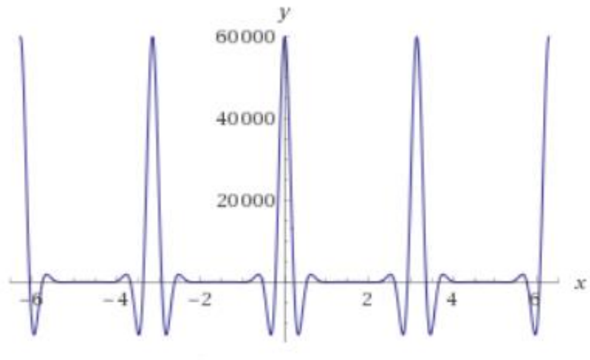
Thus, if relative to O, I were to traverse the circle [C] in the diagram above, I would see *no* change to *distance* or *time* on C, but I would along any *radial* line ℓ_θ emanating from O. And so in τ, it is acceptable to let $g_{u,v}(r)$, in its *covariant* form, be equal to r^2, where $u = v = 2$.

For the *second* component on the right-hand side of $(\ddagger)$; namely, the *quantumlike* expression

$$\xi = 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta)) ,$$

the solution $\tau = g^{u,v}(r, \theta, \sigma, \delta)$ *does* depend on θ, so as in the *first* case above, I will see a change to *distance* or *time* along any radial line ℓ_θ emanating from O; but *also* now, along C, relative to O.

Thus, my perception of the *distance* OA may be *different* than the *distance* OB, and my perception of *time* at A may be *different* than *time* at B. And furthermore, because there are *quantumlike* fluctuations along C, as shown in the picture below [p 496], it's quite possible that the angle α subtended by the arc AB will *change*, relative to O, as I traverse C (in other words, α might be subtended by a *different* arc $A'B'$ that is not equivalent to AB, as I make the traversal).



r = 11 , $\theta = x$, 2D star , *angular*

If this is so, then as we said earlier – in an (r, θ, t) layout, there may be justification for including the component $u = v = 2$ (the θ term) in our calculations for ξ, when solving (‡); as opposed to letting $g^{u,v}$ default to r^2 in τ, in *covariant* form, in this case.

$$C^{u,v} - kT^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] \qquad (‡)$$

And finally, similar remarks apply in *three* dimensions for the θ and ϕ terms in the *g*-matrix, but again, I wish to remind the reader that this is simply an interpretation of (‡) above.

It should be mentioned as well, that we expect σ to by *very* small – perhaps on the order of the cosmological constant, so the kind of discrepancies we are talking about here, very likely wouldn't show up, except at *very large* distances from O.

On the other hand, if $\sigma = 0$, this is vintage Schwarzschild, which is part of the *total* solution $\mathscr{T}$ to (‡), and here, it is well-known that discrepancies in the *radial* and *time* directions are measurable.

Indeed, the *total* solution $\mathscr{T}$, as per our earlier research notes, may be written as …

$$\mathscr{T} = g^{u,v}(r) + g^{u,v}(r, \sigma) + g^{u,v}(r, \theta, \sigma, \delta) ,$$

where the *first* term on the *right*-hand side is Schwarzschild, subject (perhaps) to a reconsideration of the *time* component for the *interior* case [pp 383-460].

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XIV

Recall that in a *physical* coordinate system [labelling as (*)], the *coupled* field equations are ...

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha))g^{u,v}(\delta\cos(\alpha)) + 2\cosh(\delta r\sin(\beta))J_0(\delta r\cos(\beta))g^{u,v}(\delta\sin(\beta)) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] \,.$$

And here, $\cos(\alpha) = \sin(\theta)\cos(\phi)$ and $\sin(\beta) = \sin(\theta)\sin(\phi)$, and the *physical* singularities associated with the *underlying* dark energy density function $\lambda(s) = \sigma / s$, are at the origin O, and at

$$S = \{(\pm\delta, 0, 0), (0, \pm\delta, 0), (0, 0, \pm\delta)\} \,.$$

As well, $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; and $C^{u,v} = R^{u,v} - ½Rg^{u,v}$, where $R^{u,v}$ is the Ricci tensor and R the Ricci scalar.

Now suppose, in our *g*-matrix, and shown below in *covariant* form (again, using *physical* coordinates),

$$\begin{bmatrix} g_{rr} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -g_{tt} \end{bmatrix}$$

we decide to include $u = v = 2$ or 3 in our calculations, when solving (*) for the *three quantumlike* components; so that for the time being, we are *excluding* the term $\sigma \cdot g^{u,v}(0)$ on the *right*-hand side of (*).

We'll assume our *three*-dimensional *perfect* star [S*], is filled with both *tangible* matter (perfect fluid) and *intangible* matter (dark energy), and that the *left*-hand side of (*) is to be interpreted as 'the remainder, after subtracting out *tangible* matter from the space-time fabric.'

Thus, when calculating $T^{u,v}$ in the *comoving* frame in (*), we would simply *diagonalize* it with pressure *p* and density ρ, from the perfect star filled with *tangible* matter.

Our solution $\tau = g^{u,v}(r, \theta, \phi, \sigma, \delta)$, however we arrive at it [component by component, or 'in one go'] now applies to *all four entries* in the *g*-matrix, so that the *default* inner block [r^2, $r^2 \sin^2 \theta$] is now replaced.

We don't know what this solution looks like [other than it will, *most* certainly, be based on *harmonics* of one type or another]; but *if* we adhere to our *associative* principle under *addition*,

relative to σ, then as $\sigma \dashrightarrow 0$, we should expect $\tau \dashrightarrow 0$ as well, in the *radial* and *time* directions; that is to say, $u = v = 1$ or 4. Let's assume this is so.

The thorny issue of what happens to the *inner* block in τ, as $\sigma \dashrightarrow 0$ [$u = v = 2$ or 3], can't be known until we do get a solution on paper, but clearly τ does *not* tend to *zero* here. A more reasonable guess is that τ reverts to the default inner block [r^2, $r^2 \sin^2 \theta$], but this is just a hunch, on my part.

To solve for the *first* component on the *right*-hand side of (*) above; that is to say,

$$C^{u,v} - kT^{u,v} \approx \sigma \cdot g^{u,v}(0)\,, \qquad (\dagger)$$

we can do it *directly*, again interpreting the *left* side of (†) as 'the remainder, after subtracting out *tangible* matter from the space-time fabric'. Our solution τ, in this case, will now be of the form $g^{u,v}(r, \sigma)$, since it *doesn't* depend on θ *or* ϕ; and here we can let the *inner* block in τ be equal to the default, which is [r^2, $r^2 \sin^2 \theta$], in *covariant* form, using *physical* coordinates.

Thus, τ becomes a solution in the *radial* and *time* directions, and again, we expect that as $\sigma \dashrightarrow 0$, it will be the case that $\tau \dashrightarrow 0$ also, according to our *associative* principle, when $u = v = 1$ or 4.

We can *also* take our *first* solution $g^{u,v}(r, \theta, \phi, \sigma, \delta)$, for the *three quantumlike* components, and let the parameter $\delta \dashrightarrow 0$ here. The solution should then reduce to $g^{u,v}(r, 6\sigma)$, since the *strength* of the singularity at the origin O, has now *increased* six-fold. By mapping 6σ to σ, we would, in theory, obtain a solution to (†); *but* the thorny issue of what happens to the *inner block* for our *first* solution $g^{u,v}(r, \theta, \phi, \sigma, \delta)$, as the parameter $\delta \dashrightarrow 0$, remains.

The *total* solution to (*) [p 499] is now of the form

$$\mathcal{T} = g^{u,v}(r) + g^{u,v}(r, \sigma) + g^{u,v}(r, \theta, \phi, \sigma, \delta)\,,$$

where $g^{u,v}(r)$ is *vintage* Schwarzschild [$\sigma = 0$], subject to a reconsideration of the *time* component in the *interior* case [pp 383-460]. And let us remember too, that when performing the addition, only terms in $g^{u,v}(r, \sigma)$ or $g^{u,v}(r, \theta, \phi, \sigma, \delta)$ that actually contain the parameter σ, or are *uniquely different* than the *default* inner block, are to be added to $g^{u,v}(r)$ in $\mathcal{T}$ above [we're mixing *covariant* and *contravariant* forms here, but clearly when the work is actually done, we'll have to settle on one or the other].

So if, for example, the *inner* block of $g^{u,v}(r, \sigma)$ was [r^2, $r^2 \sin^2 \theta$], in *covariant* form, it would be excluded from the addition, altogether. And if the *inner* block of $g^{u,v}(r, \theta, \phi, \sigma, \delta)$ contained terms *attached* to σ, like for example, the expression $(1 + \sigma) \cdot r^2$, in *covariant* form, when $u = v = 2$; only the piece $\sigma \cdot r^2$ would be included in the addition. In other words, terms that are *uniquely different* than the *default* inner block, as we said above.

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XV

Let us bring back our form for the *coupled* field equations, with a dark energy singularity at the origin O, of our *physical* coordinate system, in an (r, θ, ϕ, t) layout …

$$C^{u,v} - kT^{u,v} \approx \sigma \cdot g^{u,v}(0) . \quad (\dagger)$$

If we decide to interpret the *left*-hand side of (†) as 'the remainder, after subtracting out *tangible* matter from the space-time fabric', then in the case where σ = 0, the *right*-hand side above vanishes, and there is *no* such remainder at all. The *coupling* is broken, and (†) reduces to

$$C^{u,v} - kT^{u,v} \approx 0 , \quad (*)$$

so that the only *internal* source for the gravitational tensor is the perfect star [S*] itself, filled with *tangible* matter (typically, a perfect fluid). The corresponding *internal* solution to (*) is vintage Schwarzschild [interior / exterior], subject to a reconsideration of the *time* component, *inside* S* [pp 383-460].

On the other hand, if we add in a dark energy singularity at O, then (†) holds, and the *right*-hand side of (†) *becomes* the remainder, after subtracting out *tangible* matter from the space-time fabric. The coupling is *restored*, and the *right*-hand side of (†) thus becomes the only *external* source that has an influence on $g^{u,v}$.

The corresponding *external* solution to (†) is $\tau = g^{u,v}(r, \sigma)$ in the *radial* and *time* directions, where the *inner* block $[r^2, r^2 \sin^2 \theta]$, in *covariant* form in τ, is the default.

So the question, philosophically speaking, is whether or not the *internal* solution [Schwarzschild] is a subset of τ, which can be extracted from τ by letting σ ---› 0, given our interpretation of (†).

Remember, we have already dealt with the *internal* solution by setting σ = 0, so that (†) can be seen as 'that solution τ which exists and is driven by the *right*-hand side of (†), after subtracting out tangible matter from the space-time fabric'. Such an *external* solution [τ], should *not* contain the *internal* solution [Schwarzschild], on philosophical grounds alone, in my opinion …

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Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XVI

Let us bring back the *covariant* form for our equation shown below, with singularities at the origin O, and at $(\pm\delta, 0)$. We'll assume our *two*-dimensional *perfect* star [S*], is filled with both *tangible* matter (perfect fluid) and *intangible* matter (dark energy).

$$C_{u,v} - kT_{u,v} \approx \sigma[g_{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g_{u,v}(\delta\cos(\theta))] \qquad (\ddagger)$$

What we want to do here, is design in more detail, a template that may be used to solve (‡) above, so let's begin by interpreting the *left*-hand side as 'the remainder, after subtracting out *tangible* matter from the space-time fabric'. Thus, when calculating $T_{u,v}$ in the *comoving* frame, we would simply *diagonalize* it with pressure p and density ρ, from the perfect star filled with *tangible* matter.

Next, we'll define r to be δr, so that our equation above may be written as …

$$C_{u,v} - kT_{u,v} \approx \sigma[g_{u,v}(0) + 2\cosh(r\cos(\theta))J_0(r\sin(\theta))g_{u,v}(\delta\cos(\theta))] , \qquad (*)$$

and we'll begin by looking at the *second* component on the *right*-hand side, in the expression (*) above.

Now from page 349, we know that

$$2\sigma \cdot \cosh(r\cos(\theta))J_0(r\sin(\theta)) = 2\sigma \cdot \Sigma\, r^n P_n(\cos(\theta)) / \Gamma(n + 1) , n = 0, 2, 4 \ldots$$

where P_n is a Legendre polynomial; so that for the *second* component, we may write

$$C_{u,v} - kT_{u,v} \approx \{2\sigma \cdot \Sigma\, r^n P_n(\cos(\theta)) / \Gamma(n + 1)\}\, g_{u,v}(\delta\cos(\theta)) , \qquad (\dagger)$$

where the sum is over all *even* n = 0, 2, 4, … and so on.

Now the first thing to notice about (†), is that the summation on the *right*-hand side *separates* the variables r and θ. Also notice that the term δcos(θ), associated with $g_{u,v}$, is actually a *radial* measure that does *not* depend on r [p 461 *ff*.]. So in fact, the *right*-hand side of (†) is actually a *complete* separation of the variables r and θ.

Thus, if I want a *template* Ʈ for $g_{u,v}$, from which I can calculate the *left*-hand side of (†), I should write Ʈ in such a way that it reflects the *right*-hand side of (†). For example, I might write for Ʈ,

$$\Sigma\, c_n(u, v)\, r^n P_n(\cos(\theta)) , \qquad n = 0, 1, 2, 3, \ldots$$

where the $c_n(u, v)$ are coefficients we are solving for, in *each* line of (†); that is to say, where it is the case that $u = v = 1, 2, 3$. Such a solution might be obtained by *matching* powers of r on *both* sides of (†), for example; but regardless, Ʈ is just one of many templates we might design for $g_{u,v}$.

Now as to $g_{u,v}(\delta\cos(\theta))$ in (†), since the argument here is a *radial* measure, we would calculate $g_{u,v}(\delta\cos(\theta))$ by letting $r = \delta\cos(\theta)$ in Ƭ. Thus, (†) becomes, remembering that $r = \delta r$ …

$$C_{u,v} - kT_{u,v} \approx \{2\sigma \cdot \Sigma\, (\delta r)^n P_n(\cos(\theta)) / \Gamma(n+1)\} \cdot \{\Sigma\, c_n(u, v)\, (\delta\cos(\theta))^n P_n(\cos(\theta))\}\,, \quad (\sim)$$

where the *second* summation above corresponds to $g_{u,v}(\delta\cos(\theta))$.

Letting $\delta \dashrightarrow 0$, we see that because $P_0(\cos(\theta))$ is always 1, the *first* bracketed expression above is 2σ, while the *second* bracketed expression above is $c_0(u, v)$, which is equal to $g_{u,v}(0)$. And so, (~) reduces to

$$C_{u,v} - kT_{u,v} \approx 2\sigma \cdot g_{u,v}(0)\,,$$

which is what we would expect, when looking at (‡) on the previous page.

The template Ƭ, and shown below,

$$\Sigma\, c_n(u, v)\, (\delta r)^n P_n(\cos(\theta))\,, \quad n = 0, 1, 2, 3, \ldots$$

can now be used to calculate the *left*-hand side of (~), and perhaps by *matching* powers of r on *both* sides of (~), we may be able to solve (~) for $u = v = 1, 2, 3$.

The solution τ to (~) will be of the form $g_{u,v}(r, \theta, \sigma, \delta)$, subject to the condition that as $\sigma \dashrightarrow 0$, τ will tend to *zero* also, in the *radial* and *time* components ($u = v = 1$ or 3). And as $\sigma \dashrightarrow 0$, τ should tend to r^2 for $u = v = 2$; a requirement we can probably enforce through the template Ƭ.

As to the *first* component on the *right*-hand side of (‡), from the previous page, and shown below, the template Ƭ for $g_{u,v}$ is much simpler because there is *no* dependency on θ. And here we can let Ƭ be some *generic* function of $g_{u,v}(r)$, when $u = v = 1$ or 3. For the *inner* block ($u = v = 2$), we let $g_{u,v}(r)$ be equal to r^2, which is the default.

$$C_{u,v} - kT_{u,v} \approx \sigma \cdot g_{u,v}(0) \quad (\wedge)$$

The solution τ to (^) will be of the form $g_{u,v}(r, \sigma)$, subject to the condition that as $\sigma \dashrightarrow 0$, τ will tend to *zero* also, in the *radial* and *time* components ($u = v = 1$ or 3). And as we said above, $g_{u,v}(r)$ is equal to r^2 in τ, by default, if $u = v = 2$.

The *total* solution to (‡) [p 502] is now of the form

$$\mathscr{T} = g_{u,v}(r) + g_{u,v}(r, \sigma) + g_{u,v}(r, \theta, \sigma, \delta)\,,$$

where $g_{u,v}(r)$ is *vintage* Schwarzschild [$\sigma = 0$], subject to a reconsideration of the *time* component in the *interior* case [pp 383-460]. We only add to the *inner* block if the entry here is *uniquely* different than r^2. This will apply to $g_{u,v}(r, \theta, \sigma, \delta)$, but *not* $g_{u,v}(r, \sigma)$ in $\mathscr{T}$ above.

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XVII

Scaling up to *three* dimensions, from our work in Part XVI, presents no great theoretical difficulty. Now recall that in a *physical* coordinate system [labelling as (*)], the *coupled* field equations are ...

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha))g^{u,v}(\delta\cos(\alpha)) + 2\cosh(\delta r\sin(\beta))J_0(\delta r\cos(\beta))g^{u,v}(\delta\sin(\beta)) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] .$$

And here, $\cos(\alpha) = \sin(\theta)\cos(\phi)$ and $\sin(\beta) = \sin(\theta)\sin(\phi)$, and the *physical* singularities associated with the *underlying* dark energy density function $\lambda(s) = \sigma / s$, are at the origin O, and at

$$S = \{(\pm\delta, 0, 0), (0, \pm\delta, 0), (0, 0, \pm\delta)\} .$$

As well, $G^{u,v} = C^{u,v} - kT^{u,v}$, where $C^{u,v}$, $T^{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; and $C^{u,v} = R^{u,v} - ½Rg^{u,v}$, where $R^{u,v}$ is the Ricci tensor and R the Ricci scalar.

Using the *same* approach, as in the previous research note, we find for the *quantumlike* components in (*), that

$$C_{u,v} - kT_{u,v} \approx \{2\sigma \cdot \Sigma\, (\delta r)^n P_n(\omega) / \Gamma(n+1)\} \cdot \{\Sigma\, c_n(u, v)\, (\delta\omega)^n P_n(\omega)\} , \quad (\sim)$$

where ω is one of $\cos(\alpha)$, $\sin(\beta)$ or $\cos(\theta)$; and $g_{u,v}(\delta\omega)$ is the *second* bracketed expression on the *right*-hand side of (~) above. The template Ꚍ for $g_{u,v}$ is (where $r = \delta r$)

$$\Sigma\, c_n(u, v)\, r^n P_n(\omega) , \quad n = 0, 1, 2, 3, \ldots$$

and as δ ---› 0, we see that (~) becomes

$$C_{u,v} - kT_{u,v} \approx 2\sigma \cdot g_{u,v}(0) .$$

Thus, $c_0(u, v)$ is equal to $g_{u,v}(0)$, no matter the choice of ω, but in general, the coefficients $c_n(u, v)$ will depend on the choice of ω.

The solution τ to (~) for *each* component will be of the form $g_{u,v}(r, \theta, \phi, \sigma, \delta)$, subject to the condition that as σ ---› 0, τ will tend to *zero* also, in the *radial* and *time* directions ($u = v = 1$ or 4). And as σ ---› 0, τ should tend to $[r^2, r^2 \sin^2\theta]$ for $u = v = 2$ or 3; again, a requirement we can probably enforce through the template Ꚍ. We'll label these solutions $\tau_{\cos(\alpha)}$, $\tau_{\sin(\beta)}$ and $\tau_{\cos(\theta)}$, respectively, for the *quantumlike* components.

And notice again, as in previous notes, that the solution for the *second* component [$\omega = \sin(\beta)$] is probably just a 90 degree rotation of the solution for the *first* component [$\omega = \cos(\alpha)$], in the *x-y* plane; and similarly for the *third* component [$\omega = \cos(\theta)$], where the *first* component is rotated by 90 degrees in the *x-z* plane. The *third* component does *not* depend on ϕ, now.

As to the *first* component on the *right*-hand side of (*), from the previous page, and shown below, the template Ƭ for $g_{u,v}$ is much simpler because there is *no* dependency on θ or ϕ. And here we can let Ƭ be some *generic* function of $g_{u,v}(r)$, when $u = v = 1$ or 4. For the *inner* block ($u = v = 2$ or 3), we let $g_{u,v}(r)$ be equal to $[r^2, r^2 \sin^2\theta]$, which is the default.

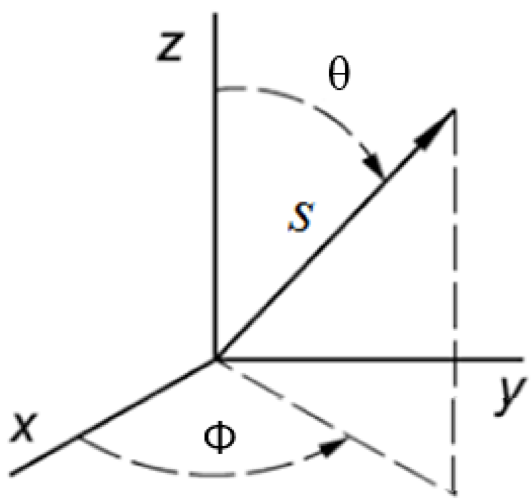
$$C_{u,v} - kT_{u,v} \approx \sigma \cdot g_{u,v}(0) \quad (\wedge)$$

The solution τ to (^) will be of the form $g_{u,v}(r, \sigma)$, subject to the condition that as σ ---› 0, τ will tend to *zero* also, in the *radial* and *time* directions ($u = v = 1$ or 4). And as we said above, $g_{u,v}(r)$ is equal to $[r^2, r^2 \sin^2\theta]$ in τ, by default, if $u = v = 2$ or 3.

The *total* solution to (*) [p 504] is now of the form

$$\mathscr{T} = g_{u,v}(r) + g_{u,v}(r, \sigma) + \tau_{\cos(\alpha)} + \tau_{\sin(\beta)} + \tau_{\cos(\theta)}$$

where $g_{u,v}(r)$ is *vintage* Schwarzschild [σ = 0], subject to a reconsideration of the *time* component in the *interior* case [pp 383-460]. We only add to the *inner* block if the entry here is *uniquely* different than $[r^2, r^2 \sin^2\theta]$. This will apply to $\tau_{\cos(\alpha)}$, $\tau_{\sin(\beta)}$ and $\tau_{\cos(\theta)}$, but *not* $g_{u,v}(r, \sigma)$ in $\mathscr{T}$ above.



Physical Coordinate System

And let us recall [pp 365-8], that our *quantumlike* dark energy components satisfy Laplace, so that in a *physical* coordinate system, for example, it is the case that

$$P_2(\sin(\theta)\cos(\phi)) = -1/2 \cdot P_2^{\,0}(\cos\theta) + 1/4 \cdot P_2^{\,2}(\cos\theta)\cos(2\phi)\,,$$

and that

$$P_3(\sin(\theta)\cos(\phi)) \;=\; 1/4 \cdot P_3^{\,1}(\cos\theta)\cos(\phi) \;-\; 1/24 \cdot P_3^{\,3}(\cos\theta)\cos(3\phi)\;.$$

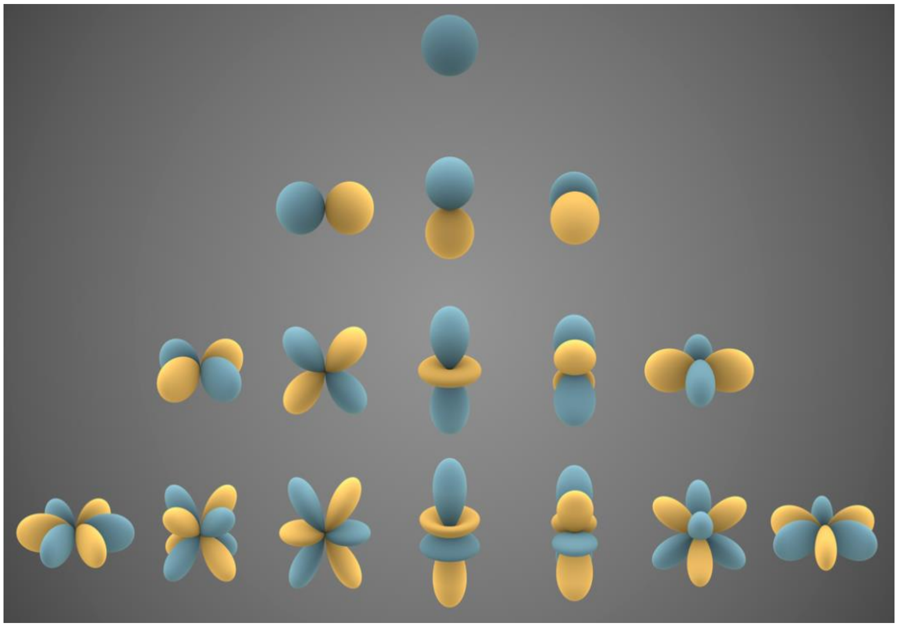
Such decompositions will exist in general, so that we shouldn't be surprised if solutions to (~), and shown below,

$$C_{u,v} \;-\; kT_{u,v} \;\approx\; \{2\sigma \cdot \Sigma\, (\delta r)^n P_n(\omega) \,/\, \Gamma(n+1)\} \cdot \{\Sigma\, c_n(u,v)\,(\delta\omega)^n P_n(\omega)\} \qquad (\sim)$$

ultimately involve *spherical* harmonics, just as we see at the *atomic* level; where ω is one of $\cos(\alpha)$, $\sin(\beta)$ or $\cos(\theta)$, and $\cos(\alpha) = \sin(\theta)\cos(\phi)$ and $\sin(\beta) = \sin(\theta)\sin(\phi)$, using *physical* coordinates.

And again, the template Ꚋ for $g_{u,v}$ is, in this case (where $r = \delta r$)

$$\Sigma\, c_n(u,v)\, r^n P_n(\omega)\,, \qquad n = 0, 1, 2, 3, \ldots$$



In the picture above, we see some *spherical* harmonics, often referred to as *orbitals*. The *center* column corresponds to Legendre polynomials with *no* superscript. or equivalently, a superscript of *zero*. Away from the center column, are the *associated* Legendre polynomials, where the superscript is *non-zero*.

OTHER CONSIDERATIONS

For the *two*-dimensional star, let us bring back our expression for $g_{u,v}(\delta\cos(\theta))$, which is …

$$\{\Sigma\, c_n(u, v)\, (\delta\cos(\theta))^n\, P_n(\cos(\theta))\}\ , \quad (*)$$

where $n = 0, 1, 2, 3, \ldots$ in this summation.

If we let $\delta \dashrightarrow 0$, then (*) reduces to $c_0(u, v)$, as we said in previous notes. On the other hand, if we let $\theta \dashrightarrow \pi / 2$ with $\delta > 0$, then (*) again reduces to $c_0(u, v)$, if we define 0^0 to be 1. In *both* cases, this value is $g_{u,v}(0)$.

Thus, relative to $g_{u,v}(0)$, approaching the origin along the line ℓ_θ, where $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; is *no* different than approaching the origin along *any* radial line, where the singularities have now been relocated to O, and thus have *no* influence on $g_{u,v}(0)$ at all [relative to θ].

In turn, this must mean that in the *first* case, where $\theta = \pi / 2$, the singularities at $(\pm\delta, 0)$ have all but *lost* their influence on $g_{u,v}(0)$, along ℓ_θ, which is consistent with our Singularity Invariance Postulate [p 461 *ff.*].

And so, this can only *bolster* our view that the template Ƭ for $g_{u,v}$, and shown below (where $r = \delta r$),

$$\Sigma\, c_n(u, v)\, r^n\, P_n(\cos(\theta)) \qquad n = 0, 1, 2, 3, \ldots$$

is probably a reasonable choice for our *coupled* field equations below, when calculating the *left*-hand side; in the case of the *quantumlike* dark energy components …

$$C_{u,v} - kT_{u,v} \approx \{2\sigma \cdot \Sigma\, r^n\, P_n(\cos(\theta)) / \Gamma(n + 1)\}\, g_{u,v}(\delta\cos(\theta))\ . \qquad (\dagger)$$

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The argument above, while interesting, might only be valid if $c_0(u, v)$ was a *constant*. In reality, though, for the *quantumlike* dark energy components, $c_n(u, v)$ ought to be a function of (θ, σ, δ) for *all* $n \geq 0$, since here we are solving for these coefficients by matching powers of r on *both* sides of (†).

Thus, letting $\delta \dashrightarrow 0$ in $c_0(u, v)$, may *not* give us the *same* result we would obtain by letting the angle $\theta \dashrightarrow \pi / 2$ in $c_0(u, v)$; and indeed, the series expansion for $g_{u,v}(\delta\cos(\theta))$ above is really only valid for $\delta > 0$ anyway, as is the template Ƭ.

Similar remarks apply in *three* dimensions, where here, we expect $c_n(u, v)$ to be a function of the variables $(\theta, \phi, \sigma, \delta)$ for *all* $n \geq 0$, for the *quantumlike* dark energy components.

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Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XVIII

More and more, I am coming around to the idea that when dealing with dark energy, and the *coupled* field equations, the *tangible* stress tensor $T^{u,v}$ is *irrelevant*. For if we revisit Part XV on page 501, and look again at the equation below; namely

$$C^{u,v} - kT^{u,v} \approx \sigma \cdot g^{u,v}(0) , \quad (\dagger)$$

which has a dark energy singularity at the origin O, what is this equation really saying ?

From page 501 … 'it is that solution τ which exists and is driven by the *right*-hand side of (†), after *subtracting* out *tangible* matter from the space-time fabric'. Well, this means a few things; firstly, that *tangible* matter plays no role in the *external* solution, since the only *external* source that can influence the gravitational tensor $g^{u,v}$ is the right-hand side of (†), itself. Only when the coupling is *broken* by letting $\sigma = 0$, do we have an *internal* source that can influence $g^{u,v}$, and that source is the perfect star itself, filled with tangible matter (typically a perfect fluid). Hence the need for a *tangible* stress tensor $T^{u,v}$, and thus the proportionality constant *k* as well, in this case (it's a kind of loopback condition where curvature and stress balance one another, up to some constant *k*).

Now I *restore* the coupling, and look for an *external* solution to (†), realizing that the *internal* solution ($\sigma = 0$) has already been dealt with. Why, in searching for this solution, would I even care about tangible matter, since by definition, it is to be removed from the space-time fabric, anyway ?

Wouldn't it be better to write (†) as

$$C^{u,v} \approx \sigma \cdot g^{u,v}(0) , \quad (*)$$

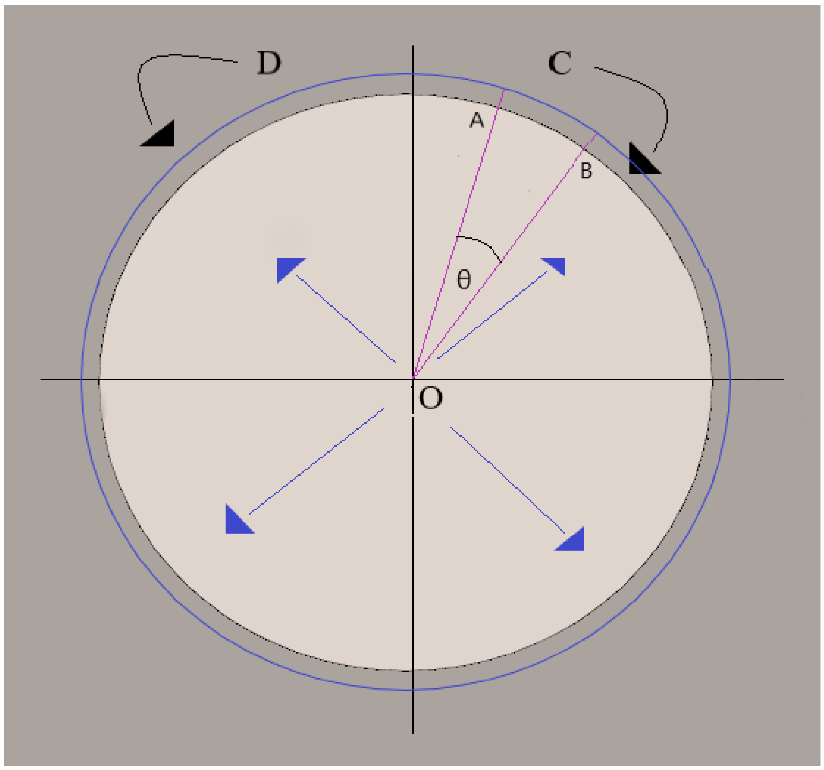
if σ is no longer *zero* ? We've certainly removed tangible matter from the space-time fabric, but we've done this now by *eliminating* the *tangible* stress tensor $T^{u,v}$, altogether !

And if we suppose that *intangible* matter (dark energy) does *not* have an associated *intangible* stress tensor, then (*) would indeed be the right formation [intangible stress tensors have their own set of problems, such as how to diagonalize them in the *comoving* frame, and what the proportionality constant *k* might actually be, in this case – problems that are only amplified when dealing with the *quantumlike* dark energy components].

So while we can leave (†) as is, we can *also* start to think about (*) as perhaps a better alternative for the dark energy components, in the *coupled* case.

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XIX

Imagine an *empty* space-time fabric $\boldsymbol{S}$, as shown in the diagram below, with a circle C of radius r, say, centered at O.



Now suppose there is a dark energy singularity at O, so that dark energy [ξ] itself, pours into $\boldsymbol{S}$ via O, according to the *underlying* dark energy density function $\lambda(s) \approx \sigma / s$ [blue arrows]. This should cause the circle C to expand out to D, say, so that the angle θ that *was* subtended by the arc AB on circle C, is now subtended by a *larger* arc on circle D.

Bringing back our *coupled* equation for this *two*-dimensional setup, which is

$$C^{u,v} - kT^{u,v} \approx \sigma \cdot g^{u,v}(0) , \quad (\dagger)$$

we may now ask, when contemplating a solution τ to (†), whether we should or shouldn't let the *inner* block in τ default to r^2, in covariant form. Up until now we have, but because the original circle C has now expanded, perhaps we *should* include the *inner* block [$u = v = 2$] in our calculations, when solving (†) [see also the notes on pages 395-6].

And similarly for the *three*-dimensional setup, where the default *inner* block is [r^2, $r^2 \sin^2 \theta$] in covariant form, using *physical* coordinates [$u = v = 2$ or 3]; but see also pages 497-8 for more on this subject.

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XX

Let us bring back our equation from the last page, and write it as …

$$C^{u,v} \approx kT^{u,v} + \sigma \cdot g^{u,v}(0) . \quad (\dagger)$$

In this form, we see there are *two* sources that can influence the space-time fabric [$\boldsymbol{S}$], by way of the gravitational tensor $g^{u,v}$. Here, $\boldsymbol{S}$ is to be associated with the Einstein curvature tensor $C^{u,v}$.

The *first* source is $kT^{u,v}$, where the *tangible* stress tensor $T^{u,v}$ is to be associated with the perfect star [S*]. The *second* source is $\sigma \cdot g^{u,v}(0)$, which is associated with *dark* energy [an *intangible* form of matter], where the dark energy singularity is located at the origin O.

I now want to solve (†), realizing that these two sources are *very* distinct from one another. But I *also* know the curvature tensor $C^{u,v}$ *doesn't care* at all which source I am dealing with, since it is a purely *mathematical* object. How then shall I proceed ?

In looking at (†), we'll *rule out* the possibility of an *intangible* stress tensor, since in (†) it doesn't appear at all. So I decide to *break* the coupling, by setting $\sigma = 0$, and solve what's left of the equation above; which gives us vintage Schwarzschild [interior / exterior], subject to a reconsideration of the *interior* time component [pp 383-460]. We'll call this solution $g^{u,v}{}_{\mathcal{M}}$, where $\boldsymbol{\mathcal{M}}$ means 'tangible matter'.

I now *restore* the coupling, by setting σ to some *non-zero* value, and want a solution to (†) in the *intangible* space, realizing again that $C^{u,v}$ doesn't care which space I am operating in. So all I have to worry about is the right-hand side of (†); and knowing now that there is *no* such thing as an *intangible* stress tensor, reason to myself that the *tangible* stress tensor $T^{u,v}$ must go, if I am to operate in the *intangible* space. I therefore set k to zero, and solve what's left of (†). We'll call this solution $g^{u,v}{}_{\boldsymbol{\xi}}$, where ξ means 'intangible matter'.

Adding the two solutions together gives us a total solution $\boldsymbol{\mathcal{T}}$, where we only add to the *inner* block if the entry here is *uniquely* different than the default [r^2, $r^2 \sin^2 \theta$], in *three* dimensions, using a *physical* coordinate system. The process can be extended to include *all* quantumlike dark energy components, in the *intangible* space, and this is how I think we should approach the broader problem of solving the *coupled* field equations, in general, like those shown at the top of page 504.

In short, I don't believe in the *intangible* space, a stress tensor of *any* kind plays a role in solving the *coupled* field equations, for the dark energy components, but *do* believe the curvature tensor $C^{u,v}$ is *impartial* to the space we are working in.

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XXI

Let us bring back our equivalency from page 470 and rewrite it, where there are singularities at the origin O and at $(\pm\delta, 0)$, for the *two*-dimensional star, in the *coupled* case. Here, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$.

Then (1) is …

$$2\kappa\int_{\delta\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\delta\alpha + iy) - g^{u,v}(\delta\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\delta\alpha + iy) + g^{u,v}(\delta\alpha - iy)]\}\, dy \,/ \sqrt{y^2 - (\delta\varepsilon)^2}$$

and (2) is …

$$G^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\alpha)J_0(\delta r\varepsilon)g^{u,v}(\delta\alpha)]\,.$$

Recalling our equivalency, for this case …

$$(1) = g^{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true}, \qquad (\sim)$$

we see that as $\delta \dashrightarrow 0$, (1) becomes

$$2\kappa\int_{0}^{\infty} \{\cos(yr)[g^{u,v}(iy) - g^{u,v}(-iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(iy) + g^{u,v}(-iy)]\}\, dy \,/\, y \qquad \ldots (*)$$

which is *always* equal to $g^{u,v}(0)$, provided $g^{u,v}$ is *well-behaved* in the *complex* plane, *real-valued* on the *x*-axis, and we can secure *convergence* along our contour γ [p 467]. Indeed, we showed it to be true when $g^{u,v}(s) = \zeta(\beta - s)$, where $\beta > 1$ [pp 330-2].

Thus, from our equivalency (~) above, it must *also* be true that

$$G^{u,v} \approx 3\sigma \cdot g^{u,v}(0)\,,$$

since α is now 0 in (*), as is ε. And, of course, this agrees with our *original* calculations on pages 197-9, using the dark energy *contour* integral for a singularity at O. Thus, we have a confirmation of our harmonic expression (1) in this case, as well as our equivalency (~), when $\delta > 0$, say.

———

As to the template Ꞇ for $g_{u,v}$, and shown below (where $r = \delta$r), we might want to sum *only* over all *even* values of $n \geq 0$, since Legendre polynomials are *even* if *n* is *even*, but *odd* if *n* is *odd*.

$$\Sigma\; c_n(u, v)\, r^n P_n(\cos(\theta)) \qquad n = 0, 2, 4, \ldots$$

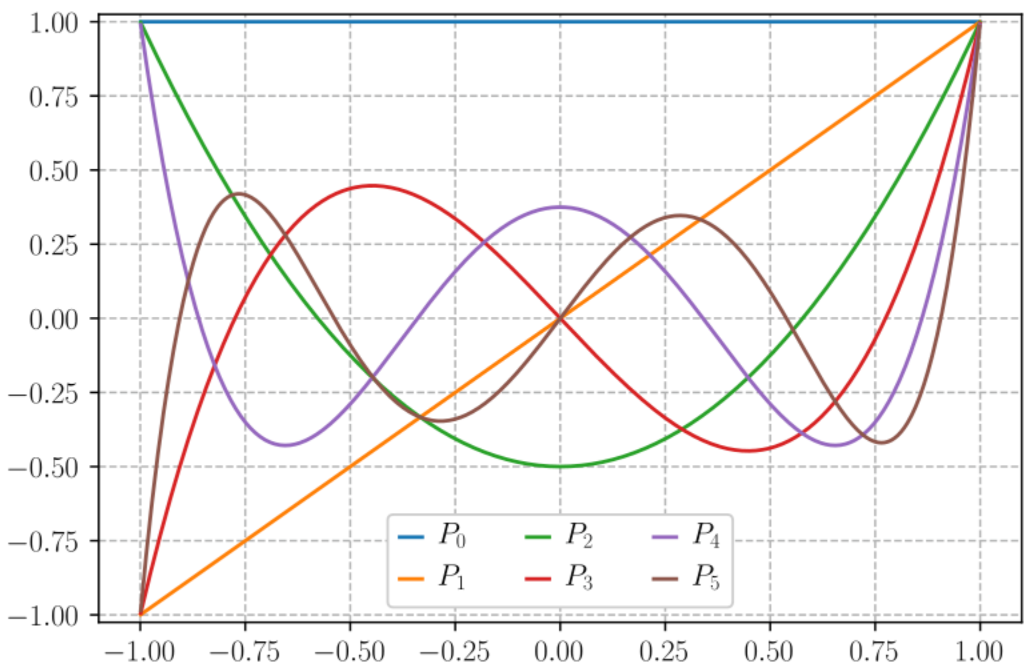
This will allow us to better align with the *quantumlike* dark energy components, which are always *even* functions, as shown below …

$$2\sigma \cdot \cosh(r\cos(\theta))J_0(r\sin(\theta)) \;=\; 2\sigma \cdot \Sigma\, r^n P_n(\cos(\theta)) \,/\, \Gamma(n+1)\;,\; n = 0, 2, 4 \ldots$$

And in doing so, this will preserve the *symmetrical* properties of the gravitational tensor $g^{u,v}$ itself, over a circle or a sphere, when searching for solutions to the *coupled* equations, such as those shown below [p 502 *ff.*] …

$$C_{u,v} - kT_{u,v} \;\approx\; \{2\sigma \cdot \Sigma\, (\delta r)^n P_n(\cos(\theta)) \,/\, \Gamma(n+1)\} \cdot \{\Sigma\, c_n(u, v)\, (\delta\cos(\theta))^n P_n(\cos(\theta))\}\;. \quad (\sim)$$

Here, the *second* summation above corresponds to $g_{u,v}(\delta\cos(\theta))$; and let us recall too, from the last research note [p 510], that we ought to set $k = 0$, when working in the *intangible* space (dark energy), and we would set $\sigma = 0$ when working in the *tangible* space.



The picture above shows some Legendre polynomials, and you can see from the diagram, which are *even* functions (n = 0, 2, 4) and which are *odd* (n = 1, 3, 5) …

Revisiting Our Form For The Field Equations of General Relativity When $\lambda(s) \approx \sigma / s$, Part XXII

Let us bring back our form for the *coupled* field equations, in *two* dimensions, with singularities at the origin O and at (±1, 0). Here, we are looking only at the *quantumlike* component, and *both* sums below are to be taken over all *even* integers ($n = 0, 2, 4, \ldots$).

$$C_{u,v} - kT_{u,v} \approx \{2\sigma \cdot \Sigma\, (r)^n P_n(\cos(\theta)) / \Gamma(n+1)\} \cdot \{\Sigma\, c_n(u, v)\, (\cos(\theta))^n P_n(\cos(\theta))\} . \quad (\sim)$$

Now since we are working in the *intangible* space (dark energy), we'll set $k = 0$, and using our template Ꞇ below, for $g_{u,v}$,

$$\Sigma\, c_n(u, v)\, r^n P_n(\cos(\theta)) \qquad n = 0, 2, 4, \ldots$$

we'll define (hypothetically), for *all* even $n \geq 0$ and γ any real number …

$$c_n(u, v) = 2\sigma\gamma / \Gamma(n+1) .$$

Defining the *first* term in curly braces, on the *right*-hand side of (~) to be $\xi(r, \theta)$, we see that the *second* term in curly braces, on the *right*-hand side of (~) is

$$2\sigma\gamma \cdot \Sigma\, (\cos(\theta))^n P_n(\cos(\theta)) / \Gamma(n+1) ,$$

and that Ꞇ = $\gamma \cdot \xi(r, \theta)$.

Thus, from (~) we have

$$C_{u,v}\, (\gamma \cdot \xi(r, \theta)) = 2\sigma\gamma \cdot \xi(r, \theta) \cdot \Sigma\, (\cos(\theta))^n P_n(\cos(\theta)) / \Gamma(n+1) , \qquad (\dagger)$$

and letting γ ---› 0, we see that the *curvature* tensor in (†) is to be evaluated against a function which is everywhere *zero*, so that $C_{u,v} = 0$. As well, the *right*-hand of (†) is also *zero* as γ ---› 0, and so, we have a confirmation of (~) and Ꞇ in this rather hypothetical case.

Some notes, which may be useful, and served as a precursor to the essay – a work in progress …

Time, Space, The Spiritual and The Physical

Because ~[time] and ~[space] are *attributes* of the Godhead G, and thus the set of laws T in the *spiritual* world, no law dependent on (s, t) makes much sense here, since ~[s, t] belongs to T. It's rather like trying to describe the north pole, where your language of choice does not contain the words 'north' or 'pole'. Here, ~ is the logical 'not' and G is made from a form of *exotic* anti-matter that is unfamiliar to us. We know nothing about it.

What is *time* ? What is *space* ? They are equivalent to one another, perhaps symmetrically so, and thus manifestations of the same thing. What about theorems like GR (General Relativity) or SR (Special Relativity) that relate the two ? Losing one component means losing the other, and conversely, if one component emerges, so does the other. Rather like a reflection of yourself in the mirror – it vanishes if you vanish or the mirror vanishes. In the anti-matter world, where the spiritual laws live and operate, there can be *no* time or space, so (s, t) as a pair only emerges in the matter world, for our benefit [in the case of a neutron star collapsing to a black hole, say, we may lose the time component altogether, as per the α-γ equations developed on pages 383-460].

Remember, according to James T's NDE, the laws are antimatter !! In *our* reality, then, should we be looking for a *mirror* to GR in the anti-matter world, and if so, what is it ? Does it come about through analytic continuation, or a Laplace transform, for example ? Is there an all-encompassing theory of GR that takes account of both (GR, matter) && (GR, anti-matter) ? Is this what I saw the hooded monks talking about in the first dream or OBE, where they mentioned using the Laplace transform, on what looked like an extension to the GR equations ?

In the second OBE, I saw more in the book of laws, maybe around page 500, but the monk told me these were 'tensor skeins' embedded in what appeared to be *braided* equations, written in a type of hieroglyphics. GR in the physical world certainly has a spiritual counterpart, so one could argue

(GR, anti-matter, ~s, ~t) <~ some transform ~> (GR, matter, s, t)

Thus, there must be a mechanism which takes you from one world to the other and back. Perhaps this is what the Laplace transform does, or maybe it's more complex than that. Also, the FTOC (Fundamental Theorem Of Creation) is probably anti-matter, since the laws are, inherently so.

How To Interpret The g-Matrix

Here are some thoughts I had on the g-matrix, long ago. Imagine a sphere, say M = S(2), embedded in E3, with a tangent plane T at a point p on M. At this point there are no coordinate systems [charts], no policies in place for moving between these charts, and so on. So any vector V in T could be attached to any policy P, once T was endowed with any number of charts.

Thus if A and B are two distinct policies for moving between charts, V can belong to *either* A or B, so long as the existence of these policies can be justified. And so, we can view T in the A frame of reference or the B frame of reference, once we decide to choose.

The *contravariant* [A] and *covariant* [B] frames are two such policies, and both exist in T quite naturally. The first, by simply studying run-of-the-mill vectors and the second, by studying the *gradient* of any scalar field on M. Each transforms differently between charts, for any vector V in T, and you could probably use or find other policies as well, but these two are solid.

So suppose V is a vector in T following policy A. Since any vector in T can belong to either policy, there must exist an operator g which takes V in the A frame to W in the B frame. Effectively, I am saying I can choose to view V in the A or the B frame by way of g. And g can't be the identity, for if so, then V is *itself* in either frame; which means that if V ---› V' is a move between charts in A, say, then V ---› V' is the *same* move between charts in B. Thus, the contravariant and covariant rules become one and the same thing, so-to-speak, which means the two spaces are really one and the same thing.

So g is a viable operator, and is responsible for moving between the contravariant and covariant frames of reference on T, and it does this smoothly and everywhere for any p in M. This is *also* the g-matrix found in relativity.

Now let V be a contravariant vector in T, and operate on it via g so we are in the covariant space, i.e. $g_{u,v} \cdot V^u$. This gives us the dual to V in the covariant frame; a vector, to be sure, but different than V. And we'll label it V_v. Now dot this with V^v , and lo and behold, you have a scalar, which is the *same* value regardless of which chart you choose, or which frame you are in !!

That is to say, the scalar $g_{u,v} \cdot V^u\, V^v$ is *invariant*, and this is the beginnings of relativity theory, since this scalar is simply the measure of distance when looking at differentials. But remember, we got here by operating on a vector V in the contravariant space via g, to produce a vector in the covariant space, and then dotted this second vector with the first, to get our scalar. Without the dual contravariant and covariant spaces, you couldn't do this.

'for every contravariant vector there exists a covariant vector and conversely'

Now that we have an invertible g-matrix that takes us between these two spaces on T, a meaningful *curvature* tensor can be established which behaves like a tensor. That is, it follows contravariant or covariant rules. And that tensor is, as per our previous research notes …

$$C^{u,v} = R^{u,v} - \tfrac{1}{2} R g^{u,v}$$

Now marry this up with a stress tensor $T^{u,v}$ and try to solve for the g-matrix, so that you can measure the distance between points on the manifold, that is to say …

$$ds^2 = g_{u,v}\, dx^u\, dx^v$$

And so, a theory is born, which we call general relativity; but in reality, it begins and ends with the policies we put in place for managing vectors in the contravariant and covariant spaces, as we noted above.

An Interesting Property Concerning Unit Spheres

If I embed a unit circle S(1) in E2, the *ratio* of volume to surface area is $\pi / 2\pi$, which is $1/2$. Similarly, if I embed a unit 2-sphere in E3, the ratio of volume to surface area is $(4\pi / 3) / 4\pi$, which is $1/3$.

For a 3-sphere embedded in E4, the Riemann metric is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 + r^2 \sin^2(\theta) \sin^2(\varphi) d\psi^2$$

where θ ranges between 0 and π, and φ, ψ range between 0 and 2π. And realizing that these four dimensions are independent, it can be shown that the volume to surface area for a unit 3-sphere embedded in E4, is actually $1/4$.

So if this trend were to continue, the ratio of volume to surface area for a unit *n*-sphere embedded in E(n + 1) would be $1 / (n + 1)$. In other words, as $n \dashrightarrow \infty$, the volume is insignificant compared to the surface area that contains it.

On The Existence and Uniqueness of The Field Equations of GR

Everything in GR begins and ends with the g-matrix, which is the mechanism by which distance between two points is measured. For g to be consistent, it must be *unique* relative to the embedding and its covariant derivative must always be *zero*.

And, indeed, this is the case for the embedding of a manifold M in Euclidean space, where the metric is Riemannian. By covariant derivative being 0, we really mean the covariant *divergence* for any of the g *basis* vectors (i.e. column or row vectors) computes to 0.

If *cov*(g) wasn't zero, it would be the equivalent of saying 'the ruler itself is changing as I try to measure the distance between two points, using infinitesimals'. Covariance, then, is a measure of *consistency*, when set to 0.

$$\nabla_u g^{u,v} = 0$$

So think of g as a matrix of ‘4 rulers’, the column vectors say, all of which contribute to the overall norm of a vector, say dx^v, which we call ds^2. The covariant divergence of each of these 'rulers' is 0. We want the same thing, ultimately, for *curvature* and *stress* tensors, if we are to have a *consistent* model of GR.

Ironically, the Riemann Curvature Tensor existed long before the Einstein Stress Tensor, and from it, of course the Einstein-Ricci Curvature Tensor C, which *is* covariant. So if you think of C as a basis of column vectors, the covariant divergence of each vector is 0. That is to say, the basis of vectors for C is neither 'expanding' nor 'shrinking' as curvature measurements are being made, at any point in M, in the infinitesimal sense. The basis itself is an *invariant*, just like it is for g.

I can now define a new 'norm' using $C_{u,v}$ instead of $g_{u,v}$. It is to curvature what g is to distance, and it should be an invariant. To wit,

$$dc^2 = C_{u,v}\, dx^u\, dx^v$$

Musings On The Law of Conservation of Sectional Volumes

If $C_{u,v}$ is the Einstein curvature tensor, we can, notationally, employ it however we wish. So we'll consider $\Delta C(u,v)/\Delta x$, in the direction of x, across the cube's *infinitesimal* interface perpendicular to x. In actuality, this is really the partial derivative + connection terms at this point; that is to say,

$$\partial C(u,v)/\partial x + \Gamma \qquad (\dagger)$$

for some element of the row vector [C(1, 1), C(1, 2), ...]. In other words, the *covariant* derivative, which we'll abbreviate as *cov*, on occasion.

Now in *differential* notation, if we multiply (†) through by the volume of the cube $\Delta x \Delta y \Delta z \Delta t$, and we're operating on C(1,1), say; we would get the following, where Γ is the connection coefficient for this element C(1,1), and ∂C(1,1) is the *partial* differential in the direction of x:

$$\partial C(1,1)\Delta y \Delta z \Delta t \;+\; \Gamma\, \Delta x \Delta y \Delta z \Delta t \;=\; \{\partial C(1,1) \;+\; \Gamma\, \Delta x\}\Delta y \Delta z \Delta t \qquad (*)$$

So the *covariant* differential ΔC(1,1) is the expression in curly braces above, and since we can rewrite ∂C(1,1) as $[\partial C(1,1)/\partial x]\Delta x$, the *right*-hand side of (*) then becomes

$$\begin{aligned}
&\{[\partial C(1,1)/\partial x]\Delta x \;+\; \Gamma\, \Delta x\}\Delta y \Delta z \Delta t \\
&\qquad = \{[\partial C(1,1)/\partial x] \;+\; \Gamma\}\Delta x \Delta y \Delta z \Delta t \\
&\qquad = \{[\partial C(1,1)/\partial \mathrm{x}] \;+\; \Gamma\} * V_{\mathrm{cube}} \\
&\qquad = \Delta C(1,1)/\Delta x \;\; * V_{\mathrm{cube}} \\
&\qquad = cov\{C(1,1)\} * V_{\mathrm{cube}}
\end{aligned}$$

You can repeat this exercise for C(1, 2), C(1, 3) and C(1, 4), so that for this vector, the the *total* net change is going to be [see also pp 178-80]:

$$[\Delta C(1,1)/\Delta x + \Delta C(1,2)/\Delta y + \Delta C(1,3)/\Delta z + \Delta C(1,4)/\Delta t] * V_{\mathrm{cube}} \;=\; 0\,,$$

because of the *Bianchi* identities. That is to say, if we *interpret* the pieces in the bracketed expression above as *sectional* volumes, when multiplied by V_{cube}, then as a whole, they are *conserved*, because from Bianchi, (where ∇_u means covariant derivative), it is the case that

$$\nabla_u C^{u,v} \;=\; 0\,.$$

Thus, $\Delta C(1,1)\Delta y\Delta z\Delta t$ *is* the *sectional* volume loss/gain along the (1, 1) direction, and similarly for the other components. Note that if there is *no* connection term, then $\Delta C(1,1)$ is actually $[\partial C(1,1)/\partial x]\Delta x$, which is what you'd expect. The differential of C(1,1), in the direction of x, is simply the *partial* derivative in x, multiplied by Δx.

The sectional volume loss/gain, if $\Gamma = 0$, computes to $[\partial C(1,1)/\partial x]\Delta x\Delta y\Delta z\Delta t$, so if there is no 'bend' to the interface perpendicular to x, then this expression is *zero*, provided there is no *shearing* [i.e. the cube becomes a rhombus, say]. But whether there is or isn't any shearing, the following will *always* be true, in the *inertial* frame, meaning, in the absence of gravity …

$$[\partial C(1,1)/\partial x + \partial C(1,2)/\partial y + \partial C(1,3)/\partial z + \partial C(1,4)/\partial t] * V_{\text{cube}} = 0$$

Calculating The Invariant Curvature Over a 2-Sphere of Radius *r*

From our curvature norm on page 518, and reproduced below, using *physical* coordinates,

$$dc^2 = C_{u,v}\, dx^u\, dx^v , \quad (\dagger)$$

it is the case that from the literature, $R_{\theta,\theta} = 1$, $R_{\varphi,\varphi} = \sin^2(\theta)$, and that the Ricci scalar is $2 / r^2$. Thus, the Einstein curvature tensor, which is covariant; namely,

$$C_{u,v} = R_{u,v} - \tfrac{1}{2} R\, g_{u,v}$$

computes to *zero* for $u = v = \theta$ or φ, since here $g_{\theta,\theta} = r^2$ and $g_{\varphi,\varphi} = r^2 \sin^2(\theta)$. And so, if *over* the sphere we were to integrate the *square root* of our norm in (†), for $u = v = \theta$ or φ, where $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$, we'd obtain a value of *zero* for the *invariant* curvature. That is, for our sphere [S],

$$\int_0^{2\pi}\int_0^{\pi} [C_{\theta,\theta} \cdot C_{\varphi,\varphi}]^{1/2}\, d\theta d\varphi = 0 .$$

So *invariant* curvature over S is to dc^2 what *invariant* area over S is to ds^2, where the latter is our distance metric

$$ds^2 = g_{u,v}\, dx^u\, dx^v .$$

On The Nature of Tangible and Intangible Vacuums

Let us bring back our equation from page 510, which is …

$$C^{u,v} \approx kT^{u,v} + \sigma \cdot g^{u,v}(0) . \qquad (\dagger)$$

In this form, we see there are *two* sources that can influence the space-time fabric [$\mathcal{S}$], by way of the gravitational tensor $g^{u,v}$. Here, $\mathcal{S}$ is to be associated with the Einstein curvature tensor $C^{u,v}$.

The *first* source is $kT^{u,v}$, where the *tangible* stress tensor $T^{u,v}$ is to be associated with the perfect star [S*]. The *second* source is $\sigma \cdot g^{u,v}(0)$, which is associated with *dark* energy [an *intangible* form of matter], where the dark energy singularity is located at the origin O.

Now suppose we let σ = 0, so we are in the *tangible* space, and our equation (†) becomes

$$C^{u,v} \approx kT^{u,v} .$$

Setting $k = 0$, we reach a *vacuum* state induced by the *tangible* space, which we'll call $V_{\mathcal{M}}$, where $\mathcal{M}$ means 'tangible matter'. And here the metric becomes

$$c^2 d\tau^2 = -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + sin^2\theta \, d\phi^2) , \qquad (\ddagger)$$

so that *even* in $V_{\mathcal{M}}$, the notion of *space* and *time* exists.

Now let's repeat the exercise, by first setting $k = 0$, so that (†) becomes

$$C^{u,v} \approx \sigma \cdot g^{u,v}(0) , \qquad (*)$$

and by letting σ = 0, we again reach a *vacuum* state induced by the *intangible* space, which we'll call V_{ξ}, where ξ means 'intangible matter'.

Now from our *associative* principle under addition, relative to σ in the *intangible* space [p 491], we believe the solution $g^{u,v}(r, \sigma)$ to (*) must be *zero* in the *radial* and *time* components, as σ ---› 0; so that in V_{ξ}, the notion of *space* and *time* does *not* exist at all.

But if this is so, then the metric (‡) cannot be correct for V_{ξ}, for without the constructs of space and time, the *right*-hand side of (‡) vanishes altogether ! And if this is true, then it means that when forming solutions to (*), the *inner* block [θ, φ components] must *also* vanish as σ ---› 0. And this is now a departure from our earlier view [p 479 *ff*.], where we discussed the idea of preserving the inner block in V_{ξ}. In hindsight, we may not be able to do this …

So when solving (*) for the dark energy singularity at O, or when solving for the *quantumlike* dark energy components, it might be best to use a *template* Ꮏ for $g^{u,v}$, which incorporates a vanishing *inner* block in the *intangible* space, as σ ---› 0.

Calculating The Invariant Curvature Over a 2-Sphere of Radius *r* For a Tangible Source

For a *tangible* source, which we'll assume to be a *perfect* star [S*], the field equations are

$$C_{u,v} \approx kT_{u,v}, \quad (\dagger)$$

and here, $T_{\theta,\theta} = pr^2$ and $T_{\varphi,\varphi} = pr^2 \sin^2(\theta)$, where p is pressure *in* S*. The pressure p is some *fixed* value at a radius $r \leq r_g$, where r_g is the radius of S*. Notice that like the curvature tensor $C_{u,v}$, the stress tensor $T_{u,v}$ is *also* fully covariant.

Thus, when calculating the *invariant* curvature $\mathcal{C}$ over the 2-sphere for $r \leq r_g$; namely

$$dc^2 = C_{u,v}\, dx^u\, dx^v ,$$

we can simply use the $T_{u,v}$ equivalents *if* $r \leq r_g$, and this leads to a value of

$$k \int_0^{2\pi}\!\!\int_0^{\pi} [T_{\theta,\theta} \cdot T_{\varphi,\varphi}]^{1/2}\, d\theta d\varphi = 4k\pi r^2 p$$

Note that if $r > r_g$, then we can't calculate $\mathcal{C}$ using this method, because (†) *only* holds true *inside* S*. On the other hand, if $p = 0$ or $k = 0$, then $\mathcal{C}$ is *zero*, from the expression just above; and this agrees with our calculation on page 519, as it should, since in *both* cases, we are now in a *tangible* vacuum $V_{\mathcal{M}}$.

Notice too, that in the *intangible* vacuum V_ξ, there is *no* such thing as *space* or *time* [p 520], so that calculating the *invariant* curvature over a 2-sphere in V_ξ makes no sense. Said another way, the *associative* principle under addition, relative to σ in the *intangible* space, applies to *all* components of the *g*-matrix here. Again, see the previous research note for more on this interesting idea.

*****

Is there an *ideal* pressure p(r), for *all* r between 0 and r_g, such that $\mathcal{C}$ is equal to the Ricci curvature scalar R for the 2-sphere, which is $2/r^2$? There is, for if we set $\mathcal{C}$ = R, this *ideal* pressure computes to

$$p(r) = 1 / 2k\pi r^4 ,$$

and may have some *physical* significance.

A small note was added to the bottom of page 504 in this update, as it was never mentioned explicitly in any of the previous releases. The *third* component mentioned there could *also* be derived by a 90 degree rotation of the *second* component [$\omega = \sin(\beta)$] in the *y*-*z* plane.

Some More On Solving The Equation $C^{u,v} \approx \sigma \cdot g^{u,v}(0)$ When $\lambda(s) \approx \sigma / s$

Let us bring back this equation, in the *intangible* space, where we have a dark energy singularity at the origin O …

$$C_{u,v} \approx \sigma \cdot g_{u,v}(0) . \quad (*)$$

Recalling one of our results from the *associative* principle under addition, relative to σ in the *intangible* space [p 482], where $g_{u,v}(r, \sigma)$ is the solution to (*); that is to say

$$2g_{u,v}(r, \sigma) = g_{u,v}(r, 2\sigma) ,$$

we can *extend* the result, just above, to *any* real number γ by writing

$$\gamma g_{u,v}(r, \sigma) = g_{u,v}(r, \gamma\sigma) . \quad (\dagger)$$

Now let σ = 1, so that we are solving (*) in this *normalized* case, where our template in the *radial* and *time* directions is simply $g_{u,v}(r)$, when $u = v = 1$ or 4. And for the *inner* block, we'll choose the default this time, namely $[r^2,\ r^2\sin^2(\theta)]$, for $u = v = 2$ or 3, using *physical* coordinates.

Thus, our *normalized* solution [σ = 1] to (*), is going to be

$$[\ g_{r,r}(r) ,\ r^2,\ r^2\sin^2(\theta) ,\ -g_{t,t}(r)\ ] , \quad (\ddagger)$$

and in this case, (†) becomes

$$g_{u,v}(r, \gamma) = \gamma g_{u,v}(r, 1) .$$

Now let γ = σ for *any* choice of σ, and we see that

$$g_{u,v}(r, \sigma) = \sigma g_{u,v}(r, 1) ,$$

so that the solution to (*), for *any* arbitrary choice of σ, is equal to σ *times* the *normalized* solution (‡). That is to say,

$$g_{u,v}(r, \sigma) = \sigma \cdot [\ g_{r,r}(r) ,\ r^2,\ r^2\sin^2(\theta) ,\ -g_{t,t}(r)\ ] . \quad (\sim)$$

Thus, when solving the *coupled* equations in the *intangible* space, for a dark energy singularity at O, *or* for the *quantumlike* components [see, for example, p 502 *ff*.], we might want to *first* find *normalized* solutions and then the more *general* solution, according to the methodology in this note.

The result (~) above, applies in *both* cases, and notice too in (~) that as σ ---› 0, $g_{u,v}(r, \sigma)$ ---› 0 for *all* elements of the *g*-matrix – in line with our conclusions on page 520. That is to say, the *intangible* vacuum V_ξ has nothing in it, or said another way, $V_\xi = \{ \emptyset \}$, the empty set.

From this, we see that the *inner* block needn't be included in the calculations [p 509], for while it is true that a sphere of radius r will *expand* or even *contract* in the presence of a dark energy singularity at O, this is now actually handled by the methodology above. Thus, for the *normalized* solution, it is acceptable to choose the *default* inner block $[r^2,\ r^2 \sin^2(\theta)]$ when solving (*) above, and then employ (~) to obtain the more general solution. And so, the remarks on page 509 are to be superseded by the comments here.

For the *quantumlike* components, the remarks on pages 497-8 still hold, in my opinion, where it is recommended that we *do* include the *inner* block in our calculations for the *normalized* solution, as opposed to using the default, which again, is $[r^2,\ r^2 \sin^2(\theta)]$. The general solution will then be patterned after (~) on the previous page.

Finally, it should be mentioned that if we are observing *space-time* events in a vacuum, then necessarily this is the *tangible* vacuum $V_{\mathcal{M}}$. For *no* such events can occur in the *intangible* vacuum V_{ξ}, as we now know.

———

Suppose now, we have a *perfect* star [S*] centered at O, where the solution in the *tangible* space is derived from the field equations

$$C_{u,v} \approx kT_{u,v}\ .$$

The solution below is going to be vintage Schwarzschild, subject to a reconsideration of the *time* component, for the *interior* case [pp 383-460],

$$[\ g_{r,r}(r)\ ,\ r^2,\ r^2 \sin^2(\theta)\ ,\ -g_{t,t}(r)\ ] \qquad (\S)$$

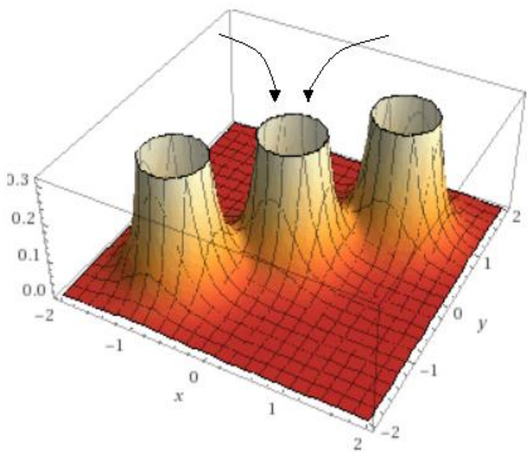
and a *total* solution, which incorporates the dark energy singularity at O, can now be produced by adding (~) to (§). And recall again, that σ in the expression (~) on the previous page, is likely to be an extremely small constant – perhaps on the order of the cosmological constant.

Thus, our *perception* of the *space-time* fabric, via the gravitational tensor, is *influenced* by *two* sources – S* in the *tangible* space, and the dark energy singularity at O, in the *intangible* space.

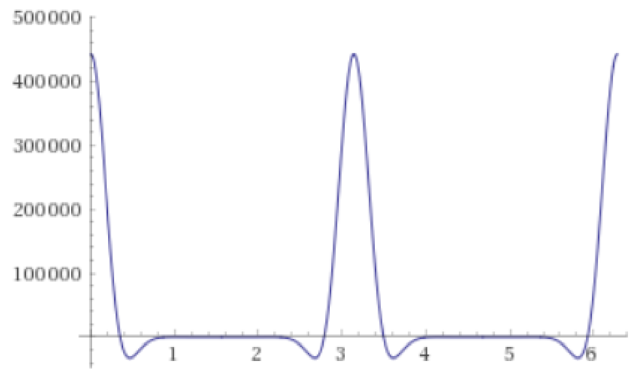
And similarly for the dark energy *quantumlike* components, and their associated dark energy singularities, in the *intangible* space. It is *these* components, in particular, that are responsible for the expansion *and* contraction of the universe, as we have said in previous research notes [e.g. pages 491-2, 496]; but we actually *perceive* this event in the *space-time* fabric, by way of sources [singularities] in the *intangible* space.

Some More On The Underlying Dark Energy Density Function $\lambda(s) \approx \sigma / s$

The underlying dark energy density function [$\lambda(s)$] is associated with the dark energy singularities, through which dark energy itself, is pouring into our universe, as per the pictures below [p 282].



dark energy tunnelling into our reality via the singularities associated with $\lambda(s)$



our perception of dark energy, obtained from $\lambda(s)$, via the Laplace inverse transform

Now dark energy, for a singularity at O, *is* the Laplace inverse of $\lambda(s) \approx \sigma / s^{\mu}$, for $\mu > 0$, and computes to

$$\sigma\{1 / \Gamma(\mu)\} \cdot r^{(\mu - 1)}, \qquad (*)$$

where r is our radius and $\Gamma(\mu)$ is the gamma function; and you can see from (*) that when $\mu = 1$, (*) equates to σ. And from our dark energy contour integral, in the *intangible* space,

$$C^{u,v} \approx \kappa \int_{\gamma} e^{sr} \lambda(s) g^{u,v} ds$$

we arrive at the *coupled* field equations, in this case, for the dark energy singularity at O; that is to say

$$C_{u,v} \approx \sigma \cdot g_{u,v}(0) .$$

Now the Laplacian in *physical* coordinates, is the expression below [labelling as (†)], and clearly (†) is satisfied by (*) if $\mu = 1$.

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \varphi^2} = 0$$

On the other hand, if $\mu > 0$ is any *real* number *other* than 1, then it is *also* clear that (†) will *not* be satisfied by (*), and so our *only* hope for an underlying dark energy density function [$\lambda(s)$], whose Laplace *inverse* satisfies (†), for *all* dark energy singularities, is when $\mu = 1$.

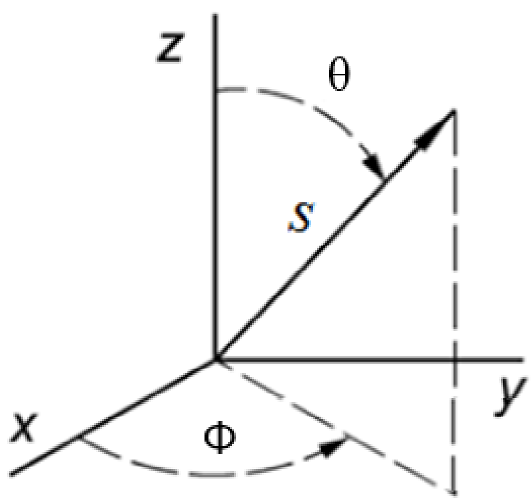
But from pages 365-9, we *also* now know that the *quantumlike* dark energy components satisfy (†), when $\mu = 1$, and so the *entire* gravitational model becomes fully *covariant*, in this case. Here again, is the layout for the field equations in the *intangible* space, taken from page 504, using *physical* coordinates, for the density function $\lambda(s) \approx \sigma / s$.

$$C^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha))g^{u,v}(\delta\cos(\alpha)) + 2\cosh(\delta r\sin(\beta))J_0(\delta r\cos(\beta))g^{u,v}(\delta\sin(\beta)) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))] .$$

And here, $\cos(\alpha) = \sin(\theta)\cos(\phi)$ and $\sin(\beta) = \sin(\theta)\sin(\phi)$, and the *physical* singularities associated with the *underlying* dark energy density function $\lambda(s) = \sigma / s$, are at the origin O, and at

$$S = \{(\pm\delta, 0, 0), (0, \pm\delta, 0), (0, 0, \pm\delta)\} .$$

As well, $C^{u,v}$ is the Einstein *curvature* tensor, and $C^{u,v} = R^{u,v} - ½ R g^{u,v}$, where $R^{u,v}$ is the Ricci tensor and R the Ricci scalar.



Physical Coordinate System

It is a minor miracle to me, at least, that we are *even* able to find such a density function [$\lambda(s)$], which ultimately leads to a fully *covariant* model in the *intangible* space, described by the equations above …

Some More On The Inner Block In The Intangible Space When $\lambda(s) \approx \sigma / s$

Bringing back our *coupled* equations from the previous page, in the *intangible* space and labelling as (*),

$$C^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha))g^{u,v}(\delta\cos(\alpha)) + 2\cosh(\delta r\sin(\beta))J_0(\delta r\cos(\beta))g^{u,v}(\delta\sin(\beta)) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta))]$$

we see, as in previous notes, there are *two* approaches to solving (*). We can try the 'in one go' method, where the template for $g^{u,v}$ encompasses *all* the components on the *right*-hand side above; or we can try the 'component by component' method, which is the better approach, in my view.

If we go with the *first* method, and opt for the default inner block $\mathcal{B} = [r^2, \; r^2\sin^2(\theta)]$, in covariant form, then this is what we will have as part of the *total* solution to (*); that is to say

$$g_{u,v} = \sigma \cdot [\, g_{r,r}(r, \theta, \phi, \delta) \, , r^2, \; r^2\sin^2(\theta) \, , -g_{t,t}(r, \theta, \phi, \delta) \,] \, , \qquad (\sim)$$

where the bracketed part in (~) is the *normalized* solution $\mathcal{N}$ to (*) [pp 522-3], when $\sigma = 1$.

On the other hand, if we go with the *second* method, and again opt for the default inner block $\mathcal{B}$, then we arrive at a *total* normalized solution $\mathcal{N}'$ to (*), by *adding* the normalized component solutions together. And here, each *normalized* component solution will contain $\mathcal{B}$, but because $\mathcal{B}$ in $\mathcal{N}$ *must* now agree with $\mathcal{B}$ in $\mathcal{N}'$, we see in the second method, that we only *add* to the *inner* block when summing *normalized* component solutions, *if* the entry here is *uniquely* different than $\mathcal{B}$, which is the default, in this case.

In other words, the sphere of radius r is *still* a sphere of radius r, for *both* methods, when looking at *normalized* solutions; because the sphere *is* $\mathcal{B}$, if $\mathcal{B} = [r^2, \; r^2\sin^2(\theta)]$.

And this is a general principle we developed in previous notes, when looking at the *second* method. However, a total solution $\mathcal{N}'$ will almost surely include the *inner* block in its calculations, for the *quantumlike* terms in (*); so that the *uniqueness* of $\mathcal{B}$ for each term [which is *no* longer the default], is virtually guaranteed, when summing the *normalized* component solutions together, in this case.

And from pages 522-3, we noted there that for the *first* term on the right-hand of (*), we could let $\mathcal{B}$ default to $[r^2, \; r^2\sin^2(\theta)]$ in $\mathcal{N}'$.

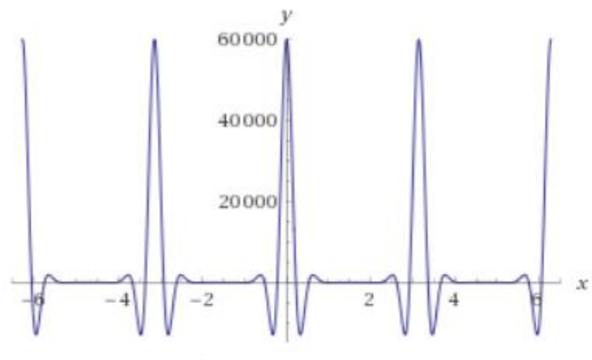
Still More On The Inner Block In The Intangible Space When $\lambda(s) \approx \sigma / s$

Let us bring back our *coupled* equations in *two* dimensions, for the *intangible* space, where there are singularities at O, and at $\{(\pm\delta, 0) , (0, \pm\delta)\}$.

$$C^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g^{u,v}(\delta\cos(\theta)) + 2\cosh(\delta r\sin(\theta))J_0(\delta r\cos(\theta))g^{u,v}(\delta\sin(\theta))] \quad (*)$$

For the *normalized* solution ($\sigma = 1$), we'll set the inner block [$\mathscr{B}$] to r^2, in *covariant* form, for the *first* component on the *right*-hand side of (*), and we'll presume $\mathscr{B}$ is *included* in the calculations for the *quantumlike* terms, again for the *normalized* solutions to (*).

Now for the *first* quantumlike term [$\mathscr{Q}_1$], the *radial* fluctuations are strongest at $\theta = 0$ and π, as per the picture below [p 496],



angular fluctuations for a given radius r

so geometrically, $\mathscr{B}$ might look more like an ellipse, in this case, as shown below in *blue*, on the next page. Thus, $\mathscr{B}$ will inevitably have a dependency on θ here.

The *black* circle is our default inner block [r^2], associated with a radius r, and we'll suppose here a *total* solution can be obtained, by summing over component solutions in (*) [see page 526].

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Note that the radial fluctuations *grow* in amplitude, as r increases, and that the *angular* fluctuations shown above, are really a *cross-sectional* view of the *radial* fluctuations, for a given radius r.

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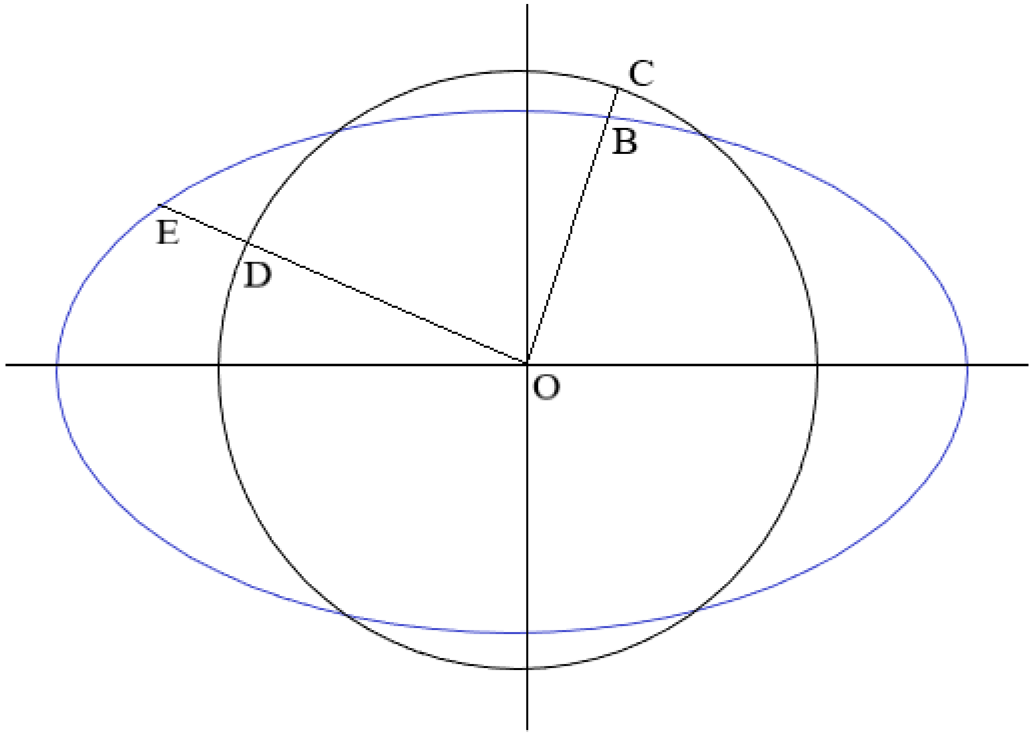


To add the two inner blocks together, in the picture above, we only add to the *default* inner block [r^2, black circle $\mathcal{C}$], if the entry [blue ellipse $\mathcal{E}$] is *uniquely* different than r^2. Thus, at E in the diagram, we would *add* DE to r^2, according to the formula (at least symbolically)

$$OE^2 - OD^2 = (OD + DE)^2 - OD^2 = 2OD \cdot DE + DE^2 ,$$

since terms are squared in $\mathcal{B}$, and OD = r. Note that if DE = 0, then we add nothing to the default inner block – it remains at r^2. Thus, the expression just above is *added* to r^2.

And similarly at B in the diagram, but here we are *subtracting* from r^2, because the ellipse is *below* $\mathcal{C}$. Thus, we would subtract BC from r^2, where OC = r, according to the formula (symbolically)

$$OC^2 - OB^2 = (OB + BC)^2 - OB^2 = 2OB \cdot BC + BC^2 ,$$

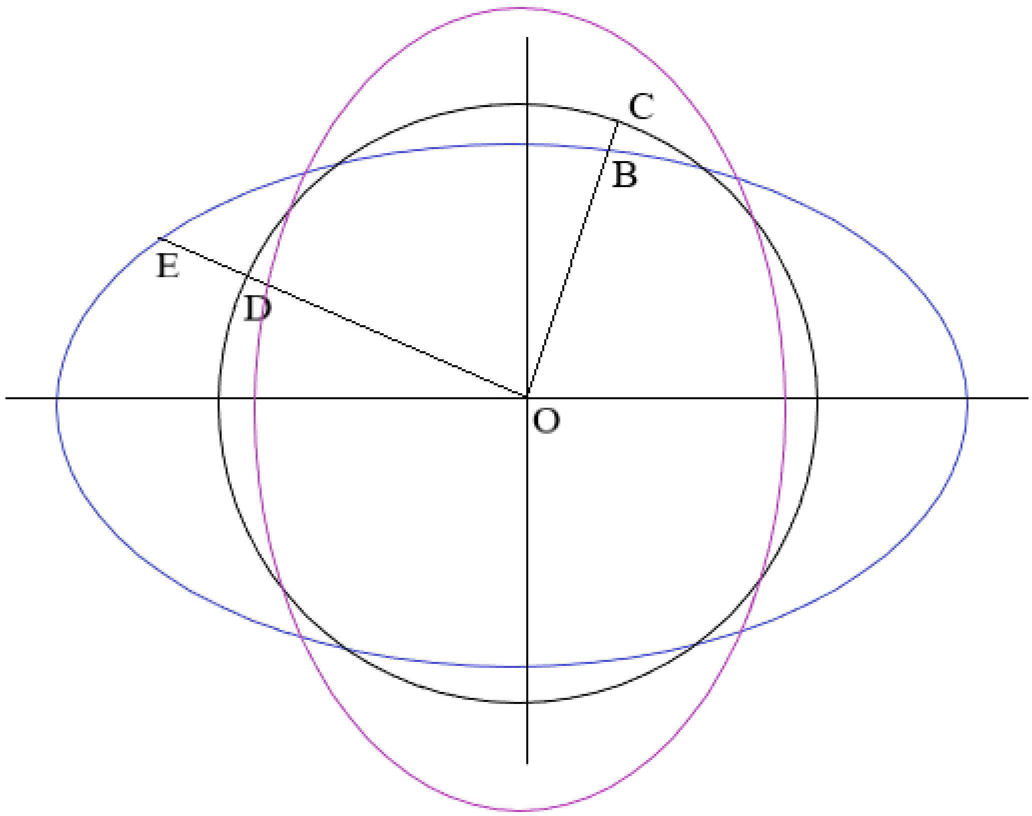
and again, if BC = 0, then we subtract nothing from the default inner block – it remains at r^2. Thus, the expression just above is *subtracted* from r^2.

A similar exercise can be carried out for the *second* quantumlike component [$\mathcal{Q}_2$] in (*), but now the *inner* block associated with this term will be the blue ellipse in the diagram above, *rotated* by 90 degrees.

———

This simple exercise seems to indicate that adding the inner block for $\mathcal{Q}_1$ [blue ellipse, $\mathcal{E}$] to the default inner block [black circle, $\mathcal{C}$], leaves us *solely* with the blue ellipse ! The default inner block

has been subsumed by $\mathcal{E}$, as it were, according to the rule of *uniqueness*, by which we add inner blocks together. And if we were to add the inner block for $\mathcal{Q}_2$ to $\mathcal{C}$, which is simply the blue ellipse rotated by 90 degrees [$\mathcal{E}'$], we'd obtain the same result; namely $\mathcal{E}'$.



In the picture above, we add the blue ellipse [$\mathcal{Q}_1$] to the black circle *uniquely*, which leaves us with just the blue ellipse. We then repeat the exercise for [$\mathcal{Q}_2$], which leaves us with just the purple ellipse. The *inner* block for the *total* normalized solution to (*) [p 527] thus becomes the two ellipses, and the black circle [default inner block] vanishes.

This is all seems somewhat counter-intuitive, but if we adhere to our principle of *uniqueness* under addition, using this method, then this is what we end up with.

———

On the other hand, if we treat the *three* inner blocks in the picture above as being *unique*, relative to each other, then under this definition of uniqueness, we could argue that *all* three should be added together, when looking for a *total* normalized solution to (*), by summing *normalized* component solutions together. Such an argument follows, on the next page …

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Bringing back our field equations from page 527, in *covariant* form,

$$C_{u,v} \approx \sigma[g_{u,v}(0) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g_{u,v}(\delta\cos(\theta)) + 2\cosh(\delta r\sin(\theta))J_0(\delta r\cos(\theta))g_{u,v}(\delta\sin(\theta))] \quad (*)$$

we see that as δ ---› 0, this resolves as

$$C_{u,v} \approx 5\sigma \cdot g_{u,v}(0) \,. \quad (\dagger)$$

And setting $\sigma = 1/5$, we obtain a *normalized* solution to (†), which *will* contain the *default* inner block $\boldsymbol{\mathscr{B}} = r^2$ [pp 522-3, 526].

So if we opt for a *total normalized* solution $\mathscr{T}$ to (*), by summing over the *normalized* component solutions for the *right*-hand side of this equation; we see immediately that *if* the *quantumlike* solutions are *removed* from $\mathscr{T}$, then necessarily the default inner block $\boldsymbol{\mathscr{B}} = r^2$ must remain, by what we just said above.

Thus when forming $\mathscr{T}$, we *must* include the default inner block $\boldsymbol{\mathscr{B}} = r^2$, when computing the *total* inner block for $\mathscr{T}$; or said another way, the black circle *and* the two ellipses, in the diagram on the previous page, matter.

———

Some clarification is needed, concerning the *normalized* solution to (†) above. We already know the *general* solution to (†) is going to be $g_{u,v}(r, 5\sigma)$, and by our associative principle under addition, relative to σ [pp 522-3], becomes

$$g_{u,v}(r, 5\sigma) = 5 \cdot g_{u,v}(r, \sigma) = 5\sigma \cdot g_{u,v}(r, 1) \,.$$

Now notice that $g_{u,v}(r, \sigma)$ *is* the *general* solution to (†), when 5σ has been replaced by σ; that is to say

$$C_{u,v} \approx \sigma \cdot g_{u,v}(0) \,, \quad (\ddagger)$$

so that, in fact, the *normalized* solution $\boldsymbol{\mathscr{N}}$ to (†) is $g_{u,v}(r, 1)$ in (‡), when $\sigma = 1$. And the *general* solution to (†) is then $5\sigma \cdot \boldsymbol{\mathscr{N}}$.

We alluded to this by setting $\sigma = 1/5$ in (†) at the top of the page, but I don't think that was enough to clearly explain what is actually going on here. Hopefully this additional note clarifies things …

Still More On The Underlying Dark Energy Density Function $\lambda(s) \approx \sigma / s$

From page 524, we stated that for a singularity at O, the Laplace inverse of $\lambda(s) \approx \sigma / s^{\mu}$, for $\mu > 0$, computes to

$$\sigma\{1 / \Gamma(\mu)\} \cdot r^{(\mu - 1)} ,$$

where r is our radius and $\Gamma(\mu)$ is the gamma function. Here, we'll show that this is indeed true.

Now from our general treatment of Laplace inversion techniques [pp 274-77], we showed that for densities of type $\lambda(s) \approx \sigma / s^{\mu}$, with $\mu > 0$, the following is a true statement,

$$\Gamma(\nu + ½)u^{-2\nu - 1} \text{ ---› Laplace Inverse ---› } \sqrt{\pi}\,(2\alpha)^{-\nu}r^{\nu}J_{\nu}(\alpha r)\ ,\ u = \sqrt{s^2 + \alpha^2}\ ,\ \mathrm{Re}(\nu) > -1/2 \qquad (*)$$

where $\Gamma()$ is the gamma function, $J_{\nu}()$ is a Bessel function of the *first* kind, of order ν, and we have $\mu = 2\nu + 1$.

And from page 233, we also know that

$$J_{\nu}(z) \text{ ---› } (z/2)^{\nu} / \Gamma(\nu + 1) \text{ as } z \text{ ---› } 0\ , \qquad (\dagger)$$

so letting α ---› 0 in (*), and making use of (†), we find that (*) becomes

$$\Gamma(\nu + ½)u^{-2\nu - 1} \text{ ---› Laplace Inverse ---› } \sqrt{\pi}\,(r/2)^{2\nu} / \Gamma(\nu + 1)\ ,\ u = s\ . \qquad (\ddagger)$$

And since $\mu = 2\nu + 1$, we can rewrite (‡) as

$$1 / s^{\mu} \text{ ---› Laplace Inverse ---› } \sqrt{\pi}\, r^{(\mu - 1)} / \{2^{(\mu - 1)}\, \Gamma(\mu / 2)\, \Gamma((\mu + 1) / 2)\}; \qquad (\wedge)$$

and now, using the Legendre *duplication* formula for the gamma function; which is

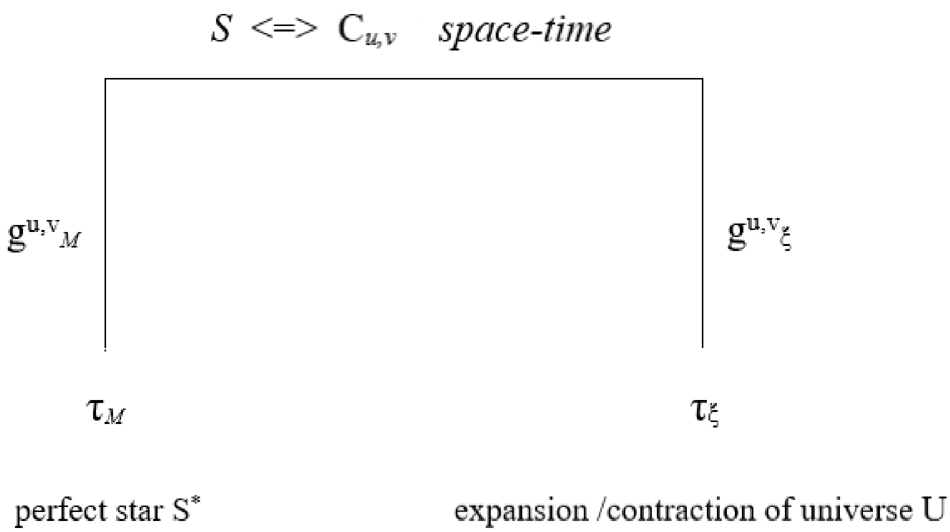
$$\Gamma(z)\, \Gamma(z + ½) = 2^{(1 - 2z)} \sqrt{\pi}\, \Gamma(2z)\ , \qquad (\sim)$$

we see that by letting $z = \mu / 2$ in (~), it is the case that on the *right*-hand side, (^) computes to

$$\{1 / \Gamma(\mu)\} \cdot r^{(\mu - 1)} .$$

And this is the desired result.

A Summary Of Where We Are With The *Coupled* Field Equations, When $\lambda(s) \approx \sigma / s$



In the diagram above, the *fabric* of space-time [$\mathcal{S}$], which is to be identified with the Einstein Curvature tensor $C_{u,v}$, is comprised of *two* spaces; namely, the *tangible* space $\tau_{\mathcal{M}}$ and the *intangible* space τ_{ξ} . An observer $\mathcal{O}$ in $\mathcal{S}$ can witness events sourced from *either* $\tau_{\mathcal{M}}$ or τ_{ξ}, like a *perfect* star [S*] influencing our perception of *space* and *time* in $\tau_{\mathcal{M}}$, or dark energy singularities in τ_{ξ}, that are ultimately responsible for the *expansion* and *contraction* of our universe $\mathcal{U}$.

The complete *notational* form for the *coupled* field equations, taken from page 504, in *physical* coordinates, is [labelling as (*)]

$$
\begin{aligned}
C_{u,v} \approx kT_{u,v} + \sigma[g_{u,v}(0) + 2\cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha))g_{u,v}(\delta\cos(\alpha)) &+ \\
2\cosh(\delta r\sin(\beta))J_0(\delta r\cos(\beta))g_{u,v}(\delta\sin(\beta)) &+ \\
2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g_{u,v}(\delta\cos(\theta))] &.
\end{aligned}
$$

And here, $\cos(\alpha) = \sin(\theta)\cos(\phi)$ and $\sin(\beta) = \sin(\theta)\sin(\phi)$, and the *physical* singularities associated with the *underlying* dark energy density function $\lambda(s) = \sigma / s$, are at the origin O, and at

$$S = \{(\pm\delta, 0, 0), (0, \pm\delta, 0), (0, 0, \pm\delta)\} .$$

As well, $C_{u,v}$ and $T_{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; and $C_{u,v} = R_{u,v} - ½Rg_{u,v}$, where $R_{u,v}$ is the Ricci tensor and R the Ricci scalar.

To operate in $\tau_{\mathcal{M}}$, set $\sigma = 0$ in (*) and solve for $g_{u,v}$, where the *tangible* source is S*. This will yield the vintage Schwarzschild solution, subject now to a reconsideration of the *time* component, for the *interior* case [pp 383-460]. The inner block $\mathcal{B}$ is the default $[r^2,\ r^2\sin^2(\theta)]$, in this scenario.

To operate in τ_ξ , set $k = 0$ in (*) and solve for $g_{u,v}$, component by component, with $\sigma = 1$, to obtain four *normalized* solutions. The *first* normalized solution is associated with $g_{u,v}(0)$, on the *right*-hand side of (*), and here the *inner* block $\mathscr{B}$ is the default; namely $[r^2 ,\ r^2 \sin^2(\theta)]$.

For the remaining *three* quantumlike components in (*), you can choose the inner block $\mathscr{B}$ for *each* to be the *default*, or you can include the *inner* block for *each*, in the calculations. If you choose the former, then the inner block for the *total* normalized solution $\mathcal{N}$ to (*), across *all* four *normalized* component solutions, added together, will *also* be the default; that is to say $[r^2 ,\ r^2 \sin^2(\theta)]$.

If you decide to include the *inner* block in the calculations for the *three* quantumlike components, then these inner blocks will be *rotations* of one another, just like the *normalized* component solutions; and thus *uniquely* different, relative to each other, *and* relative to the *default* inner block, also. As such, *all* four *inner* blocks are to be *added* together, in this case, when calculating $\mathcal{N}$, and the *general* solution to (*) in τ_ξ is now $\sigma \cdot \mathcal{N}$.

———

Several principles were developed along the way, to help with interpreting (*) correctly in τ_ξ. Among them, the Singularity Invariance Postulate [p 461 *ff.*], the *associative* principle under addition relative to σ [p 480 *ff.*], *normalized* solutions ($\sigma = 1$), the principle of *uniqueness* when adding normalized *inner* blocks together in τ_ξ, the Laplace *inverse* of the dark energy density function $\lambda(s) \approx \sigma / s$, which *must* satisfy the Laplace operator (as it does, pp 524-5), the *intangible* vacuum V_ξ, which has *nothing* in it, and the *generalized* solution, which is $\sigma \cdot \mathcal{N}$, as noted above.

And perhaps *most* significant of all, a recognition that (*) *must* be solved separately in both $\tau_{\mathcal{M}}$ and τ_ξ, and that in τ_ξ, a *sum* over *normalized* component solutions is probably the best approach …

———

Let us bring back our *coupled* field equations below, in *physical* coordinates [p 532], and label them as (*). In the *intangible* space we have,

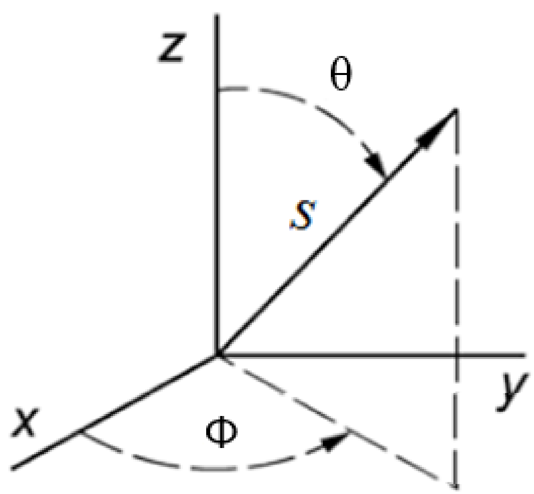
$$C_{u,v} \approx \sigma[g_{u,v}(0) + 2\cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha))g_{u,v}(\delta\cos(\alpha)) + 2\cosh(\delta r\sin(\beta))J_0(\delta r\cos(\beta))g_{u,v}(\delta\sin(\beta)) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g_{u,v}(\delta\cos(\theta))]\ .$$

And here, $\cos(\alpha) = \sin(\theta)\cos(\phi)$ and $\sin(\beta) = \sin(\theta)\sin(\phi)$, and the *physical* singularities associated with the *underlying* dark energy density function $\lambda(s) = \sigma / s$, are at the origin O, *and* at

$$S = \{(\pm\delta, 0, 0), (0, \pm\delta, 0), (0, 0, \pm\delta)\}\ .$$

As well, $C_{u,v}$ and $T_{u,v}$ are the Einstein *curvature* and relativistic *stress* tensors, respectively; and $C_{u,v} = R_{u,v} - ½Rg_{u,v}$, where $R_{u,v}$ is the Ricci tensor and R the Ricci scalar.

Now we want to emphasize again, that the gravitational tensors on the *right*-hand side of (*) are *uniquely* bound to their corresponding dark energy components, and that the argument to $g_{u,v}$ is a *radial* measure along the line $\boldsymbol{\ell}_{\theta\phi}$. The *last* quantumlike component in (*) has *no* dependency on ϕ, which is actually a *manifestation* of the coordinate system we are working in.



Physical Coordinate System

If the *normalized* component solutions ($\sigma = 1$) for the *quantumlike* terms in (*) are $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$, respectively; then $\mathcal{N}_2$ is a 90° rotation of $\mathcal{N}_1$ in the *x-y* plane, and $\mathcal{N}_3$ is a 90° rotation of $\mathcal{N}_1$ in the *x-z* plane *or* a 90° rotation of $\mathcal{N}_2$ in the *y-z* plane. The same is true for their *inner* blocks.

An Interesting Identity Concerning $\mathcal{N}$ Solutions

The *normalized* [$\mathcal{N}$] solution for the equation below, in the *intangible* space, is obtained by setting $\sigma = 1$.

$$C_{u,v} \approx \sigma \cdot g_{u,v}(0) \qquad (*)$$

We write this solution as $g_{u,v}(r, 1)$, and the *general* solution to (*) is simply $\sigma \cdot \mathcal{N}$. Now from page 522, we know from the *associative* principle under addition, relative to σ, that

$$g_{u,v}(r, \gamma) = \gamma g_{u,v}(r, 1) \,.$$

Thus, for any $|\gamma| > 0$, it must be the case that

$$g_{u,v}(r, \gamma) \cdot g_{u,v}(r, 1/\gamma) = g_{u,v}(r, 1)^2 = \mathcal{N}^2 \,, \qquad (\dagger)$$

or said another way, for all $|\gamma| > 0$,

$$\mathcal{N} = \sqrt{g_{u,v}(r,\gamma) \cdot g_{u,v}(r,1/\gamma)} \,.$$

If we define

$$g_{u,v}(r, 0) = \lim g_{u,v}(r, \gamma) \text{ as } \gamma \dashrightarrow 0 \text{ and } g_{u,v}(r, \infty) = \lim g_{u,v}(r, 1/\gamma) \text{ as } \gamma \dashrightarrow 0 \,,$$

then it is *also* the case from (†), that for *all* $r > 0$,

$$g_{u,v}(r, 0) \cdot g_{u,v}(r, \infty) = \mathcal{N}^2 \,.$$

In particular then, and from (*); for the *normalized* ($\sigma = 1$) *default* inner block $\mathcal{B} = r^2$ in a *two* dimensional model, where $u = v = 2$; since we already know $g_{u,v}(r, 0) = 0$ and $g_{u,v}(r, \infty) = \infty$, we can translate the expression just above, as

$$0 \cdot \infty = r^4 = \mathcal{B}^2 \,.$$

Such *non-standard* operations may prove to be useful down the road, as we learn more about how to solve these coupled equations in the *intangible* space, and how we might interpret ‘in the limit’ operations, as well …

A Little More On Rotating $\mathcal{N}$ Solutions

Let us bring back our *coupled* field equations below, in *physical* coordinates [p 534], and label them as (*). In the *intangible* space we have,

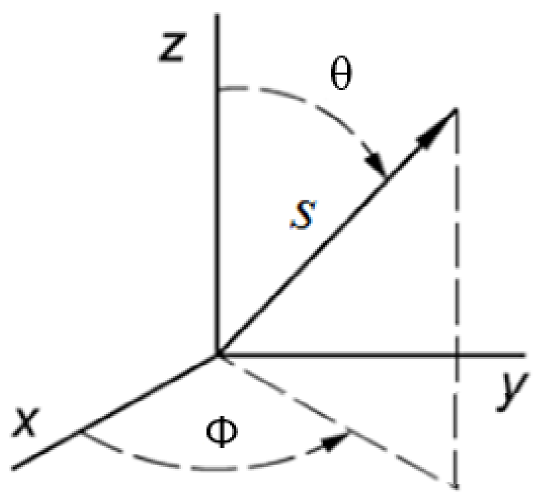
$$C_{u,v} \approx \sigma[g_{u,v}(0) + 2\cosh(\delta r\cos(\alpha))J_0(\delta r\sin(\alpha))g_{u,v}(\delta\cos(\alpha)) + 2\cosh(\delta r\sin(\beta))J_0(\delta r\cos(\beta))g_{u,v}(\delta\sin(\beta)) + 2\cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))g_{u,v}(\delta\cos(\theta))] \; .$$

And here, $\cos(\alpha) = \sin(\theta)\cos(\phi)$ and $\sin(\beta) = \sin(\theta)\sin(\phi)$, and the *physical* singularities associated with the *underlying* dark energy density function $\lambda(s) = \sigma / s$, are at the origin O, *and* at

$$S = \{(\pm\delta, 0, 0), (0, \pm\delta, 0), (0, 0, \pm\delta)\} \; .$$

And again, from page 534 we said …

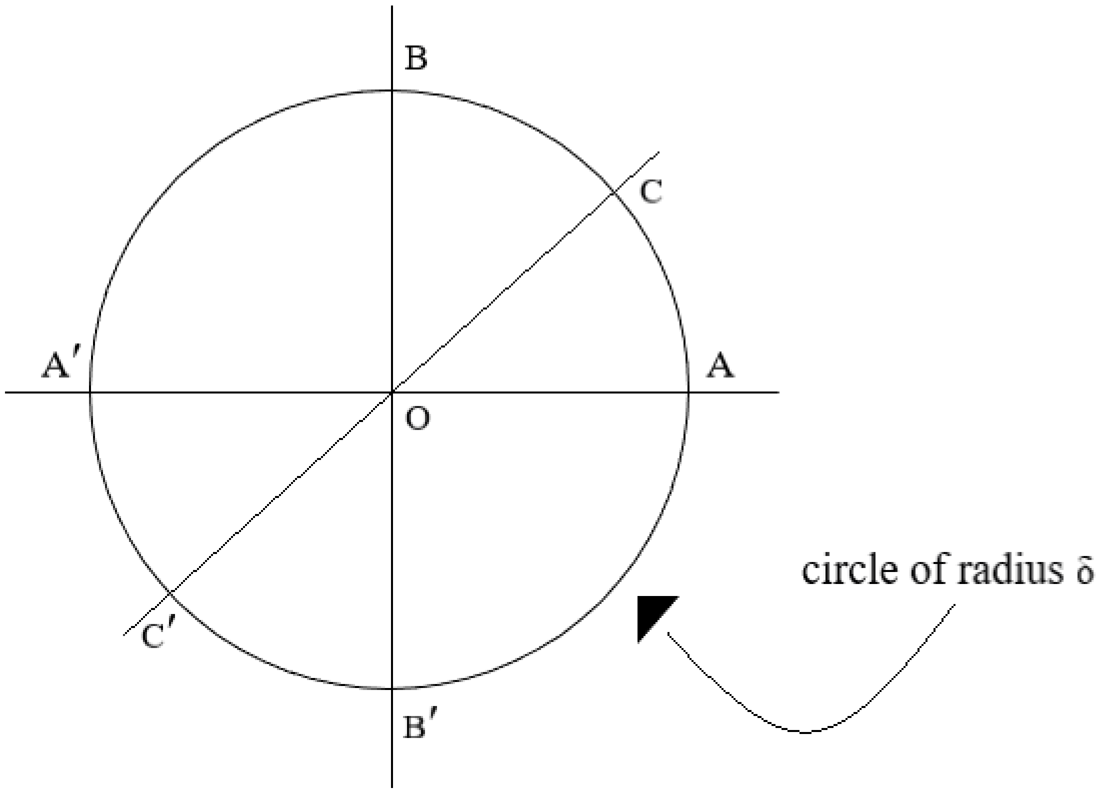
If the *normalized* component solutions ($\sigma = 1$) for the *quantumlike* terms in (*) are $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$, respectively; then $\mathcal{N}_2$ is a 90° rotation of $\mathcal{N}_1$ in the *x-y* plane, and $\mathcal{N}_3$ is a 90° rotation of $\mathcal{N}_1$ in the *x-z* plane *or* a 90° rotation of $\mathcal{N}_2$ in the *y-z* plane. The same is true for their *inner* blocks.



Physical Coordinate System

The rotation of $\mathcal{N}_1$ into $\mathcal{N}_2$, in the *x-y* plane, is actually a rotation of the ϕ variable in $\mathcal{N}_1$ – the θ variable in $\mathcal{N}_1$ is left alone. The rotation of $\mathcal{N}_1$ into $\mathcal{N}_3$ in the *x-z* plane, necessarily means that ϕ *must* be set to *zero* in $\mathcal{N}_1$, since $\mathcal{N}_3$ has *no* dependency on ϕ, *and* the rotation itself, must map the singularities located at $(\pm\delta, 0, 0)$ to $(0, 0, \pm\delta)$. This can only happen if $\phi = 0$, and similarly, the rotation of $\mathcal{N}_2$ into $\mathcal{N}_3$ in the *y-z* plane, requires that we set $\phi = 90^{\circ}$ in $\mathcal{N}_2$.

Dark energy singularities likely come in *pairs*, save for the origin O of our coordinate system [see, for example, pages 168-96]; so in the diagram below imagine a *ring* of singularities, symmetrically positioned on the circle $\mathcal{C}$ of radius δ, for the *two* dimensional model.



The *pair* of singularities at [A, A′], for example, would correspond to (±δ, 0), and we could suppose there are *n* such pairs on $\mathcal{C}$, where *n* is any integer *greater* than *zero*.

The question now becomes one of how to solve the *coupled* equations in the *intangible* space, when *n* becomes large, or even *very* large. If we decide to use the 'in one go' method, we'll soon find out that it becomes an *intractable* problem in this case, because in the equation below, there will be *n* terms for the *quantumlike* entries that have to be incorporated into a *single* template for $g_{u,v}$.

$$C_{u,v} \approx \sigma[g_{u,v}(0) + \xi_1 \cdot g_{u,v}(\mathrm{arg}_1) + \xi_2 \cdot g_{u,v}(\mathrm{arg}_2) + \ldots] \quad (\dagger)$$

On the other hand, if we opt for a sum over *normalized* component solutions for the quantumlike terms [where ξ_i is the *i*-th dark energy component]; then we only need solve for $i = 1$ in (†) [which we'll call $\mathcal{N}_1$], since *all other* solutions will be a *rotation* of $\mathcal{N}_1$, in the *two* dimensional plane. And here, we can let $i = 1$ correspond to (±δ, 0).

And to me, at least, this makes much more sense, and is a strong argument for choosing the *second* method over the *first*, especially if *n* is very large.

Rotating Our Harmonic Expression For The Equivalency Theorem In The *Coupled* Case

Let us recall our *equivalency* theorem, in the *intangible* space, from pages 468-9, and reproduced here; where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and $\boldsymbol{\mathcal{E}} = g^{u,v}(\alpha) \cdot J_0(r\varepsilon)$ …

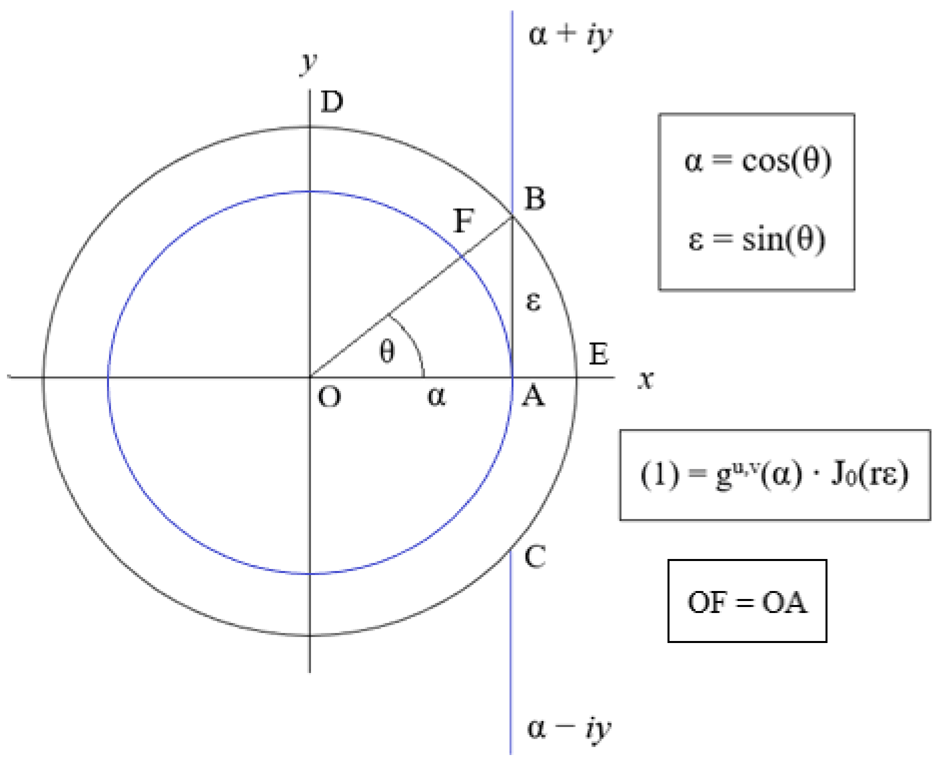
$$(1) = g^{u,v}(\alpha) \cdot J_0(r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true} \quad (*)$$

Here, the following expression is (1),

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\alpha + iy) - g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) + g^{u,v}(\alpha - iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

and the expression below is (2), where the dark energy singularities are at O and at (±1, 0) …

$$C^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)]$$



From our diagram above, if we rotate B into E on the *unit* circle, then $\alpha = 1$ and $\varepsilon = 0$ in (1), and logic tells us we should see it the *same* way for the arguments associated with expression $\boldsymbol{\mathcal{E}}$. Similarly, if we rotate B into D, then $\alpha = 0$ and $\varepsilon = 1$ in (1), and again, logic tells us we should see it the *same* way for the arguments associated with $\boldsymbol{\mathcal{E}}$. Seen this way, within the context of rotations, it becomes easier to see how the argument to $g^{u,v}$ in $\boldsymbol{\mathcal{E}}$; namely α, becomes a *radial* measure [OF] along the line $\boldsymbol{\ell}_\theta$; whilst in (1), α and y are always Euclidean coordinates in the *complex* plane.

Calculating The Harmonic Expression For The Equivalency Theorem When $\theta = \pi / 2$

Let us recall our *equivalency* theorem from the previous page, in the *intangible* space, where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and $\mathcal{E} = g^{u,v}(\alpha) \cdot J_0(r\varepsilon)$ …

$$(1) = g^{u,v}(\alpha) \cdot J_0(r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true} \qquad (*)$$

Here, the following expression is (1),

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g^{u,v}(\alpha + iy) - g^{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g^{u,v}(\alpha + iy) + g^{u,v}(\alpha - iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

and the expression below is (2), where the dark energy singularities are at O and at $(\pm 1, 0)$ …

$$C^{u,v} \approx \sigma[g^{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g^{u,v}(\alpha)] .$$

And from page 502, let us choose our template Ʈ for $g^{u,v}$ to be the following, where the sum is over all *even* integers, and $c_n(u, v)$ is a function of the variable θ, *only*.

$$\Sigma\, c_n(u, v)\, r^n P_n(\cos(\theta)) , \qquad n = 0, 2, 4, \ldots$$

Setting $\theta = \pi / 2$, we see that $\alpha = 0$ and $\varepsilon = 1$ in (1), and that Ʈ becomes $c_0(u, v)$ when $r = \cos(\theta)$, since this is now $g^{u,v}(\alpha)$ we are calculating in the equivalency (*), *and* $P_0(0) = 1$. But in (1), where again, $\alpha = 0$ and $\varepsilon = 1$, we are actually calculating $g^{u,v}(\pm iy)$ in Ʈ, and this computes to

$$\Sigma\, c_n(u, v)\, (iy)^n P_n(0) , \qquad n = 0, 2, 4, \ldots \qquad (\dagger)$$

Now since we're summing over all *even* integers in (†), the *first* bracketed term in (1) associated with cos(yr), vanishes, leaving us with the expression ($\alpha = 0$, $\varepsilon = 1$, $\kappa = 1/2\pi\boldsymbol{i}$)

$$4\kappa\boldsymbol{i}\, \Sigma \int_{\varepsilon}^{\infty} c_n(u, v)\, (-1)^{n/2} P_n(0)\, y^n \sin(yr)\, dy \,/\sqrt{y^2 - \varepsilon^2} , \qquad n = 0, 2, 4, \ldots$$

Integrals of this type, just above, will *only* converge if $c_n(u, v) = 0$ for *all* even $n > 0$, in which case the expression above evaluates to $c_0(u, v) \cdot J_0(r)$, and this agrees with $\mathcal{E}$ in (*) when $\alpha = 0$, $\varepsilon = 1$. A validation of our template Ʈ, you might say. Otherwise, no such validation of Ʈ is possible, in which case we have to look for other forms for Ʈ.

When $\theta = 0$, so that $\alpha = 1$ and $\varepsilon = 0$, we know from past research that (1) computes to $g^{u,v}(\alpha)$, and using our template Ʈ above, this is

$$\Sigma\, c_n(u, v)\, r^n P_n(\cos(\theta)) \;=\; \Sigma\, c_n(u, v)\,, \qquad n = 0, 2, 4, \ldots \qquad (\ddagger)$$

since r = 1 here, and $P_n(\cos(\theta)) = 1$ for all $n \geq 0$, when $\theta = 0$. And in the *complex* plane, for our template Ƭ, r computes to $(\alpha^2 + (iy)^2)^{1/2}$, so that (1) becomes ($\alpha = 1$, $\varepsilon = 0$, $\kappa = 1/2\pi\boldsymbol{i}$)

$$4\kappa\boldsymbol{i}\, \Sigma \int_{\varepsilon}^{\infty} c_n(u, v)\, P_n(1)\, (\alpha^2 - y^2)^{n/2} \sin(yr)\, dy \,/\sqrt{y^2 - \varepsilon^2}\,, \qquad n = 0, 2, 4, \ldots$$

Just as on the previous page, integrals of this type will *only* converge if $c_n(u, v) = 0$ for *all* even $n > 0$, in which case the expression above evaluates to $c_0(u, v)$, and this agrees with (‡) when $\alpha = 1$ and $\varepsilon = 0$. Again, a validation of our template Ƭ, you might say. Otherwise, no such validation of Ƭ is possible, in which case we have to look for other forms for Ƭ.

OTHER CONSIDERATIONS

When considering a template like

$$Ƭ \;=\; \Sigma\, c_n(u, v)\, r^n P_n(\cos(\theta))\,,$$

it is probably more correct to take the *absolute* value (or *modulus*) of r, when doing our calculations, because the argument to $g^{u,v}$ in (*), on the previous page [namely $\alpha = \cos(\theta)$], could well be *negative*. So if we rewrite our template as

$$Ƭ \;=\; \Sigma\, c_n(u, v)\, |\, r\, |^n P_n(\cos(\theta))\,, \qquad (\S)$$

and now carry this over to the *complex* plane, then for an expression like $g^{u,v}(\alpha \pm iy)$ in (1) on page 539, the modulus of r in our template computes to $(\alpha^2 + y^2)^{1/2}$. But either way, whether we do or do not use the modulus of r for Ƭ, our conclusions do not change – convergence in the integrals can't be achieved if $n > 0$.

STILL MORE CONSIDERATIONS

If we suppose that θ tracks r in the *complex* plane $\mathcal{C}$, as it does in the *two* dimensional plane $\mathcal{R}^2$, since r is a *radial* measure along the line ℓ_θ; then for our *quantumlike* component in (2) [p 539], using the template (§) above, the *sum* over integrals at the top of the page becomes ($\alpha = 1$, $\varepsilon = 0$, and $\kappa = 1/2\pi\boldsymbol{i}$) …

$$4\kappa\boldsymbol{i}\, \Sigma \int_{\varepsilon}^{\infty} c_n(\arccos \alpha\,/\,(\alpha^2 + y^2)^{1/2})\, P_n(\alpha\,/\,(\alpha^2 + y^2)^{1/2})\, (\alpha^2 + y^2)^{n/2} \sin(yr)\, dy\,/\sqrt{y^2 - \varepsilon^2}\,, \qquad (\sim)$$

where the sum extends over $n = 0, 2, 4, \ldots$ *and* c_n under the integral sign is $c_n(u, v)$ in (§), which we'll suppose, is an *even* function of θ.

Now suppose also that in (~), c_n is *zero* for all *even* $n > 0$. Then (~) reduces to

$$4\kappa \boldsymbol{i}\int_{\varepsilon}^{\infty} c_0(\mathit{arccos}\ \alpha / (\alpha^2 + y^2)^{1/2})\ P_0(\alpha / (\alpha^2 + y^2)^{1/2})\ \sin(yr)\ dy \,/\sqrt{y^2 - \varepsilon^2}\ , \qquad (\wedge)$$

and this must compute to $c_0(u, v)$ in (‡), where $\theta = 0$ [p 540]. Reducing (^) even further, by noting that P_0 is *always* 1, and $4\kappa\boldsymbol{i} = 2/\pi$, we see that (^) becomes ($\alpha = 1$, $\varepsilon = 0$) …

$$4\kappa \boldsymbol{i}\int_{\varepsilon}^{\infty} c_0(\mathit{arccos}\ \alpha / (\alpha^2 + y^2)^{1/2})\ \sin(yr)\ dy \,/\sqrt{y^2 - \varepsilon^2}\ . \qquad (\|)$$

Now since we *also* know that ($\alpha = 1$, $\varepsilon = 0$)

$$4\kappa \boldsymbol{i}\int_{\varepsilon}^{\infty} \sin(yr)\ dy \,/\sqrt{y^2 - \varepsilon^2} \ = \ 1\ ,$$

it is thus, reasonable to conclude, that *arccos* $\alpha / (\alpha^2 + y^2)^{1/2}$ in (‖) must be 0 as well, since again, (‖) must compute to $c_0(u, v)$ in (‡), where $\theta = 0$. But if this is so, then

$$\alpha / (\alpha^2 + y^2)^{1/2} \ = \ \cos(0) \ = 1\ ,$$

and since $\alpha = 1$, it must be the case that $y = 0$. In other words, we are actually in the *two* dimensional plane $\mathcal{R}^2$ *along* the x-axis, so that θ does *not* track r in the complex plane $\mathcal{C}$, after all.

So when computing the template below in $\mathcal{C}$

$$Ƭ \ = \ \Sigma\ c_n(u, v)\ |\mathrm{r}|^{\mathrm{n}}\, P_n(\cos(\theta))\ , \qquad (\S)$$

all we have to worry about is mapping the argument associated with $g^{u,v}(\alpha \pm iy)$ to $|\mathrm{r}|$; that is to say, $\alpha \pm iy$ … ..

A Few Additional Notes On The Harmonic Expression For Our Equivalency Theorem

Let us recall our *equivalency* theorem from page 539, in the *intangible* space, where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and here we'll recast it in *covariant* form …

$$(1) = g_{u,v}(\alpha) \cdot J_0(r\varepsilon) \text{ if and } only \text{ if (2) is true} \qquad (*)$$

Now the following expression is (1),

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\alpha + iy) - g_{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\alpha + iy) + g_{u,v}(\alpha - iy)]\}\,dy\,/\sqrt{y^2 - \varepsilon^2}$$

and the expression below is (2), where the dark energy singularities are at O and at (±1, 0) …

$$C_{u,v} \approx \sigma[g_{u,v}(0) + 2\cosh(r\alpha)J_0(r\varepsilon)g_{u,v}(\alpha)]\ .$$

The *first* thing to note is that (1) *only* applies to the *quantumlike* component in (2); that is to say

$$C_{u,v} \approx 2\sigma \cdot \cosh(r\alpha)J_0(r\varepsilon)g_{u,v}(\alpha)\ ,$$

and the *second* thing to note is that in (1), $\alpha \pm iy$ is to be interpreted as a *radial* measure in the *complex* plane $\mathcal{C}$, calculated from the template Ꞇ for $g_{u,v}$ in $\mathcal{R}^2$, like the one shown below.

$$Ꞇ = \Sigma\, c_n(u, v)\,|r|^n P_n(\cos(\theta))\ , \qquad n = 0, 2, 4, \ldots \qquad (\S)$$

For example, if Ꞇ was (§) above, then in $\mathcal{C}$, $g_{u,v}(\alpha \pm iy)$ would become (§), where $|r| = (\alpha^2 + y^2)^{1/2}$, since we now know that θ does *not* track r in the complex plane [p 541]. Thus, when calculating (1), we are *always* doing so in the *direction* of r, for *any* $g_{u,v}$, where here, in the *two* dimensional model, $u = v = 1$, 2, or 3 (r, θ, t).

Now if we associate $u = v = 2$ with the inner block $\mathcal{B}$, and choose the *default*, so that $g_{u,v} = r^2$, then in $\mathcal{C}$, $g_{u,v}(\alpha \pm iy)$ equates to $|r|^2 = (\alpha^2 + y^2)$, and we know in this case that (1) will *not* converge. We do this, of course, when calculating the *normalized* solution, where $\sigma = 1$.

Thus, (1) is telling us, perhaps in a rather indirect sense, that the default for $\mathcal{B}$ may *not* be the correct choice, which means, in turn, that the *inner* block should be included in the calculations, according to some template Ꞇ, just as we do for $u = v = 1$ or 3. And similarly for the *three* dimensional model, for the *quantumlike* components.

As to the *first* component in (2), on the *right*-hand side, the equation is

$$C_{u,v} \approx \sigma \cdot g_{u,v}(0)\ ,$$

and this has nothing to do with the equivalency theorem (*), above, as we said earlier. Thus, choosing the *default* inner block for the *normalized* solution ($\sigma = 1$), is acceptable.

A Few Additional Notes On The Harmonic Expression For Our Equivalency Theorem, Part II

From Part I [p 542] let us allow α *and* $\varepsilon \dashrightarrow 0$, so that (1) is equal to $g_{u,v}(0)$, and our equivalency reads, where *initially* the dark energy singularities are at O and at $(\pm 1, 0)$ …

$$(1) = g_{u,v}(0) \text{ if and } \textit{only} \text{ if (2) is true .} \quad (*)$$

Here (2) is

$$C_{u,v} \approx 3\sigma \cdot g_{u,v}(0) \,,$$

since the dark energy singularities at $(\pm 1, 0)$ have now *converged* to O.

Now (1) is ($\alpha = 0$, $\varepsilon = 0$, $\kappa = 1/2\pi\boldsymbol{i}$),

$$2\kappa \int_{\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\alpha + iy) - g_{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\alpha + iy) + g_{u,v}(\alpha - iy)]\}\; dy \,/ \sqrt{y^2 - \varepsilon^2}$$

so that for the case of the *inner* block $\mathscr{B}$, if we allow this to become the *default*; namely r^2, in the case of the *normalized* solution $\mathscr{N}$ to the following expression, where $\sigma = 1$ …

$$C_{u,v} \approx \sigma \cdot g_{u,v}(0) \,; \quad (\dagger)$$

then (1) will *not* converge for $g_{u,v} = r^2$, $u = v = 2$. Again, from the last page, $g_{u,v}(\alpha \pm iy)$ equates to $|r|^2 = y^2$ here, and (1) becomes ($\alpha = 0$, $\varepsilon = 0$, $\kappa = 1/2\pi\boldsymbol{i}$)

$$4\kappa\boldsymbol{i} \int_{\varepsilon}^{\infty} y \cdot \sin(yr)\; dy \,.$$

On the other hand, it seems most reasonable to let $\mathscr{B}$ default to r^2 in ($\dagger$) when $\sigma = 1$, using our *physical* intuition, so this could well be a case where the *physical* overrides the *mathematical*. In other words, while we would like (1) to converge as much as possible, where more generally, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$; there may be times when we can't make it happen, and have to defer to our intuition, instead. The example in this note may be such a case … ..

Seeing The Harmonic Expression Via The Laplace Inverse Of a Generating Function $f(s)$

Consider the following template Ꞇ for $g_{u,v}$ in (*) below, where again $c_n(u, v)$ is an *even* function of θ, and from page 542, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$ [the dark energy singularities for (*) are at $(\pm 1, 0)$].

$$C_{u,v} \approx 2\sigma \cdot \cosh(r\alpha)J_0(r\varepsilon)g_{u,v}(\alpha) \qquad (*)$$

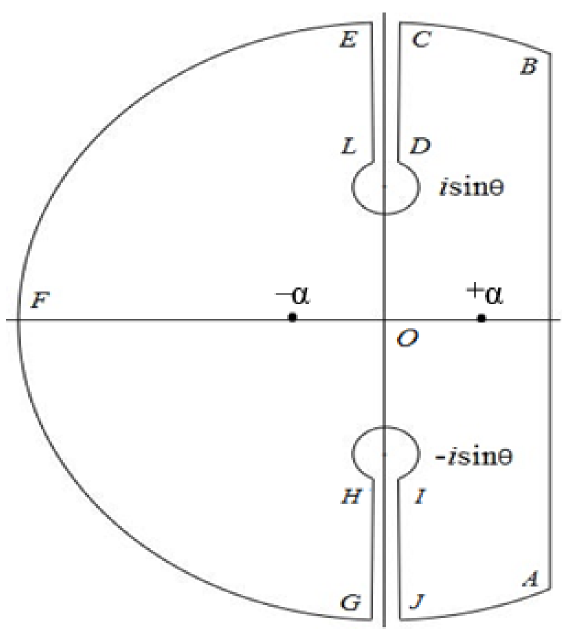
$$Ꞇ = \Sigma\, c_n(u, v)\,|r|^{-n}, \quad n = 0, 2, 4, \ldots \qquad (\S)$$

Then the harmonic expression (1) from page 542 becomes the following, and these integrals do *converge* for every *even* integer $n \geq 0$ ($\kappa = 1/2\pi\boldsymbol{i}$) …

$$4\kappa\boldsymbol{i}\, \Sigma \int_{\varepsilon}^{\infty} c_n(u, v)\, (\alpha^2 + y^2)^{-n/2} \sin(yr)\, dy \,/\sqrt{y^2 - \varepsilon^2}\,, \quad n = 0, 2, 4, \ldots \qquad (\dagger)$$

Now define the function $f(s, n)$ below, where s is a *complex* variable, and let us calculate the Laplace *inverse* of f along AB, using the contour that follows …

$$f(s, n) = 1 / \{(s^2 - \alpha^2)^{n/2}\sqrt{s^2 + \varepsilon^2}\}\,, \quad n = 0, 2, 4, \ldots$$



$\alpha = \cos(\theta)$

$\varepsilon = \sin(\theta)$

Adopting the methods on pages 274-77, we find that the Laplace inverse of $f(s, n)$ computes to

$$4\kappa\boldsymbol{i}\, (-1)^{n/2} \int_{\varepsilon}^{\infty} (\alpha^2 + y^2)^{-n/2} \sin(yr)\, dy \,/\sqrt{y^2 - \varepsilon^2} + R_n \quad n = 0, 2, 4, \ldots$$

which we'll define to be $\mathcal{L}^{-}$ $(f(s, n))$. The term R_n is the *residue* calculation from the *poles* at the points $(\pm\alpha, 0)$, and for $n = 0$, $R_n = 0$; and for $n = 2$, $R_n = \sinh(\alpha r) / \alpha\sqrt{\alpha^2 + \varepsilon^2}$.

Thus, when $n = 0$, $\mathcal{L}^{-}$ $(f(s, n))$ agrees with the *first* term in (†) [omitting R_n and $c_n(u, v)$], and when $n = 2$, $\mathcal{L}^{-}$ $(f(s, n))$ agrees with the *second* term in (†), up to sign [again, omitting R_n and $c_n(u, v)$]. And, in fact, this will be true for *all even* integers $n \geq 0$.

And so, we see how we can develop a *generating* function $f(s, n)$, and operate on it with the Laplace inverse [$\mathcal{L}^{-}$] to reproduce the integral components in (†) on the previous page. Such a technique may be useful to us down the road, as we search for templates that may work for (*) …

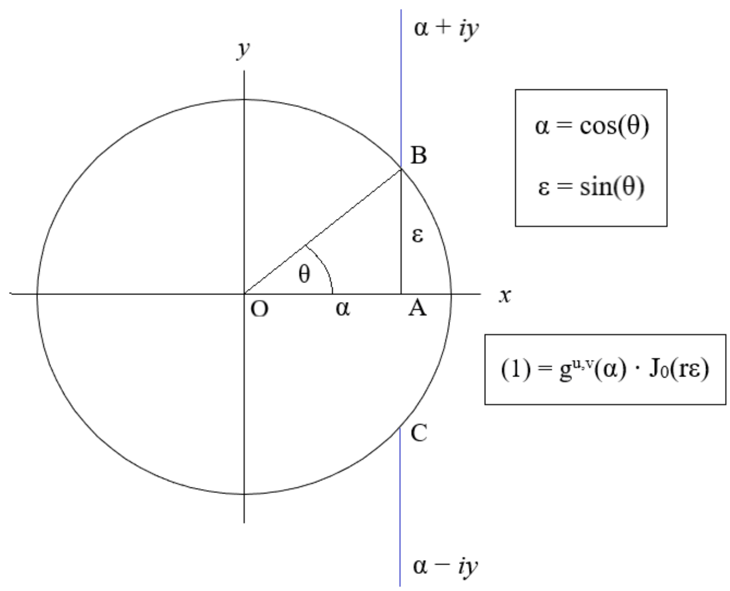
Understanding The Template Ꚍ In $\mathscr{R}^2$ Versus The Harmonic Expression In $\mathcal{C}$

Let us bring back our *hypothetical* template in $\mathscr{R}^2$, which is the *two* dimensional plane, and shown below, where $c_n(u, v)$ is an *even* function of θ *only* [pp 539-41] …

$$Ꚍ \;=\; \Sigma\, c_n(u, v)\, |r|^n P_n(\cos(\theta))\,, \quad n = 0, 2, 4, \ldots \quad (\S)$$

And let us recall our harmonic expression in the complex plane $\mathcal{C}$, in the *intangible* space, where here, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and we'll label this as (1) …

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\alpha + iy) \;-\; g_{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\alpha + iy) \;+\; g_{u,v}(\alpha - iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$



On the *unit* circle in the picture above, the template Ꚍ applies *equally* well, whether we are in $\mathscr{R}^2$ or $\mathcal{C}$, because, for example, at the point B, $|r| = 1$ and the angle is θ, in *both* spaces. But as we move *away* from B (or its counterpart at C), along the imaginary line ℓ_α in $\mathcal{C}$, heading *north* or *south*; and use the template Ꚍ to perform the integration in (1); the argument to $g_{u,v}$ in (1) becomes the *radial* expression $|r| = (\alpha^2 + y^2)^{1/2}$, as we know. Thus, the argument $\boldsymbol{a} = \alpha \pm iy$ to $g_{u,v}$, inherits this piece quite *naturally* from Ꚍ in $\mathscr{R}^2$, from a *radial* perspective.

But the corresponding angle $\gamma = \arccos \alpha / (\alpha^2 + y^2)^{1/2}$ associated with $\boldsymbol{a}$, *cannot* be inherited from Ꚍ, and is actually *separate* and *apart* from the angle θ you see in Ꚍ, *unless* you are on the *unit* circle in the picture above [pp 540-41].

So when interpreting $g_{u,v}(\alpha \pm iy)$ in (1), where we are in the *complex* plane, it is to be thought of as $g_{u,v}(r, \gamma)$ in $\mathscr{R}^2$, where here, $r = (\alpha^2 + y^2)^{1/2}$ and $\gamma = \arccos \alpha / (\alpha^2 + y^2)^{1/2}$. And this is the best we can do with the interpretation, yet it *still* allows us to comment on whether or not (1) *converges*.

Understanding The Template Ꚍ In $\mathcal{R}^2$ Versus The Harmonic Expression In $\mathcal{C}$, Part II

Let us begin this note, by making an observation concerning r versus |r|, using the Riemann zeta function [ζ(s)]. To wit, from pages 330-32, using a *variant* of our harmonic expression (1), from the previous page, we have from Wolfram …

$$\int_0^{300} -\frac{i\,(-\cos(y)\,(-\zeta(2-i\,y)+\zeta(2+i\,y))+i\sin(y)\,(\zeta(2-i\,y)+\zeta(2+i\,y)))}{\pi\,y}\,dy = 1.64477$$

$$\int_0^{300} -\frac{i\,(-\cos(y)\,(-\zeta(|2-i\,y|)+\zeta(|2+i\,y|))+i\sin(y)\,(\zeta(|2-i\,y|)+\zeta(|2+i\,y|)))}{\pi\,y}\,dy = 1.51926$$

The *first* expression above is the evaluation of $\zeta(2) = \pi^2/6 \sim 1.64493$, and you can see how closely the two match. However, in the *second* expression, where we take the *modulus* of the argument to ζ(s), you can see that we do *not* get a match with ζ(2), even though $\zeta(2) = \zeta(|2|)$.

From page 542, let us allow α *and* ε ---› 0, so that (1) *is always* equal to $g_{u,v}(0)$ [pp 330-2], and our equivalency reads, where *initially* the dark energy singularities are at O and at (±1, 0) …

$$(1) = g_{u,v}(0) \text{ if and } \textit{only} \text{ if (2) is true .} \quad (*)$$

Here (2) is

$$C_{u,v} \approx 3\sigma \cdot g_{u,v}(0)\,,$$

since the dark energy singularities at (±1, 0) have now *converged* to O.

Now (1) is ($\alpha = 0$, $\varepsilon = 0$, $\kappa = 1/2\pi \boldsymbol{i}$),

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\alpha + iy) - g_{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\alpha + iy) + g_{u,v}(\alpha - iy)]\}\,dy\,/\sqrt{y^2 - \varepsilon^2}$$

and the *normalized* solution $\mathcal{N}$ to (2) [$C_{u,v} \approx \sigma \cdot g_{u,v}(0)$ and $\sigma = 1$] will be a function of r *only*; so that when evaluating $g_{u,v}(\alpha \pm iy)$ in (1), in this case [$\alpha = 0$, $\varepsilon = 0$, $\kappa = 1/2\pi\boldsymbol{i}$], it *behooves* us to take a lesson learned from the ζ integrals, at the top of the page.

For just like ζ(s), which is a function of *one* variable and *real-valued* along the *x*-axis, so is the *g*-matrix associated with $\mathcal{N}$. That is to say, any element $g_{u,v}$ in *g* will *also* be *real-valued*, and a function of r only.

Thus, when evaluating $g_{u,v}(\alpha \pm iy)$ in (1), in this case [$\alpha = 0$, $\varepsilon = 0$, $\kappa = 1/2\pi\boldsymbol{i}$], we should calculate $g_{u,v}(\pm iy)$ *as is*, as opposed to calculating $g_{u,v}(|\pm iy|)$. For to do the latter, would be a mistake in my view, given what we've learned in this research note.

And $g_{u,v}(\pm iy)$ can be calculated from the the elements of *g* directly, which is $\boldsymbol{\mathcal{N}}$, or if we were using a template Ꞇ, we could use that too, where the variable r in Ꞇ becomes $\pm iy$.

Now if we move on to the more *general* form of (1), where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and decide to use a *hypothetical* template in $\boldsymbol{\mathcal{R}}^2$ such as

$$Ꞇ = \Sigma\, c_n(u, v)\, r^n P_n(\cos(\theta))\,, \quad n = 0, 2, 4, \ldots \qquad (\S)$$

where $c_n(u, v)$ is an *even* function of θ *only*; then the *same* principles will apply, so that in (1) we evaluate $g_{u,v}(\alpha \pm iy)$ by letting $r = \alpha \pm iy$ in (§).

Or if we found a normalized $\boldsymbol{\mathcal{N}}$ solution ($\sigma = 1$) to

$$C_{u,v} \approx 2\sigma \cdot \cosh(r\alpha)J_0(r\varepsilon)g_{u,v}(\alpha)\,,$$

we could use that too, when evaluating (1); where again, it is $r = \alpha \pm iy$ in the $\boldsymbol{\mathcal{N}}$ solution $g_{u,v}(r, \theta)$, that we are calculating in (1), just as it is when using Ꞇ.

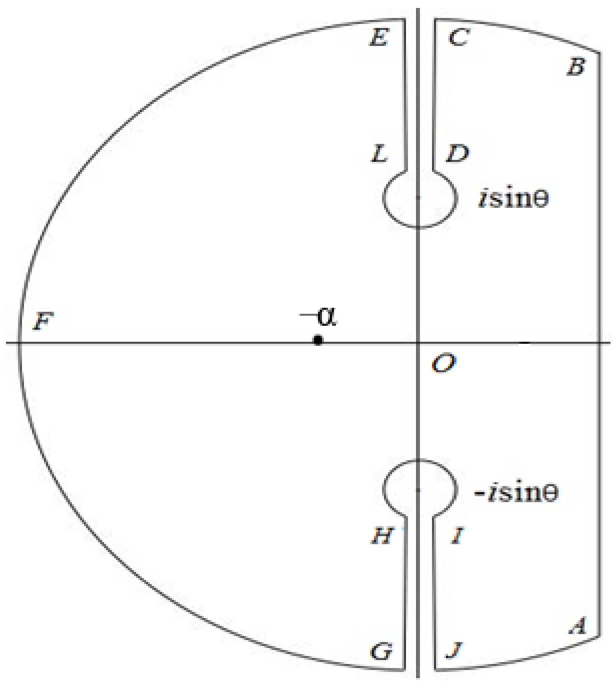
And this is now a *departure* from previous notes in this section [p 539 *ff*.], where we suggested using $r = |\alpha \pm iy|$ in our calculations. Again, I think it would be a mistake to do so, based on what we have learned in this research note.

But does this mean our study of convergence issues for (1) [p 539 *ff*.] must be tossed, just because we opted for $|r|$ versus r in (1) ? No, in my opinion, because there is still some *residual* value in studying convergence of these integrals, in the former case.

A Harmonic Expression For 1 / r

Consider the following function, where we are evaluating *f* over the contour shown below, using the Laplace inverse …

$$f(s) \ = \ 1 \,/\, \{(s + \alpha)\, \sqrt{s^2 + \varepsilon^2}\}\ ,$$



$\alpha > 0$

$\varepsilon = \sin(\theta)$

For the time being, we'll presume $\alpha > 0$, so that a *simple* pole exists at $-\alpha$ in *f*. Using the methods on pages 262-4, we find that our harmonic expression computes to ($\varepsilon = 0$, $\kappa = 1/2\pi\boldsymbol{i}$),

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\alpha + iy) \ - 1/(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[1/(\alpha + iy) \ + \ 1/(\alpha - iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2} \ + \ R \ \ (*)$$

where $R = 2e^{-r\alpha} / \alpha$, is a *doubling* of the residue from the *simple* pole, itself. Here is a snapshot showing the integral above, when $\alpha = 1$ and $r = 1$ in the cos() and sin() terms, as well as R, but r can be *any* real number > 0 …

$$\int_0^{100} \frac{\cos(y)\left(\frac{1}{1+i\,y} - \frac{1}{1-i\,y}\right) + i\sin(y)\left(\frac{1}{1+i\,y} + \frac{1}{1-i\,y}\right)}{(i\,\pi)\,y}\,dy = \frac{i\,\mathrm{Ci}(-100+i) - i\,\mathrm{Ci}(100+i) + \mathrm{Si}(-100+i) + 2\,e\,\mathrm{Si}(100) - \mathrm{Si}(100+i)}{e\,\pi} \approx 0.264274$$

In this case R ~ 0.735759, and when added to the number just above [.264274], the result is 1. But you can see from the *integrand* in (*), that we are dealing with the harmonic expression

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\alpha + iy) - g_{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\alpha + iy) + g_{u,v}(\alpha - iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

where now, $g_{u,v} = 1/r$, where r represents the *radial* direction in $g_{u,v}$. From this, we conclude that when $r = 1$, $g_{u,v}(r)$ computes to 1, which is equal to $g_{u,v}(\alpha)$, which in turn, is equal to (*). That is to say ($\alpha = 1$, $\varepsilon = 0$, r = 1, $r = 1$, $\kappa = 1/2\pi\boldsymbol{i}$, $R = 2e^{-r\alpha}/\alpha$), it is the case that

$g_{u,v}(\alpha)$ =

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\alpha + iy) - 1/(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[1/(\alpha + iy) + 1/(\alpha - iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2} + R\,,$$

and this demonstrates *clearly*, that when going over to the complex plane $\mathcal{C}$ from $\mathcal{R}^2$, the radius r in $g_{u,v}$ maps to $\alpha \pm iy$, which we spoke about in the previous research note [pp 547-8]. That is to say, $g_{u,v}(\alpha \pm iy) = 1/(\alpha \pm iy)$ in this case, since $g_{u,v} = 1/r$ in $\mathcal{R}^2$.

Let's take another snapshot, which we see below …

$$\int_0^{100} -\frac{i\left(\left(-\frac{1}{2-i\,y}+\frac{1}{2+i\,y}\right)\cos(2\,y)+i\left(\frac{1}{2-i\,y}+\frac{1}{2+i\,y}\right)\sin(2\,y)\right)}{\pi\,y}\,dy = 0.481712$$

In this case ($\alpha = 2$, $\varepsilon = 0$, r = 2, $r = 2$, $\kappa = 1/2\pi\boldsymbol{i}$, $R = 2e^{-r\alpha}/\alpha$), we find that R evaluates to ~ .0183156, and when added to the number above, the result is 0.5. And this is precisely $g_{u,v}(r)$, which is equal to $g_{u,v}(\alpha)$, since again, $\alpha = r = 2$, and $g_{u,v} = 1/r$. Yet another confirmation of our harmonic expression, you might say.

Finally, let's look at another snapshot, where $\alpha < 0$ ($\alpha = -3$, $\varepsilon = 0$, r = 1, $r = -3$, $\kappa = 1/2\pi\boldsymbol{i}$) …

$$\int_0^{100} -\frac{i\left(\left(-\frac{1}{-3-i\,y}+\frac{1}{-3+i\,y}\right)\cos(y)+i\left(\frac{1}{-3-i\,y}+\frac{1}{-3+i\,y}\right)\sin(y)\right)}{\pi\,y}\,dy = -0.333298 + 0\,i$$

In this case, the *simple* pole is at $-\alpha > 0$, and we can *omit* it by setting $0 < \mathrm{Re(AB)} < -\alpha$ in our contour. Thus, R is now *zero*, and *only* the harmonic expression above matters. It is approximately –⅓, which agrees with $g_{u,v}(r)$, which is equal to $g_{u,v}(\alpha)$, since again, $g_{u,v} = 1/r$, and $\alpha = r = -3$.

A Harmonic Expression For 1 / r^2

This case is similar to the previous research note, so we'll be more brief here and quote the results. And here, we are evaluating *f* over the *same* contour as in the previous note, using the Laplace inverse, where *f* is …

$$f(s) \;=\; 1 \,/\, \{(s+\alpha)^2 \sqrt{s^2+\varepsilon^2}\}\ .$$

Now with $\alpha > 0$, a pole of *order* 2 exists at $-\alpha$ in *f*, and using the methods on pages 262-4, we find that our harmonic expression [labelling as (*)] computes to ($\varepsilon = 0$, $\kappa = 1/2\pi\boldsymbol{i}$),

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\alpha+iy)^2 \;-1/(\alpha-iy)^2] + \boldsymbol{i}\sin(yr)\,[1/(\alpha+iy)^2 \;+\; 1/(\alpha-iy)^2]\}\,dy \,/\sqrt{y^2-\varepsilon^2} \;+\; R$$

where $R = 2e^{-r\alpha}\,(1+\alpha r)\,/\,\alpha^2$, is a *doubling* of the residue from the pole, itself. Here is a snapshot showing the integral above, when $\alpha = 1$ and $r = 1$ in the cos() and sin() terms, as well as R, but r can be *any* real number > 0 …

$$\int_0^{100} -\frac{i\left(\left(-\frac{1}{(1-i\,y)^2}+\frac{1}{(1+i\,y)^2}\right)\cos(y)+i\left(\frac{1}{(1-i\,y)^2}+\frac{1}{(1+i\,y)^2}\right)\sin(y)\right)}{\pi\,y}\,dy =$$
$$-0.471517+0\,i$$

In this case R ~ 1.471517, and when added to the number just above [–.471517], the result computes to 1.

Now here, $g_{u,v} = 1/r^2$, where *r* represents the *radial* direction in $g_{u,v}$. From this, we conclude that when $r = 1$, $g_{u,v}(r)$ computes to 1, which is equal to $g_{u,v}(\alpha)$, which in turn, is equal to (*). Now let's look at the next snapshot …

$$\int_0^{100} -\frac{i\left(\left(-\frac{1}{(2-i\,y)^2}+\frac{1}{(2+i\,y)^2}\right)\cos(2\,y)+i\left(\frac{1}{(2-i\,y)^2}+\frac{1}{(2+i\,y)^2}\right)\sin(2\,y)\right)}{\pi\,y}\,dy =$$
$$0.204211$$

In this case, $\alpha = 2$ and $r = 2$ in the cos() and sin() terms, as well as R, and R computes to .045789. And when added to the number just above [.204211], the result is 0.25, which agrees with $g_{u,v}(r)$ and $g_{u,v}(\alpha)$, since again, $g_{u,v} = 1/r^2$, and $\alpha = r$. Now for the last snapshot, which follows on the next page …

$$\int_0^{100} -\frac{i\left(\left(-\frac{1}{(-1-i\,y)^2}+\frac{1}{(-1+i\,y)^2}\right)\cos(y)+i\left(\frac{1}{(-1-i\,y)^2}+\frac{1}{(-1+i\,y)^2}\right)\sin(y)\right)}{\pi\,y}\,dy = 1.$$

In this case, $\alpha = -1$ and $r = 1$ in the cos() and sin() terms, and the pole is now at $-\alpha > 0$; so we can *omit* it by setting $0 < \mathrm{Re(AB)} < -\alpha$ in our contour. Thus, R is now *zero*, and *only* the harmonic expression above matters. It computes to 1, which agrees with $g_{u,v}(r)$, which is equal to $g_{u,v}(\alpha)$, since again, $g_{u,v} = 1/r^2$, and $\alpha = r = -1$.

———

Let us recall our *equivalency* theorem, in the *intangible* space, where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and $\mathcal{E} = g_{u,v}(\alpha) \cdot J_0(r\varepsilon)$ …

$$(1) = g_{u,v}(\alpha) \cdot J_0(r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true} \qquad (*)$$

Here, the following expression is (1),

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\alpha + iy) \;-\; g_{u,v}(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\alpha + iy) \;+\; g_{u,v}(\alpha - iy)]\}\,dy\,/\sqrt{y^2 - \varepsilon^2}$$

and the expression below is (2), where the dark energy singularities are at O and at (±1, 0) …

$$C_{u,v} \approx \sigma[g_{u,v}(0) \;+\; 2\cosh(r\alpha)J_0(r\varepsilon)g_{u,v}(\alpha)]\,.$$

———

In general, when quoting an equivalency theorem, like the one above, the assumption has *always* been that $g_{u,v}$ is *well-behaved* in the *complex* plane $\mathcal{C}$. But it may not be, as we have seen in this research note, and the preceding one. In such cases, a residue R will show up in the calculations, and it now becomes part of the harmonic expression. But even so, the equivalency theorem still holds …

Testing The Template $1/r$ In Our Harmonic Expression For A Quantumlike Component

In this case, we use the *same* methods as outlined on pages 549-50, for the function below, where the equivalency theorem, as stated on the last page, applies.

$$f(s) = 1 / \{(s + \alpha)\sqrt{s^2 + \varepsilon^2}\} .$$

However, the *difference* now is that $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$ are *both* variables, so when calculating the residue in the Laplace inverse for f, at the pole $-\alpha$ [where $\alpha > 0$]; it will compute to $e^{-r\alpha} / \sqrt{\alpha^2 + \varepsilon^2}$. It is *this* residue that we *double*, and add to the harmonic expression above, and so obtain, for the *complete* expression, in the case of the *quantumlike* component,

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\alpha + iy) - 1/(\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[1/(\alpha + iy) + 1/(\alpha - iy)]\}\,dy\,/\sqrt{y^2 - \varepsilon^2} + R \quad (*)$$

where $R = 2e^{-r\alpha}$, since $\alpha^2 + \varepsilon^2 = 1$.

Now let's look at a snapshot of the integral in (*), when $\theta = \pi / 4$, so that α and ε are $\sqrt{2}/2$ (approximately 0.707106), and the template for $g_{u,v}$ is $1/r$. And here, $r = 1$ in the cos() and sin() terms, as well as R, but r can be *any* real number > 0 …

$$\int_{\frac{1}{\sqrt{2}}}^{100} \frac{\cos(y)\left(\frac{1}{0.707106+i\,y} - \frac{1}{0.707106-i\,y}\right) + i\sin(y)\left(\frac{1}{0.707106+i\,y} + \frac{1}{0.707106-i\,y}\right)}{(i\,\pi)\sqrt{y^2 - 0.5}}\,dy =$$

$$0.189332$$

In this particular case, R computes to ~ .986137, and when added to the number above [.189332], we obtain 1.17547.

Now in order for the equivalency to hold on page 552, for the *quantumlike* component, (*) *must* agree with $g_{u,v}(\alpha) \cdot J_0(r\varepsilon)$, where here, $\alpha = \varepsilon = \cos(\theta) = \sqrt{2}/2$, and $r = 1$. And since $g_{u,v}(r) = 1 / r$, this means $g_{u,v}(\alpha) = 1 / \alpha = \sqrt{2}$. On the other hand, $J_0(r\varepsilon) = J_0(\sqrt{2}/2) \sim .878852$, so that

$$g_{u,v}(\alpha) \cdot J_0(r\varepsilon) \sim 1.24288 .$$

The two numbers [1.17547 and 1.24288] do *not* agree, so that we may conclude $g_{u,v} = 1/r$ is not an acceptable template, for the *normalized* solution ($\sigma = 1$) to the *quantumlike* term in (2) [p 552]; that is to say,

$$C_{u,v} \approx 2\sigma \cdot \cosh(r\alpha)J_0(r\varepsilon)g_{u,v}(\alpha) .$$

And this is how we can use the equivalency theorem to test the *viability* of our templates …

Is $g_{u,v} = 1/(r + \Delta)$ A Suitable Template For $C_{u,v} \approx \sigma g_{u,v}(0)$, Where $\Delta > 0$

Let us bring back the more *general* equivalency from page 511, where there are singularities at the origin O and at $(\pm\delta, 0)$, in the *intangible* space. Here, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and $\delta > 0$.

Then if $g_{u,v}$ is *well-behaved* in the *complex* plane $\mathcal{C}$, (1) is …

$$2\kappa\int_{\delta\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\delta\alpha + iy) - g_{u,v}(\delta\alpha - iy)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\delta\alpha + iy) + g_{u,v}(\delta\alpha - iy)]\}\,dy \,/\, \sqrt{y^2 - (\delta\varepsilon)^2}$$

and (2) is …

$$C_{u,v} \approx \sigma[g_{u,v}(0) + 2\cosh(\delta r\alpha)J_0(\delta r\varepsilon)g_{u,v}(\delta\alpha)]\,.$$

And let us recall our equivalency, for this case, where $\boldsymbol{\mathcal{E}} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$ …

$$(1) = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true .} \qquad (\sim)$$

Now since $g_{u,v} = 1 / (r + \Delta)$ is *not* well-behaved in the complex plane $\mathcal{C}$, we'll take the Laplace inverse of the following expression, in order to append to (1) the proper *residue*, when $\alpha > 0$ …

$$f(s) = 1 / \{(s + \delta\alpha + \Delta)\sqrt{s^2 + (\delta\varepsilon)^2}\}\,.$$

Note that whether or not Δ is equal to *zero*, r maps to $\delta\alpha \pm iy$ in $\mathcal{C}$; since again, in $\mathcal{R}^2$, it is the case that $g_{u,v}(r) = 1 / (r + \Delta)$.

Taking the Laplace inverse of f leads to the following *complete* harmonic expression [pp 262-4], which we'll label as (*) …

$$2\kappa\int_{\delta\varepsilon}^{\infty} \{\cos(yr)[1 / (\delta\alpha + \Delta + iy) - 1 / (\delta\alpha + \Delta - iy)] + \boldsymbol{i}\sin(yr)\,[1 / (\delta\alpha + \Delta + iy) + 1 / (\delta\alpha + \Delta - iy)]\}\,dy \,/\, \sqrt{y^2 - (\delta\varepsilon)^2} + R\,,$$

where $R = 2e^{-(\delta\alpha + \Delta)r} / \sqrt{\delta^2 + 2\delta\alpha\Delta + \Delta^2}$, is a *doubling* of the residue associated with the *simple* pole that is at $-(\delta\alpha + \Delta)$ in $f(s)$. And again, α, δ, Δ are all > 0, and recall too that $\alpha^2 + \varepsilon^2 = 1$.

Now let $\delta \dashrightarrow 0$, so that in this special case, (*) must equal $1/\Delta$ from our previous notes [pp 549-50], since this is now $g_{u,v}(0)$, and therefore (2) below must hold, by equivalency. That is to say, (2) is …

$$C_{u,v} \approx 3\sigma \cdot g_{u,v}(0)\,,$$

since the dark energy singularities at $(\pm\delta, 0)$ have converged to O. And again, $g_{u,v} = 1 / (r + \Delta)$ in $\mathcal{R}^2$, and thus we may conclude that this template is valid for $C_{u,v} \approx \sigma \cdot g_{u,v}(0)$, where $\sigma = 1$. Said another way, $g_{u,v}$ does not 'blow up' at the origin $[r = 0]$ in $\mathcal{R}^2$.

As an example, suppose $\delta = 0$ and $\Delta = 1$ in (*) on the previous page, so that (*) becomes (r = 1 in the cos() and sin() terms, as well as R; but r can be *any* real number > 0) …

$$2\kappa\int_{0}^{\infty} \{\cos(yr)[1/(\Delta + iy) - 1/(\Delta - iy)] + \boldsymbol{i}\sin(yr)\,[1/(\Delta + iy) + 1/(\Delta - iy)]\}\,dy / y + R\,,$$

where here, $R = 2e^{-(\delta\alpha + \Delta)r} / \sqrt{\delta^2 + 2\delta\alpha\Delta + \Delta^2} = 2e^{-r} \sim 0.735759$. And from the snapshot below we see what the integral, just above, computes to …

$$\int_0^{100} \frac{\cos(y)\left(\frac{1}{1+i\,y} - \frac{1}{1-i\,y}\right) + i\sin(y)\left(\frac{1}{1+i\,y} + \frac{1}{1-i\,y}\right)}{(i\,\pi)\,y}\,dy =$$

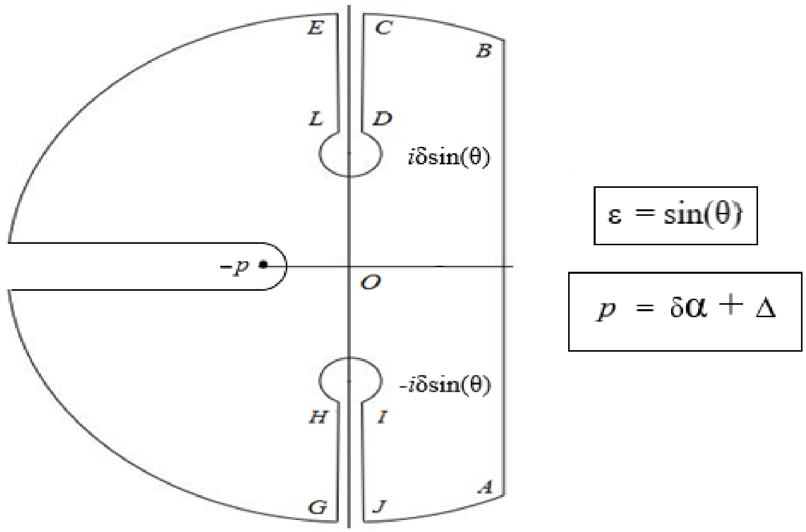
$$\frac{i\,\mathrm{Ci}(-100+i) - i\,\mathrm{Ci}(100+i) + \mathrm{Si}(-100+i) + 2\,e\,\mathrm{Si}(100) - \mathrm{Si}(100+i)}{e\,\pi} \approx 0.264274$$

Adding the two numbers together [.735759 and .264274], we obtain 1 … the desired result, since it is the case that $g_{u,v}(r) = 1 / (r + \Delta) = 1$, when $r = 0$ …

Is $g_{u,v} = 1/(\sqrt{r + \Delta})$ A Suitable Template For $C_{u,v} \approx \sigma g_{u,v}(0)$, Where $\Delta > 0$

In this case we use the function *f* below, associated with the following contour γ, and take the Laplace inverse of *f*, along AB, where α, δ, Δ are all > 0 …

$$f(s) = 1 / \{\sqrt{s + \delta\alpha + \Delta} \cdot \sqrt{s^2 + (\delta\varepsilon)^2}\}$$



At the outset, it should be said that *f* has *no* poles; only the branching points at $\pm i\delta\sin(\theta)$, and at $-p$. As well, the *phases* in γ are π on the arms CD and GH, *relative* to the *y*-axis; and –π on LE and IJ. And for the horizontal arms sweeping around $-p$, the top arm has a phase of π, *relative* to the *x*-axis; and the bottom arm has a phase of –π. Finally, the Laplace inverse, itself, is defined to be the following integral, along AB, where $\kappa = 1/2\pi \boldsymbol{i}$ …

$$\kappa \int e^{sr} f(s)\, ds$$

Computing along the *vertical* branches (or arms) yields [labelling as (*)] …

$$2\kappa \int_{\delta\varepsilon}^{\infty} \{\cos(yr)[1 / (\sqrt{\delta\alpha + \Delta + iy}) - 1 / (\sqrt{\delta\alpha + \Delta - iy})] + \boldsymbol{i}\sin(yr)[1 / (\sqrt{\delta\alpha + \Delta + iy}) + 1 / (\sqrt{\delta\alpha + \Delta - iy})]\}\, dy / \sqrt{y^2 - (\delta\varepsilon)^2}$$

And, after letting δ ---› 0, the *horizontal* arms, added together and *doubled*, compute to [labelling as (†)] …

$$4\kappa \boldsymbol{i} \int_{0}^{\infty} exp(-r(x + \Delta)) / (x + \Delta)\sqrt{x}\, dx$$

Now we'll look at some examples we've prepared, for (*) and (†), respectively. In the *first* one, below (δ = 0, Δ = 1, r = 1, 4κ***i*** = 2 / π),

$$\int_0^{300} -\frac{i\left(\left(-\frac{1}{\sqrt{1-i\,y}}+\frac{1}{\sqrt{1+i\,y}}\right)\cos(y)+i\left(\frac{1}{\sqrt{1-i\,y}}+\frac{1}{\sqrt{1+i\,y}}\right)\sin(y)\right)}{\pi\,y}\,dy = 0.68549$$

$$\int_0^{300} \frac{2\,e^{-1-x}}{\pi\sqrt{x}\,(1+x)}\,dx \approx 0.314598\ldots$$

And the sum of these two numbers [1.000088] is ~ 1, which agrees with $g_{u,v} = 1 / (\sqrt{r+\Delta})$, when $r = 0$, since again, Δ = 1.

In the second one, below (δ = 0, Δ = 2, r = 1, 4κ***i*** = 2 / π),

$$\int_0^{500} -\frac{i\left(\left(-\frac{1}{\sqrt{2-i\,y}}+\frac{1}{\sqrt{2+i\,y}}\right)\cos(y)+i\left(\frac{1}{\sqrt{2-i\,y}}+\frac{1}{\sqrt{2+i\,y}}\right)\sin(y)\right)}{\pi\,y}\,dy = 0.642814$$

$$\int_0^{500} \frac{2\,e^{-2-x}}{\pi\sqrt{x}\,(2+x)}\,dx \approx 0.0643471\ldots$$

And the sum of these two numbers [.7071611] is ~ $\sqrt{2}$ / 2 [~ .7071067], which agrees with our template $g_{u,v} = 1 / (\sqrt{r+\Delta})$, when $r = 0$, since again, Δ = 2.

Thus we may conclude that this template is *also* valid for $C_{u,v} \approx \sigma \cdot g_{u,v}(0)$, where σ = 1. Said another way, $g_{u,v}$ does not 'blow up' at the origin [$r = 0$] in $\mathcal{R}^2$.

———

The *actual* sum of the upper and lower *horizontal* arms is *half* the value quoted in (†), on the preceding page. However, just like residues, we *double* this value, before adding it to the harmonic expression (*), also on the previous page. For more on the methodology here, see pages 262-4.

Some corrections to both the diagram *and* the text were made in this release, for this last research note [pp 556-7].

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Is $g_{u,v} = 1/(r+\Delta)$ A Suitable Template For $C_{u,v} \approx \sigma g_{u,v}(0)$, Where $\Delta > 0$, Part II

In Part I [pp 554-5] we developed a a harmonic expression for the *general* equivalency theorem, which is reproduced here [labelling as (*)], where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$ …

$$2\kappa\int_{\delta\varepsilon}^{\infty} \{\cos(yr)[1/(\delta\alpha+\Delta+iy) - 1/(\delta\alpha+\Delta-iy)] + \boldsymbol{i}\sin(yr)[1/(\delta\alpha+\Delta+iy) + 1/(\delta\alpha+\Delta-iy)]\}\, dy / \sqrt{y^2-(\delta\varepsilon)^2} + R\,,$$

And here, $R = 2e^{-(\delta\alpha+\Delta)r} / \sqrt{\delta^2+2\delta\alpha\Delta+\Delta^2}$, is a *doubling* of the residue associated with the *simple* pole that is at $-(\delta\alpha+\Delta)$ in $f(s)$. And again, α, δ, Δ are all > 0, and recall too that $\alpha^2+\varepsilon^2 = 1$.

$$f(s) = 1/\{(s+\delta\alpha+\Delta)\sqrt{s^2+(\delta\varepsilon)^2}\}\,.$$

We *also* validated (*) when $\delta \dashrightarrow 0$ [pp 554-5], and now wish to do the same thing when $\varepsilon \dashrightarrow 0$. In this case, since $\alpha^2+\varepsilon^2 = 1$, R reduces to $2e^{-(\delta+\Delta)r} / (\delta+\Delta)$, and (*) will evaluate to $1/(\delta+\Delta)$, from our previous research notes. And this is equal to $g_{u,v}(\delta)$, which agrees with our *general* equivalency theorem, stated on page 554, since again, α is now 1.

As an example, with ($\alpha = \delta = \Delta = 1$, r = 2, $\varepsilon = 0$, $\kappa = 1/2\pi\boldsymbol{i}$, $r = 1$, $R = 2e^{-2r}/2$), the integral in (*) computes to …

$$\int_0^{100} -\frac{i\left(\left(-\frac{1}{2-i\,y}+\frac{1}{2+i\,y}\right)\cos(2\,y)+i\left(\frac{1}{2-i\,y}+\frac{1}{2+i\,y}\right)\sin(2\,y)\right)}{\pi\,y}\,dy = 0.481712$$

And R evaluates to ~ .0183156, and when added to the number above, the result is ~ 0.5. And this is *precisely* $g_{u,v}(r)$, which is equal to $g_{u,v}(\delta)$, since again, $\delta = r = 1$, and $g_{u,v}(r) = 1/(r+\Delta)$.

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A line on page 554 was rewritten to state correctly, that the following is true …

'Note that whether or not Δ is equal to *zero*, r maps to $\delta\alpha \pm iy$ in $\mathcal{C}$; since again, in $\mathcal{R}^2$, it is the case that $g_{u,v}(r) = 1/(r+\Delta)$'. This is the correct interpretation of the mapping, in my opinion, which we use in this research note, as well …

―――

A Harmonic Expression For $g_{u,v} = 1 / (r\sin(\theta) + \Delta)$ Where $\Delta > 0$

Let us bring back the more *general* equivalency from page 511, where there are singularities at the origin O and at $(\pm\delta, 0)$, in the *intangible* space. Here, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and $\delta > 0$; and we will suppose the template $g_{u,v}$ is now a function of (r, θ). To wit, $g_{u,v}(r, \theta) = 1 / (r\sin(\theta) + \Delta)$, where $0 < \theta < \pi / 2$, and α, δ, Δ are all > 0.

Then the general harmonic expression (1), where R is a *doubling* of the residue associated with $f(s)$ in the *complex* plane $\mathcal{C}$, is as follows …

$$f(s) \;=\; 1 / \{((s + \delta\alpha)\cdot \sin(\theta) + \Delta)\, \sqrt{s^2 + (\delta\varepsilon)^2}\}\;;$$

$$2\kappa\int_{\delta\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\delta\alpha + iy, \theta) \;-\; g_{u,v}(\delta\alpha - iy, \theta)] \;+\; \boldsymbol{i}\sin(yr)\,[g_{u,v}(\delta\alpha + iy, \theta) \;+\; g_{u,v}(\delta\alpha - iy, \theta)]\}\; dy \,/\, \sqrt{y^2 - (\delta\varepsilon)^2} \;+\; R\,, \qquad (1)$$

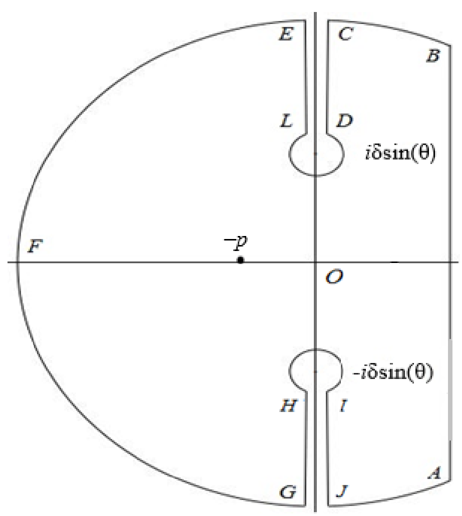
and (2) is …

$$C_{u,v} \approx \sigma[g_{u,v}(0) \;+\; 2\cosh(\delta r\alpha)J_0(\delta r\varepsilon)g_{u,v}(\delta\alpha)]\,.$$

And let us recall our equivalency, for this case, where $\boldsymbol{\mathcal{E}} = g_{u,v}(\delta\alpha)\cdot J_0(\delta r\varepsilon)$ …

$$(1) = g_{u,v}(\delta\alpha)\cdot J_0(\delta r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true} \qquad (\sim)$$

The *first* thing to note here, is that $g_{u,v}(\delta\alpha \pm iy, \theta) = 1 / (r\sin(\theta) + \Delta)$, where $r = \delta\alpha \pm iy$ in $\mathcal{C}$, and the *second* thing we want to point out is the construction of $f(s)$.



$p > 0$

$\varepsilon = \sin(\theta)$

It has a *pole* at $-p = -(\Delta + \delta\alpha\sin(\theta)) / \sin(\theta)$, and if we now study the *first* term in the denominator of $f(s)$, which is $(s + \delta\alpha)\cdot \sin(\theta) + \Delta$; we see that as we traverse the *vertical* branches in the contour above, when taking the Laplace inverse of $f(s)$; the variable s is equal to $\pm iy$, which gives us

agreement with what we see in the integrand of (1). Combined with *second* term in the denominator of $f(s)$, gives us the *full* integrand in (1), which is what we want.

Lastly, since the pole $-p$ is equal to $-(\Delta + \delta\alpha \sin(\theta)) / \sin(\theta)$ in $f(s)$, the corresponding *residue* computes to $exp(-p\mathrm{r}) / \sqrt{p^2 + (\delta\varepsilon)^2}$, noting again that $\alpha^2 + \varepsilon^2 = 1$; since we are taking the Laplace inverse of $f(s)$ along AB, where $\kappa = 1/2\pi \boldsymbol{i}$; that is to say,

$$\kappa \int e^{s\mathrm{r}} f(s)\, ds \, .$$

A *doubling* of this residue, which we call R, is now added to the integral in (1), which gives us the *complete* harmonic expression above, on the previous page.

If, on the other hand, $f(s)$ had *branching* points instead of poles, we would calculate in a similar fashion; this time *doubling* an integration along *both* arms associated with the branch points, and adding this value R to the integral in (1) [see, for example, pages 556-7].

Finally, in (2) on the previous page, we have the term $g_{u,v}(\delta\alpha)$, and in $\mathscr{R}^2$, because $\delta\alpha$ is a *radial* measure, this computes to $1 / (r\sin(\theta) + \Delta)$, where $r = \delta\alpha$.

We can now go ahead and test the template, for the *quantumlike* component in (2), just as we did on page 553, and accept or reject it, accordingly. In this particular case, we'll let $\theta = \pi/4$, so that α and ε are $\sqrt{2}/2$ (approximately 0.707106). And we'll set $\delta = \Delta = \mathrm{r} = 1$, so that the integral in (1) becomes …

$$\int_{\frac{1}{\sqrt{2}}}^{400} \frac{\cos(y)\left(\frac{1}{1.5+0.707\,i\,y} - \frac{1}{1.5-0.707\,i\,y}\right) + i\sin(y)\left(\frac{1}{1.5+0.707\,i\,y} + \frac{1}{1.5-0.707\,i\,y}\right)}{(i\,\pi)\sqrt{y^2-0.5}}\,dy =$$

0.474422

The value of R in (1) computes to ~ .107218, and when added to the number above, one obtains ~ .581640. On the other hand, $g_{u,v}(\delta\alpha)\cdot J_0(\delta r\varepsilon)$ computes to ~ .585902. The numbers are *astonishingly* close, and you have to wonder, for this *random* choice of θ, whether $1 / (r\sin(\theta) + \Delta)$ might be a suitable template for the *quantumlike* component in (2) [perhaps part of a series expansion, say].

$\dfrac{2\,e^{-3/\sqrt{2}}}{\sqrt{5}}$	$\dfrac{2\,J_0\left(\frac{1}{\sqrt{2}}\right)}{3}$
Decimal approximation	Decimal approximation
0.1072178943663382	0.585901612180728
value of R	value of $g_{u,v}(\delta\alpha)\cdot J_0(\delta r\varepsilon)$

If we were to repeat the testing, say with $\theta = \pi / 6$, then the integral in (1) becomes, along with R and $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$ [$\delta = \Delta = r = 1$, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$] …

$$\int_{0.5}^{500} \frac{\cos(y)\left(\frac{1}{1.433+0.5\,i\,y} - \frac{1}{1.433-0.5\,i\,y}\right) + i\sin(y)\left(\frac{1}{1.433+0.5\,i\,y} + \frac{1}{1.433-0.5\,i\,y}\right)}{(i\,\pi)\sqrt{y^2 - 0.25}}\,dy =$$

$$0.596044$$

$$\frac{2\exp(-2.866)}{\sqrt{2.866^2 + 0.25}}$$

Result

0.0391341…

value of R

$$\frac{1}{1.433}\,J_0(0.5)$$

Result

0.654899…

value of $g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$

Adding the numbers .596044 and .0391341 together yields .635178, which compares favorably with $\mathcal{E}$ = .654899. And here is the testing for $\theta = \pi / 3$ …

$$\int_{\frac{\sqrt{3}}{2}}^{500} \frac{\cos(y)\left(\frac{1}{1.433+0.866\,i\,y} - \frac{1}{1.433-0.866\,i\,y}\right) + i\sin(y)\left(\frac{1}{1.433+0.866\,i\,y} + \frac{1}{1.433-0.866\,i\,y}\right)}{(i\,\pi)\sqrt{y^2 - 0.75}}$$

$$dy = 0.397824$$

$$2 \times \frac{\exp(-1.65469)}{\sqrt{1.65469^2 + 0.75}}$$

Result

0.204701…

value of R

$$\frac{1}{1.433}\,J_0\left(\frac{\sqrt{3}}{2}\right)$$

Result

0.572999…

value of $g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$

Adding the numbers .397824 and .204701 together yields .602525, which compares favorably with $\mathcal{E}$ = .572999. Although the comparisons don't agree, in *both* cases, there is enough evidence now to support the following *potential* hypothesis, based on these two tests … ..

> For some smooth function $h(\theta)$, which can be approximated by $\sin(\theta)$, where $0 < \theta < \pi / 2$, it is the case that $1 / (r \cdot h(\theta) + \Delta)$ is a suitable candidate for the *quantumlike* component in (2) [p 559].

In the case where $\theta = \pi/2$, a pole exists at $-\Delta$ in $f(s)$, since $\alpha = 0$ and $\varepsilon = 1$, now. And with $\delta = \Delta = r = 1$, the harmonic expression is shown below, along with the term $R = \sqrt{2}\,/\,e$ [labelling as (*)] …

$$f(s) \;=\; 1 \,/\, \{((s + \delta\alpha)\cdot \sin(\theta) + \Delta)\,\sqrt{s^2 + (\delta\varepsilon)^2}\}$$

$$2\kappa \int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\Delta + iy) \;-\; 1/(\Delta - iy)] + \boldsymbol{i}\sin(yr)\,[1/(\Delta + iy) \;+\; 1/(\Delta - iy)]\}\; dy \,/\sqrt{y^2 - \varepsilon^2} \;+\; R$$

And since $g_{u,v}(r, \theta) = 1 \,/\, (r\sin(\theta) + \Delta)$, and $\mathcal{E} = g_{u,v}(\delta\alpha)\cdot J_0(\delta r\varepsilon)$, we see that $\mathcal{E} = J_0(1)$. Here are the snapshots from Wolfram …

$$\int_1^{500} -\frac{i\left(\left(-\frac{1}{1-i\,y} + \frac{1}{1+i\,y}\right)\cos(y) + i\left(\frac{1}{1-i\,y} + \frac{1}{1+i\,y}\right)\sin(y)\right)}{\pi\sqrt{-1+y^2}}\,dy = 0.308547$$

$$\frac{\sqrt{2}}{e}$$

Decimal approximation

0.520260095022888

value of R

$$J_0(1)$$

Decimal approximation

0.765197686557966

value of $g_{u,v}(\delta\alpha)\cdot J_0(\delta r\varepsilon)$

Adding the numbers .308547 and .520260 together yields .828077, which compares *somewhat* favorably with $\mathcal{E}$ = .765197, though it is clear, there is *no* agreement here. Yet I would still maintain, that the *potential* hypothesis from the previous page, is still valid, but becomes somewhat *weaker*, as $\theta \dashrightarrow \pi/2$.

In the case where $\theta = 0$, *no* pole exists in the *first* term of the denominator for $f(s)$, since $\alpha = 1$ and $\varepsilon = 0$ now, and $\Delta > 0$. Thus $R = 0$, and $g_{u,v}(r, \theta) = 1 \,/\, \Delta$, both in $\mathcal{R}^2$ and $\mathcal{C}$, so that the harmonic expression reduces to

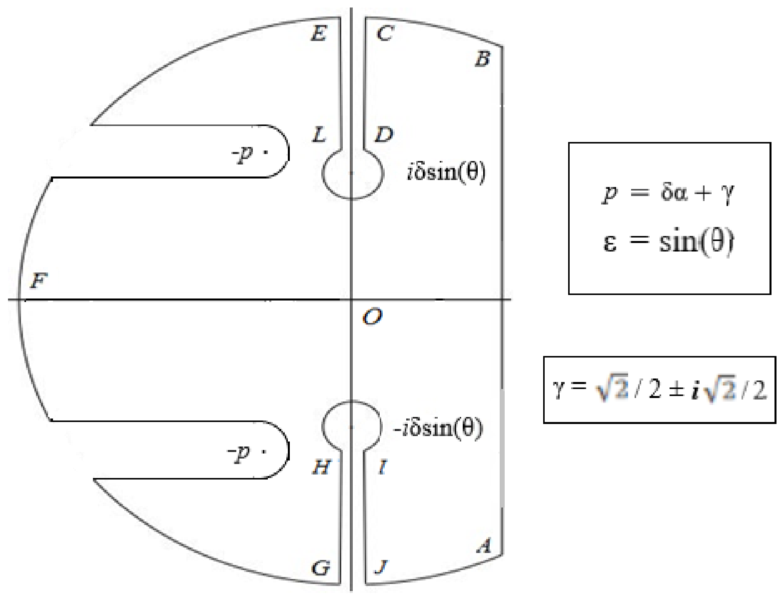
$$4\kappa\boldsymbol{i} \,/\, \Delta \cdot \int_{\varepsilon}^{\infty} \sin(yr)\; dy \,/\, y \;=\; 1 \,/\, \Delta$$

And $\mathcal{E} = g_{u,v}(\delta\alpha)\cdot J_0(\delta r\varepsilon) = g_{u,v}(\delta\alpha)$, which *must* equal $1 \,/\, \Delta$ in $\mathcal{R}^2$, for *any* choice of the argument to $g_{u,v}$, because $g_{u,v}(r, \theta) = 1 \,/\, \Delta$. Thus, the agreement is *exact* here, as expected; since again, we have assumed $\delta = \Delta = r = 1$ …

Is $1 / (\sqrt{r^4 + \Delta^4})$ A Suitable Template For $C_{u,v} \approx \sigma g_{u,v}(0)$, Where $\Delta > 0$

In this case we use the function f below, associated with the following contour γ, and take the Laplace inverse of f along AB, where α, δ, Δ are all > 0 (initially). And here $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$, and we'll assume throughout that $\Delta = 1$, so that the branch points $[-p]$ for the *first* term in the denominator of f are at $-\gamma$, as shown in the picture below, if $\delta = 0$.

$$f(s) = 1 / \{\sqrt{(s + \delta\alpha)^4 + \Delta^4} \cdot \sqrt{s^2 + (\delta\varepsilon)^2}\}$$



For the *upper* pair of *horizontal* branch lines ($\delta = 0$, $\Delta = 1$, $-p = -\sqrt{2}/2 + i\sqrt{2}/2$), the *sum* of the integrals here computes to

$$2\kappa \int_0^{\infty} exp(-r(x + p)) / \{(x + p)\sqrt{(x + p)^4 + \Delta^4}\}dx \qquad \textit{upper branch lines } \mathcal{U}$$

and for the *lower* pair of branch lines ($\delta = 0$, $\Delta = 1$, $-p = -\sqrt{2}/2 - i\sqrt{2}/2$), the integrals sum to

$$-2\kappa \int_0^{\infty} exp(-r(x + p)) / \{(x + p)\sqrt{(x + p)^4 + \Delta^4}\}dx \qquad \textit{lower branch lines } \mathcal{L}$$

The integral associated with the harmonic expression $\mathcal{H}$ [*vertical* branch lines in the contour], reduces to ($\delta = 0$, $\Delta = 1$, r = 1; range 0 to 500 for computational purposes)

$$\int_0^{500} \frac{2\sin(y)}{\pi y \sqrt{1 + y^4}} \, dy = 0.767585$$

And to compute R, which we add to $\mathcal{H}$, we simply *add* the values from $\mathcal{U}$ and $\mathcal{L}$ above; and *double* the result. Here are the snapshots, where only a *small* range in the integration is needed …

$$\int_0^{12} \frac{\exp(-(x + 0.707 - 0.707\,i))}{(i\,\pi)\left(\sqrt{(x + 0.707 - 0.707\,i)^4 + 1}\;(x + 0.707 - 0.707\,i)\right)}\,dx =$$
$$0.0593568 + 0.0687528\,i$$

value of $\mathcal{U}$

$$\int_0^{12} -\frac{\exp(-(x + 0.707 + 0.707\,i))}{(i\,\pi)\left(\sqrt{(x + 0.707 + 0.707\,i)^4 + 1}\;(x + 0.707 + 0.707\,i)\right)}\,dx =$$
$$0.0593568 - 0.0687528\,i$$

value of $\mathcal{L}$

In *both* cases ($\delta = 0$, $\Delta = 1$, r = 1), we see a *real* and *imaginary* component here, and when added together and *doubled*, the result for R is ~ .237427. If we now add this number to $\mathcal{H}$, which computes to .767585, the sum is ~ 1.00500.

The result is *very* close to the theoretical value of $g_{u,v}(r) = 1 / (\sqrt{r^4 + \Delta^4})$, which is 1, when $r = 0$; and so we may conclude that $1 / (\sqrt{r^4 + \Delta^4})$ is a suitable template for $C_{u,v} \approx \sigma g_{u,v}(0)$, when Δ is *greater* than *zero*.

We'll do one more example ($\delta = 0$, $\Delta = 1$, r = ½), and here are the snapshots for $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$, respectively …

$$\int_0^{500} \frac{2\sin\left(\frac{y}{2}\right)}{\pi\, y\, \sqrt{1 + y^4}}\,dy = 0.476315$$

$$\int_0^{12} \frac{\exp(-0.5\,(x + 0.707 - 0.707\,i))}{(i\,\pi)\left(\sqrt{(x + 0.707 - 0.707\,i)^4 + 1}\;(x + 0.707 - 0.707\,i)\right)}\,dx =$$
$$0.131872 + 0.0640548\,i$$

$$\int_0^{12} -\frac{\exp(-0.5\,(x + 0.707 + 0.707\,i))}{(i\,\pi)\left(\sqrt{(x + 0.707 + 0.707\,i)^4 + 1}\;(x + 0.707 + 0.707\,i)\right)}\,dx =$$
$$0.131872 - 0.0640548\,i$$

Adding $\mathcal{U}$ and $\mathcal{L}$ together, and then *doubling* this sum, gives us R ~ .527488. Now adding R to $\mathcal{H}$ gives us a value of .527488 + .476315 ~ 1.00380. Again, very nearly 1, which is $g_{u,v}(0)$.

We can *sharpen* our estimates by replacing .707 in the integrals above with $\sqrt{2}/2$. In so doing, $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{L}}$ become, respectively ($\delta = 0$, $\Delta = 1$, r = 1),

$$\int_0^{12} -\frac{i\, e^{-(1-i)/\sqrt{2}-x}}{\pi\left(\frac{1-i}{\sqrt{2}}+x\right)\sqrt{1+\left(\frac{1-i}{\sqrt{2}}+x\right)^4}}\, dx = 0.0581038 + 0.0702193\, i$$

$$\int_0^{12} \frac{i\, e^{-(1+i)/\sqrt{2}-x}}{\pi\left(\frac{1+i}{\sqrt{2}}+x\right)\sqrt{1+\left(\frac{1+i}{\sqrt{2}}+x\right)^4}}\, dx = 0.0581038 - 0.0702193\, i$$

Adding the numbers above, and *doubling* the result gives us R ~ .232415; and adding this to the value of $\boldsymbol{\mathcal{H}}$, which is ~ .767585, gives us *exactly* 1.

For the case ($\delta = 0$, $\Delta = 1$, r = ½), $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{L}}$ compute to, respectively,

$$\int_0^{12} -\frac{i\, e^{1/2\left(-(1-i)/\sqrt{2}-x\right)}}{\pi\left(\frac{1-i}{\sqrt{2}}+x\right)\sqrt{1+\left(\frac{1-i}{\sqrt{2}}+x\right)^4}}\, dx = 0.130921 + 0.0666319\, i$$

$$\int_0^{12} \frac{i\, e^{1/2\left(-(1+i)/\sqrt{2}-x\right)}}{\pi\left(\frac{1+i}{\sqrt{2}}+x\right)\sqrt{1+\left(\frac{1+i}{\sqrt{2}}+x\right)^4}}\, dx = 0.130921 - 0.0666319\, i$$

Adding the numbers above, and *doubling* the result gives us R ~ .523684; and adding this to the value of $\boldsymbol{\mathcal{H}}$, which is ~ .476315, gives us ~ .999999. Again, a match with $g_{u,v}(0)$, which equals 1.

It is amazing to me, anyway, just how accurate these results are, given the *complex* expressions we are actually dealing with, and as such, confirms the methodology we are using, to obtain them …

A Harmonic Expression For $g_{u,v} = 1 / (r\sin(\theta) + \Delta)$ Where $\Delta > 0$, Part II

Let us bring back the more *general* equivalency from page 511, where there are singularities at the origin O and at $(\pm\delta, 0)$, in the *intangible* space. Here, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and $\delta > 0$; and we will suppose the template $g_{u,v}$ is now a function of (r, θ). To wit, $g_{u,v}(r, \theta) = 1 / (r\sin(\theta) + \Delta)$, where $0 < \theta < \pi / 2$, and α, δ, Δ are all > 0.

Then the general harmonic expression (1), where R is a *doubling* of the residue associated with $f(s)$ in the *complex* plane $\mathcal{C}$, is as follows …

$$f(s) = 1 / \{((s + \delta\alpha)\cdot \sin(\theta) + \Delta) \sqrt{s^2 + (\delta\varepsilon)^2}\} ;$$

$$2\kappa\int_{\delta\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\delta\alpha + iy, \theta) - g_{u,v}(\delta\alpha - iy, \theta)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\delta\alpha + iy, \theta) + g_{u,v}(\delta\alpha - iy, \theta)]\}\, dy / \sqrt{y^2 - (\delta\varepsilon)^2} + R , \quad (1)$$

and (2) is, for the *quantumlike* component,

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)g_{u,v}(\delta\alpha) . \quad (*)$$

And let us recall our equivalency, for this case, where $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$ …

$$(1) = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon) \text{ if and } \textit{only} \text{ if (2) is true} \quad (\sim)$$

Now from Part I [pp 559-62], we showed that the following *potential* hypothesis is *likely* true, but becomes somewhat *weaker* as as θ ---› $\pi / 2$. In other words, as θ ---› $\pi / 2$, *h(θ)* still exists, but can no longer be approximated by *sin(θ)*. And this was based on testing when $\delta = \Delta = r = 1$.

> For some smooth function *h(θ)*, which can be approximated by *sin(θ)*, where $0 < \theta < \pi / 2$, it is the case that $1 / (r \cdot h(\theta) + \Delta)$ is a suitable candidate for the *quantumlike* component in (2) above.

Now if we let Ƭ = $1 / (r \cdot h(\theta) + \Delta)$, then we may conclude that Ƭ is the *limiting* case of a *broader* template Ƭ′, which satisfies (~) for *all* $\delta, \Delta, r > 0$, and becomes Ƭ as δ, Δ, r ---› 1. So while we may never know what Ƭ′ actually is, we can *infer* its existence by realizing that Ƭ is really just a special case of Ƭ′. And so, the *general* form for the *quantumlike* component (*), can be justified, accordingly.

Generalizing The Harmonic Expression $g_{u,v} = 1 / (r \cdot h(\theta) + \Delta)$ Where $\Delta > 0$

Following up on the last research note, we now wish to *generalize* the template for $g_{u,v}$, so that it incorporates the variables r, θ, Δ in the harmonic expression (1), as shown below. And here, we suppose that Ƭ $= 1 / (r \cdot h(\theta) + \Delta)$ satisfies (1) $= g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$ for $\delta = \Delta = \mathrm{r} = 1$, as per the last research note, with $0 \le \theta \le \pi / 2$, so that the *quantumlike* form

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha) \qquad (*)$$

holds, in this *very* specific case $[\delta = \Delta = \mathrm{r} = 1]$. Note that in $\mathcal{R}^2$, the *italicized* variable r in Ƭ maps back to r; and that r maps over to $\delta\alpha \pm iy$ in (1), and that $g_{u,v}(\delta\alpha)$ is calculated by setting $r = \delta\alpha$ in Ƭ. As well, in the *generating* function $f(s)$, r maps to $s + \delta\alpha$ in the complex plane $\mathcal{C}$. And here, we have $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and we'll suppose δ, Δ are > 0, for the time being.

$$2\kappa \int_{\delta\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\delta\alpha + iy, \theta) - g_{u,v}(\delta\alpha - iy, \theta)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\delta\alpha + iy, \theta) + g_{u,v}(\delta\alpha - iy, \theta)]\}\, dy \,/\, \sqrt{y^2 - (\delta\varepsilon)^2} + \mathrm{R} \qquad (1)$$

With these things in mind, we can now *generalize* Ƭ by writing Ƭ′ $= 1 / h(r, \theta, \Delta)$, so that $f(s)$ now becomes, where $r = s + \delta\alpha$ …

$$f(s) = 1 / \{h(r, \theta, \Delta) \cdot \sqrt{s^2 + (\delta\varepsilon)^2}\}\,. \qquad (\dagger)$$

And from (†), R can be calculated, by *doubling* the *residue* or *residues* associated with the *first* term in the denominator of $f(s)$, when taking its Laplace inverse; *or* by calculating along branch lines, if we are dealing with *branch* points. That is to say, for the Laplace inverse

$$\kappa \int e^{sr} f(s)\, ds\,,$$

for some appropriate contour γ, we examine the equation $h(s + \delta\alpha, \theta, \Delta)$ for *poles* or *branch* points, and compute accordingly, just as we have done all along, in this tutorial [p 547 *ff*.].

In the harmonic expression (1), we note that $g_{u,v}(\delta\alpha \pm iy, \theta, \Delta)$ becomes Ƭ′ $= 1 / h(r, \theta, \Delta)$, where here we have $r = \delta\alpha \pm iy$ in $\mathcal{C}$, and when moving back to $\mathcal{R}^2$,

$$g_{u,v}(\mathrm{r}, \theta, \Delta) = \text{Ƭ}' = 1 / h(\mathrm{r}, \theta, \Delta)\,;$$

so that r maps to r in Ƭ′. And this is the form for $g_{u,v}(\mathrm{r}, \theta, \Delta)$ when calculating $C_{u,v}$ in (*), so that again, $g_{u,v}(\delta\alpha)$ is calculated by setting $\mathrm{r} = \delta\alpha$ in $g_{u,v}(\mathrm{r}, \theta, \Delta)$. Clearly, any solution to (*) will be a function of $(\mathrm{r}, \theta, \delta, \Delta)$, but in Ƭ′, I see no need to carry δ in the argument list; yet this could change.

No doubt, the correct template Ƭ′, for *all* δ, Δ, $\mathrm{r} > 0$, is some kind of *infinite* series, that reduces to Ƭ as δ, Δ, r ---› 1 [p 566]; but whether we will ever find Ƭ′ is an *open* question, to say the least. But a *simple* generalization of Ƭ might look like $1 / (r^{\Delta} \cdot h(\theta) + \delta \cdot \Delta)$, where here, we *do* include δ.

And, an example of an *infinite* series that reduces to Ꚍ as δ, Δ, r ---› 1, where δ *is* included, might be

$$Ꚍ^* = \lim n \dashrightarrow \infty \;\; f(n, r, \delta, \Delta) \cdot \sum_{k=1}^{n} (1 / \{r^{k\Delta} \cdot h(\theta) + (\delta \cdot \Delta)^k\}) \, ,$$

where *r* becomes r in $\mathscr{R}^2$. And here, we shall define $f(n, r, \delta, \Delta) = 1 / n$, if and *only* if $\delta = \Delta = r = 1$; otherwise, we shall define $f(n, r, \delta, \Delta) = (\delta \cdot \Delta - 1)$, for *all* δ, Δ, $r > 0$. And to ensure convergence of the *infinite* series above, we'll suppose in this latter case that $\delta > 0$, and that $\Delta > 1 / \delta$ [recall that the location of the dark energy singularities is at (±δ, 0)].

Furthermore, we will assume $h(\theta)$ is some smooth, bounded, and *non*-negative function in the range $0 \le \theta \le \pi / 2$. Thus, with δ, Δ, $r = 1$, $Ꚍ^*$ becomes $1 / (h(\theta) + 1)$, which agrees with Ꚍ, in this case; otherwise $Ꚍ^*$ converges to something meaningful, for *all* δ, Δ, $r > 0$, where it is *no* longer the case that δ, Δ, $r = 1$.

Note that in $Ꚍ^*$, where it is *not* the case that δ, Δ, $r = 1$ [meaning *all three* variables can't equal 1, *simultaneously*]; if we then set $r = 1$, then as δ, Δ ---› 1, $f(n, r, \delta, \Delta) = (\delta \cdot \Delta - 1)$ ---› 0. And this is the *same* as setting $f(n, r, \delta, \Delta) = 1 / n$, with δ, Δ, $r = 1$, and letting n ---› ∞ in $Ꚍ^*$. Thus, there is a *smooth* transition here between the two cases, and indeed, we can actually *define* the transition in exactly this way.

We can even go further, by considering templates of type $Ꚍ' = 1 / (r^{\Delta} \cdot h(r, \theta) + \delta \cdot \Delta)$, so long as in $\mathscr{R}^2$, where r = r, it is the case that r and θ remain *independent* of one another. For example, we might let the function $h(r, \theta) = h(r \cdot \theta)$, so that as δ, Δ, r ---› 1 in $\mathscr{R}^2$, $h(\mathrm{r}, \theta) = h(\theta)$ and we return to the template Ꚍ, where Ꚍ is defined as $1 / (r \cdot h(\theta) + \Delta)$. However, such an extension complicates further the integrand in (1), in so much as *r* maps over to $\delta\alpha \pm iy$ for Ꚍ′, and in the *generating* function $f(s)$, *r* maps over to $s + \delta\alpha$ for Ꚍ′, so that

$$f(s) = Ꚍ' \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2}$$

Yet, ultimately, we may need to go down this path, for when dealing with the *quantumlike* component, as shown below, we note that the argument to the *cosh* and J_0 functions has these 'cross terms', where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$.

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha)$$

As an example, if we had a template like $Ꚍ'' = 1 / \sin(r\theta)$, and wanted to see if the *integral* in (1) below converged [$g_{u,v} = Ꚍ''$], we would let $r = \delta\alpha \pm iy$ in Ꚍ″, and calculate, accordingly. The result follows on the next page, where r = 2, δ = 1, θ = π / 4, and α = cos(θ), ε = sin(θ).

$$2\kappa \int_{\delta\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\delta\alpha + iy, \theta) - g_{u,v}(\delta\alpha - iy, \theta)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\delta\alpha + iy, \theta) + g_{u,v}(\delta\alpha - iy, \theta)]\}\, dy \,/ \sqrt{y^2 - (\delta\varepsilon)^2} + R \qquad (1)$$

$$\int_{\frac{1}{\sqrt{2}}}^{100} \frac{1}{(i\,\pi)\sqrt{y^2-0.5}} \cos(2\,y) \left(\left(\frac{1}{\sin\left(\frac{1}{4}\,(0.707+i\,y)\,\pi\right)} - \frac{1}{\sin\left(\frac{1}{4}\,(0.707-i\,y)\,\pi\right)}\right) + i\sin(2\,y) \left(\frac{1}{\sin\left(\frac{1}{4}\,(0.707+i\,y)\,\pi\right)} + \frac{1}{\sin\left(\frac{1}{4}\,(0.707-i\,y)\,\pi\right)}\right)\right) dy = 0.158878$$

The good news here is that we *do* have convergence, and so can now begin to consider templates like Ʈ″, and indeed, *infinite* series based on Ʈ″, like say

$$Ʈ^{*} \;=\; \lim n \dashrightarrow \infty \;\; f(n, r, \delta, \Delta) \cdot \sum_{k=1}^{n} (1 \,/\, \{r^{k\Delta} \cdot h(r \cdot \theta) + (\delta \cdot \Delta)^{k}\}) \;;$$

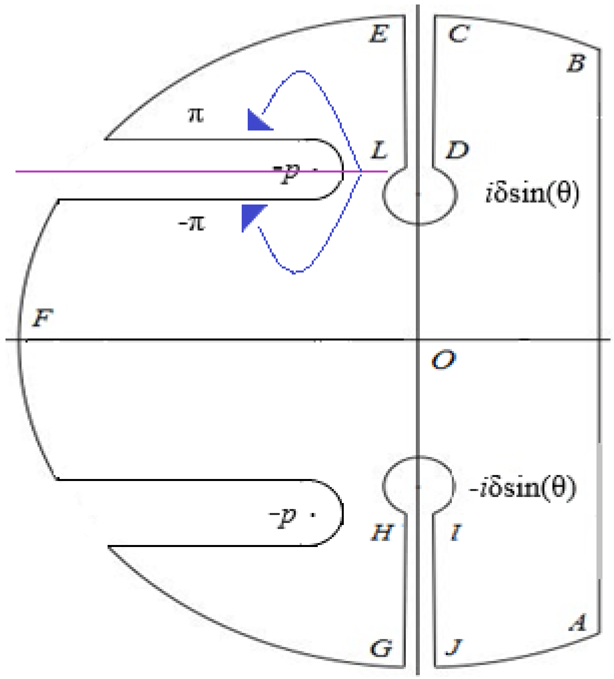
where in this case, we might let $h(r \cdot \theta) = \sin(r\theta)$, when *testing* [the summation just above, however, must *still* converge in $\mathcal{R}^2$]. And note again that $Ʈ^*$ reduces to Ʈ, as δ, Δ, r ---› 1 in $\mathcal{R}^2$, since r = r here.

Clearly the construction of $Ʈ^*$ is purely hypothetical, but it does tell us that such constructions are possible, and could lead to a template that actually *is* a suitable candidate for the *quantumlike* component, as shown below …

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha)$$

Is $1 / (\sqrt{r^4 + \Delta^4})$ A Suitable Template For $C_{u,v} \approx \sigma g_{u,v}(0)$, Where $\Delta > 0$, Part II

In this note, we want to revisit the calculations for the contour below, to see how *phasing* comes about, and ultimately, to see how the integrals along the *upper* and *lower* branch lines were developed.



$p = \delta\alpha + \gamma$

$\varepsilon = \sin(\theta)$

$\gamma = \sqrt{2}/2 \pm i\sqrt{2}/2$

To start, we take the Laplace inverse of $f(s)$, as shown below,

$$f(s) = 1 / \{\sqrt{(s + \delta\alpha)^4 + \Delta^4} \cdot \sqrt{s^2 + (\delta\varepsilon)^2}\}$$

where the Laplace inverse, itself, is

$$\kappa \int e^{sr} f(s)\, ds\ .$$

When we arrive at the *upper* branch lines $\mathcal{U}$, that are marked with the *phases* π and –π, the *first* integration runs from $s = -\infty + i\sqrt{2}/2$ to $-p$, where $-p = -\sqrt{2}/2 + i\sqrt{2}/2$. Setting $s' = s + p$, the integration now runs from $-\infty$ to 0, and letting $s' = -x$, we arrive at the expression below, at least up to *sign*, after *doubling* off of *either* branch line in $\mathcal{U}$, where $\Delta = 1$, and δ ---› 0.

$$2\kappa \int_0^{\infty} exp(-r(x + p)) / \{(x + p)\sqrt{(x + p)^4 + \Delta^4}\}dx \qquad \textit{upper branch lines } \mathcal{U}$$

I say up to *sign*, because the choice of *which* branch line to double off of, is really up to us, if we ignore *phases*. We can now do the same for the *lower* branch lines $\mathcal{L}$, where $-p = -\sqrt{2}/2 - i\sqrt{2}/2$.

Note that in taking this approach, we wind up sweeping around the origin O, in *clockwise* fashion, in the *right*-half of the *complex* plane $\mathcal{C}$, and then heading back out to ∞, where the *top* branch line now has a phase of –π, and the *bottom* branch line a phase of π. We'll call these branch lines $\mathcal{U}'$ when dealing with $\mathcal{U}$, and $\mathcal{L}'$ when dealing with $\mathcal{L}$, which you can see in the diagram on page 572.

But *nowhere* in this exercise in Part I, do we actually *see* the *phases* come into play, for the integration above, yet this can be seen by *factoring* the *square root* term as follows, for $\mathcal{U}'$ …

Let us write the *square root* term as

$$\sqrt{(x+p)^4 + \Delta^4} = \sqrt{(-1)\{(xp'+1)^4 - \Delta^4\}}\,, \qquad (*)$$

where $p = exp(-i\pi/4)$, $p' = exp(i\pi/4)$, $\Delta = 1$, and (-1) inside the *square root* is actually p^4.

Then it becomes clear, for the *top* branch line in $\mathcal{U}'$, where the phase is now –π, that $\sqrt{(-1)} = -i$; and similarly for the *bottom* branch line in $\mathcal{U}'$, where the phase is now π, that $\sqrt{(-1)} = i$. Hence, we *double* the integration along the *top* branch line, to arrive at the following expression for $\mathcal{U}'$…

$$2\kappa\int_0^\infty exp(-r(x+p)) \,/\, \{(x+p)\sqrt{(-1)\{(xp'+1)^4 - \Delta^4\}}\}dx \qquad \textit{upper branch lines } \mathcal{U}'$$

And a similar thing can be done for the *lower* branch lines $\mathcal{L}$, associated with $-p = -\sqrt{2}/2 - i\sqrt{2}/2$, where $p = exp(i\pi/4)$, $p' = exp(-i\pi/4)$, $\Delta = 1$, and (-1) inside the *square root* is actually p^4.

$$2\kappa\int_0^\infty exp(-r(x+p)) \,/\, \{(x+p)\sqrt{(-1)\{(xp'+1)^4 - \Delta^4\}}\}dx \qquad \textit{lower branch lines } \mathcal{L}'$$

Here are the snapshots for $\mathcal{U}'$ and $\mathcal{L}'$, respectively, where we use the identity (*) in the integrations above, and it is the case that ($\delta = 0$, $\Delta = 1$, r = 1) …

$$\int_0^{12} \frac{\exp(-(x + 0.707 - 0.707\, i))}{\pi\left((x + 0.707 - 0.707\, i)\sqrt{(x\,(0.707 + 0.707\, i) + 1)^4 - 1}\right)}\, dx =$$

$$0.0581305 + 0.0702366\, i$$

$$\int_0^{12} \frac{\exp(-(x + 0.707 + 0.707\, i))}{\pi\left((x + 0.707 + 0.707\, i)\sqrt{(x\,(0.707 - 0.707\, i) + 1)^4 - 1}\right)}\, dx =$$

$$0.0581305 - 0.0702366\, i$$

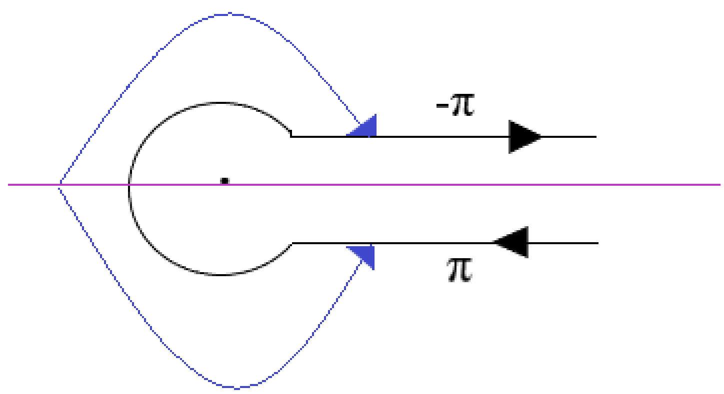
Notice that in *both* cases, we are *doubling* the integration from the *top* branch line where the phase is –π, so that $\sqrt{(-1)} = -i$, and thus with $\kappa = 1/2\pi i$, the constant becomes $2\kappa \cdot 1 / \sqrt{(-1)} = 1/\pi$.

Now let's compare these results with what we found in Part I [p 565], where we did *not* make the substitution in (*) above, and so did *not* deal with phases *directly* ($\delta = 0$, $\Delta = 1$, $r = 1$) …

$$\int_0^{12} -\frac{i\, e^{-(1-i)/\sqrt{2}-x}}{\pi\left(\frac{1-i}{\sqrt{2}}+x\right)\sqrt{1+\left(\frac{1-i}{\sqrt{2}}+x\right)^4}}\,dx = 0.0581038 + 0.0702193\, i$$

$$\int_0^{12} \frac{i\, e^{-(1+i)/\sqrt{2}-x}}{\pi\left(\frac{1+i}{\sqrt{2}}+x\right)\sqrt{1+\left(\frac{1+i}{\sqrt{2}}+x\right)^4}}\,dx = 0.0581038 - 0.0702193\, i$$

The two are nearly *identical*, and would be *closer* still, if not *exact*, were we to replace .707 in the integrations on the last page, with $\sqrt{2}/2$.



The picture above shows the branch lines for $\mathcal{U}'$ or $\mathcal{L}'$, where the *black* dot represents the origin O in $\mathcal{C}$. Notice how the phases have reversed themselves from the diagram on page 570. The integrations on page 571 now sweep around the origin, in *clockwise* fashion, in the *right*-half of the *complex* plane $\mathcal{C}$, and head out to ∞ …

And we'd like to say again, that the *first* integral in Part I [p 563], associated with $\mathcal{U}$, is actually a *doubling* of the integration on the *top* line in the diagram above; while the *second* integral in Part I [p 563], associated with $\mathcal{L}$, is actually a *doubling* of the integration along the *bottom* line in the diagram above [which causes a *sign* change]. However, nowhere here do we actually use *phasing*; rather, a more intuitive approach, and that is because when I wrote Part I, I wasn't sure how to rewrite the term $\sqrt{(x+p)^4 + \Delta^4}$ that we now see in (*), on the previous page, to take advantage of phases. Yet both approaches [Parts I and II], oddly enough, seem to be acceptable …

Generalizing The Harmonic Expression $g_{u,v} = 1 / (r \cdot h(\theta) + \Delta)$ Where $\Delta > 0$, Part II

Following up on Part I [pp 567-9], we suppose that Ꞇ = $1 / (r \cdot h(\theta) + \Delta)$ satisfies our general equivalency theorem [$\mathcal{G}$]; namely, (1) = $g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$ for $\delta = \Delta = r = 1$, with $0 \le \theta \le \pi / 2$, so that the *quantumlike* form

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)g_{u,v}(\delta\alpha) \qquad (*)$$

holds, in this *very* specific case for Ꞇ [$\delta = \Delta = r = 1$].

$$2\kappa\int_{\delta\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\delta\alpha + iy, \theta) - g_{u,v}(\delta\alpha - iy, \theta)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\delta\alpha + iy, \theta) + g_{u,v}(\delta\alpha - iy, \theta)]\}\, dy / \sqrt{y^2 - (\delta\varepsilon)^2} + R \qquad (1)$$

Now suppose Ꞇ′ is some gravitational template that satisfies $\mathcal{G}$ for *all* δ, Δ, r; so that (*) holds more generally, in this case. Then in particular, Ꞇ′ satisfies $\mathcal{G}$ if δ, Δ, r all equal 1, and so Ꞇ′ must reduce to Ꞇ in this case, if the solution to (*) is *unique*.

On the other hand, suppose without loss of generality, that δ, Δ = 1. Then a simple *continuity* argument suggests some Ꞇ′ exists that satisfies $\mathcal{G}$ for *r* greater than 1, but arbitrarily *close* to 1, and hence is a suitable candidate for (*), for all $0 \le \theta \le \pi / 2$, say. Ꞇ′ will look similar to Ꞇ, and becomes Ꞇ as *r* ---› 1.

Now let $r' > 1$ be the *largest* value of the radius, for which a Ꞇ′ can be found that satisfies $\mathcal{G}$ in the range $1 \le r \le r'$, for all $0 \le \theta \le \pi / 2$, say. Then again, a simple *continuity* argument suggests some Ꞇ* exists for $r > r'$, that satisfies $\mathcal{G}$, where *r* is *close* to *r′*, and becomes Ꞇ′ as *r* ---› *r′*, from above.

Thus, we can cover off the *whole* of $\mathcal{R}^2$ in this way, by travelling out to ∞, or heading toward the origin O. And so, we can 'stitch' together a set of templates, smoothly, that satisfy $\mathcal{G}$ for all $r \ge 0$. Said another way, there is *no* reason to believe there is an *upper* bound *less* than ∞, or a *lower* bound *greater* than *zero*, at which the 'stitching' terminates.

And this further *justifies* the form (*) for the *quantumlike* component over all of $\mathcal{R}^2$, when there are dark energy singularities at (±δ, 0), despite the fact that we may never know what Ꞇ′ really is; only that it exists.

———

To solve (*), we'll begin by assuming δ, Δ = 1, and we'll solve it for Ꞇ = $1 / (r \cdot h(\theta) + \Delta)$, which will tell us what $h(\theta)$ actually is, when δ, Δ, *r* = 1; since here we know $\mathcal{G}$ is satisfied by Ꞇ, in this case. Now from our *continuity* argument, there must exist a Ꞇ′ = $1 / (\boldsymbol{h}(r, \theta) + \Delta)$, which is the *smooth* extension of Ꞇ, that satisfies $\mathcal{G}$ over *all* of $\mathcal{R}^2$, and thus is a suitable candidate for (*).

Furthermore, it must be the case that Ꞇ′ becomes Ꞇ as δ, Δ, *r* ---› 1, which implies, in particular, that $\boldsymbol{h}(r, \theta)$ ---› $r \cdot h(\theta)$, as we reach this limit. So solve (*), using Ꞇ′, to find $\boldsymbol{h}(r, \theta)$ and validate the special case when δ, Δ, *r* ---› 1. This may be a useful methodology for obtaining a solution to (*).

Whether a *single* template such as Ʈ or Ʈ′ will suffice when solving (*) is debatable; for even if we assume the *inner* block [$\mathscr{B}$] is the *default* r^2, we're still left with solving *two* equations; namely, the *radial* and *timelike* components corresponding to $u,v = 1$ and $u,v = 3$, respectively.

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha) \qquad (*)$$

And since the general equivalency theorem [$\mathcal{G}$] doesn't tell us [and indeed, can't tell us] whether the template is *radial* or *timelike*, it is up to us to decide.

If a *single* template can handle both cases, then we can proceed to solve (*) … according to the remarks above. And if we need *two* templates that satisfy $\mathcal{G}$, then I would suggest we write Ʈ as

$$Ʈ_{u,v} = 1 / (r \cdot h_{u,v}(\theta) + \Delta_{u,v}) \,,$$

to indicate this difference. And similarly for Ʈ′, where for the *radial* and *timelike* components, we would write

$$Ʈ'_{u,v} = 1 / (\boldsymbol{h}_{u,v}(r, \theta) + \Delta_{u,v}) \,.$$

In both cases, however, we would insist that $Ʈ'_{u,v}$ becomes $Ʈ_{u,v}$ as δ, $\Delta_{u,v}$, r ---› 1, which implies, in particular, that $\boldsymbol{h}_{u,v}(r, \theta)$ ---› $r \cdot h_{u,v}(\theta)$, as we reach this limit. Again, it is understood that both $Ʈ_{u,v}$ and $Ʈ'_{u,v}$ satisfy $\mathcal{G}$.

Similar remarks apply when dealing with a dark energy singularity at the origin O, where here, we have

$$C_{u,v} \approx \sigma \cdot g_{u,v}(0) \,. \qquad (\dagger)$$

A number of templates were studied for this equation [p 547 *ff*.], which is much easier to deal with than (*), so it should be easier to verify that they satisfy $\mathcal{G}$; in the event we need *two* templates for the *radial* and *timelike* directions, respectively, when solving (†).

Generalizing The Equivalency Theorem To Three Dimensions

In *two* dimensions, the generalized equivalency theorem [$\mathcal{G}$] states that (1) = $g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$ *if* and only *if* (*) holds, where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, and the singularities are at $(\pm\delta, 0)$ …

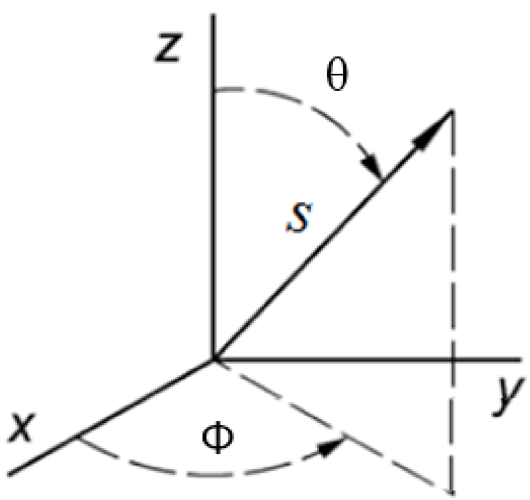
$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha) \qquad (*)$$

$$2\kappa \int_{\delta\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\delta\alpha + iy, \theta) - g_{u,v}(\delta\alpha - iy, \theta)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\delta\alpha + iy, \theta) + g_{u,v}(\delta\alpha - iy, \theta)]\}\, dy \,/\, \sqrt{y^2 - (\delta\varepsilon)^2} + R \qquad (1)$$

In *three* dimensions, (*) becomes, in *physical* coordinates [see diagram below]

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\cos(\gamma)) J_0(\delta r\sin(\gamma)) g_{u,v}(\delta\cos(\gamma)) , \qquad (\dagger)$$

where here, $\cos(\gamma) = \sin(\theta)\cos(\phi)$, and the *physical* singularities are located at $(\pm\delta, 0, 0)$.



Thus, when going over to *three* dimensions, α in (1) above maps to $\cos(\gamma)$, and ε maps to $\sin(\gamma)$. And if $Ꞇ' = 1 / h(r, \theta, \phi, \Delta)$ is our *template* in $\mathcal{R}^3$, then we would replace the gravitational expressions $g_{u,v}(\delta\alpha \pm iy, \theta)$ in (1) with $Ꞇ'$, where r maps to $\delta\alpha \pm iy$ in $Ꞇ'$; and again, α is now $\cos(\gamma)$ in (1). And in $g_{u,v}(\delta\cos(\gamma))$ in (†), r becomes $\delta\cos(\gamma)$ in $Ꞇ'$, since $Ꞇ'$ is the gravitational template $g_{u,v}$ in $\mathcal{R}^3$.

As to the *generating* function $f(s)$, it becomes, where $r = s + \delta\alpha$ in $Ꞇ'$, and $\alpha = \cos(\gamma)$ and $\varepsilon = \sin(\gamma)$;

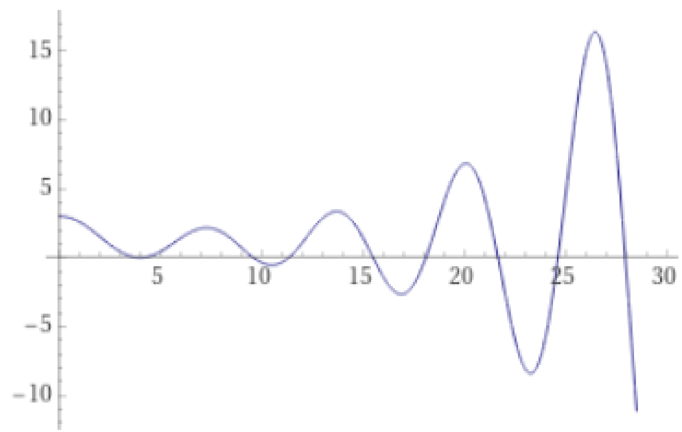
$$f(s) = Ꞇ' \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2} \,.$$

Interpreting *Radial* Versus *Time* In The Quantumlike Component

For the *quantumlike* form (*) shown below, we can speculate on what it means to *perceive* things in the *radial* direction [u,v = 1] versus the *time* direction [u,v = 3] in $\mathcal{R}^2$, where there are dark energy singularities at (±δ, 0), and we are in the *intangible* space τ_ξ.

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha) \qquad (*)$$

From page 496, we see what a typical *radial* wave pattern looks like in τ_ξ, for the dark energy component above; namely $\xi = 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon)$, where here σ, δ = 1, and α = cos(θ), ε = sin(θ).



r = *x*-axis , θ = 80° , *radial*

Such a wave ξ should possess both energy [E] and momentum [μ], and so just like in the *tangible* space $\tau_{\mathcal{M}}$ [pp 532-4], where the equation $C_{u,v} \approx kT_{u,v}$ applies for a perfect star [S*]; and μ corresponds to *pressure* (spacelike) in S* via $T_{u,v}$, while E corresponds to *density* (timelike) in S* via $T_{u,v}$; we can form a similar analogy for ξ in the *intangible* space τ_ξ, when looking at (*).

As such, when solving (*), we ought to form templates $Ʈ_{u,v}$ around this analogy, where one is for the *radial* component [μ], and the other for the *time* component [E]. Such an approach was noted on pages 573-4, and may prove to be feasible, based on the suitable template Ʈ = 1 / ($r \cdot h(\theta)$ + Δ).

In Waner's book *Introduction to Differential Geometry and General Relativity* [6th printing], he uses the templates $exp(2\Lambda(r))$ and $-exp(2\varphi(r))$, for the *radial* and *time* components, respectively, in the solution to $C_{u,v} \approx kT_{u,v}$, for a *perfect* star, and shown below. Curiously, the *radial* and *time* components in the solution g_{**} are *inverses* of one another, up to sign.

Schwarzschild Metric

$$g_{**} = \begin{bmatrix} \frac{1}{1-2M/r} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -(1-2M/r) \end{bmatrix}$$

But if, indeed, we've reached a point where the *coupled* equations (*) below *can* be justified over all of $\mathscr{R}^2$, according to our methodology $\mathscr{M}$ on pages 573-4, and previous research as well [p 316 *ff*.] …

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha)\,, \quad (*)$$

then we are free to choose our templates Λ and φ for (*), in the *radial* and *time* directions, however we want. They might *both* be functions of (r, θ, δ, Δ), and might also incorporate the *momentum* and *energy* of the ξ-wave, itself. Either way, the assumption here is that (*) is the *correct* form, whether it is induced by $\mathscr{M}$ or some other line of reasoning found elsewhere in this essay [see, for example, pp 461-479].

Thus, we don't necessarily have to adhere to the approach mentioned on pages 573-4, though it seems to be our safest bet at this point; for it guarantees us a template or *family* of templates that may imply the truth of (*). But whatever the approach, the generalized equivalency theorem [$\mathscr{G}$] should hold for the *normalized* solution $g_{u,v}$ to (*), since $\mathscr{G}$ is an 'if and *only* if' proposition [for more on normalized solutions, see pp 522-3].

In particular, therefore, if (*) is true for some $g_{u,v}$ [u,v = 1 or 3], then it must be the case that (1) = $g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$, provided the integral below *converges* …

$$2\kappa \int_{\delta\varepsilon}^{\infty} \{\cos(yr)[g_{u,v}(\delta\alpha + iy, \theta) - g_{u,v}(\delta\alpha - iy, \theta)] + \boldsymbol{i}\sin(yr)\,[g_{u,v}(\delta\alpha + iy, \theta) + g_{u,v}(\delta\alpha - iy, \theta)]\}\, dy \,/ \sqrt{y^2 - (\delta\varepsilon)^2} + R \qquad (1)$$

And we can go further now, and solve the *three*-dimensional analog; namely,

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\cos(\gamma)) J_0(\delta r\sin(\gamma)) g_{u,v}(\delta\cos(\gamma))\,, \quad (\dagger)$$

where here, $\cos(\gamma) = \sin(\theta)\cos(\phi)$, and the *physical* singularities are located at (±δ, 0, 0).

Again, we should feel confident that the templates Λ and φ, extended to *three* dimensions for the *radia*l and *time* components, *do* exist and will produce a solution $g_{u,v}$ to (†), which satisfies the *three*-dimensional analog of the general equivalency theorem [p 575].

———

Finally, it should be said that *suitable* templates *derived* or *inferred* via $\mathscr{G}$, are simply functional expressions that are *consistent* with the form for the *right*-hand side of (*) or (†), or with the form for the *right*-hand side of $C_{u,v} \approx \sigma \cdot g_{u,v}(0)^{(\S)}$, in the case of a dark energy singularity at the origin O. To see if they are really *solutions* could only be known if, for example, they satisfied (*) or (†), or in the simplest case, (§) …

A Simple Case Study

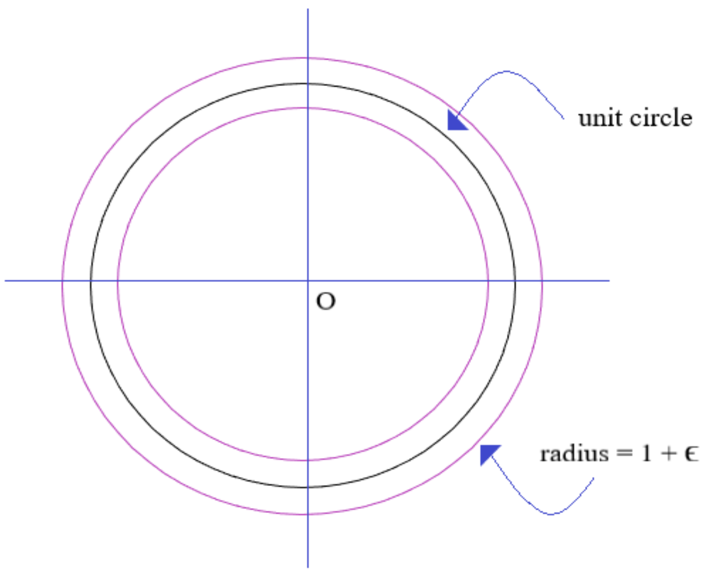
In our model, things are *static*. While it is true that the *quantumlike* radial waves [ξ] associated with (*) below, emanate from the origin O, they are *not* fluctuating with *time*. Nor are the dark energy singularities at $(\pm\delta, 0)$ *rotating* or *spinning* on a circle of radius $\delta > 0$, centered at O.

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha) \qquad (*)$$

$$\xi = 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon)$$

Thus, a diagonal solution $\mathcal{D}$ to (*) in $\mathcal{R}^2$ can be justified, where in this study we will allow the *inner* block [$\mathcal{B}$] to default to r^2 [for large r, however, it would be wise to include $\mathcal{B}$ in the calculations, according to some template, as per previous research notes (see, for example, pp 480-513)].

In the diagram below, we see a *thin* annulus $\mathcal{A}$ containing the unit circle C_1, so that the radius r is trapped between $1 - \epsilon$ and $1 + \epsilon$, where ϵ is *arbitrarily* small and greater than 0.



We'll attempt to solve (*) in $\mathcal{A}$, using the templates described on pages 573-4, and reproduced here; that is to say, for $u,v = 1$ or 3, where δ, $\Delta_{u,v} = 1$ and $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$,

$$Ꚍ_{u,v} = 1 / (r \cdot h_{u,v}(\theta) + \Delta_{u,v}) \,.$$

We choose templates of this type, because from our *potential* hypothesis [p 566], it is known that they are *consistent* with the *right*-hand side of (*), from the generalized equivalency theorem [$\mathcal{G}$].

But, you may ask, is there really more than one such $h_{u,v}(\theta)$? The answer is likely *yes*, because such templates arise from a study of $Ꚍ = 1 / (r\sin(\theta) + \Delta)$ [pp 559-62]; but we could just as easily have chosen functions *other* than *sin*(θ) in Ꚍ, thus leading to other *potential* hypotheses. And so, a

family $\mathcal{F}$ of suitable templates of type {$\mho_{u,v}$} emerges, which may be solutions to (*) in $\mathcal{A}$ for u,v = 1 or 3.

Now solve (*) in $\mathcal{A}$ for the *normalized* solution $\mathcal{N}$ (σ = 1), using the templates $\mho_{u,v}$ [u,v = 1 or 3, and $\mathcal{B} = r^2$], and then let ϵ ---› 0, so that the radius r ---› 1. This will tell us what $h_{u,v}(\theta)$ is along C_1, and thus what the solution $g_{u,v}$ is along C_1, as well [it is simply $\mho_{u,v}$, where the functions $h_{u,v}(\theta)$ are now known]. The general solution will then be $\sigma \cdot \mathcal{N}$.

———

It should be said that in (*), and reproduced below, *any* $C_{u,v}$ is calculated from {$\mho_{u,v}$}, and $g_{u,v}(\delta\alpha)$ is calculated by letting $r = \delta\alpha$ in $\mho_{u,v}$. As well, if $\mathcal{B} = r^2$, then u,v = 2, and here the solution $\mho_{u,v} = g_{u,v} = \mathcal{B}$ holds.

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)g_{u,v}(\delta\alpha) \qquad (*)$$

———

To interpret the result along C_1, we note that since this is a *diagonal* solution [$\mathcal{D}$], $g_{u,v}$ should be seen as a measure of the *ruler* tick in the direction of r, for any given θ, if u,v = 1; and as a measure of the *clock* tick in the direction of r, for any given θ, if u,v = 3; and as a measure of the *angular* tick in the direction of θ, for any given r, if u,v = 2. By 'measure', we mean 'reflection of', in some sense of the word.

Now we can solve the broader problem over *all* of $\mathcal{R}^2$, by considering templates like the following [pp 573-4], where again δ, $\Delta_{u,v}$ = 1 [u,v = 1 or 3, $\mathcal{B} = r^2$] …

$$\mho'_{u,v} = 1 / (\boldsymbol{h}_{u,v}(r, \theta) + \Delta_{u,v}) \, .$$

In both cases, however, we would insist that $\mho'_{u,v}$ becomes $\mho_{u,v}$ as r ---› 1, which implies, in particular, that $\boldsymbol{h}_{u,v}(r, \theta)$ ---› $r \cdot h_{u,v}(\theta)$, as we reach this limit. Again, it is understood that both $\mho_{u,v}$ and $\mho'_{u,v}$ satisfy $\mathcal{G}$, where u,v = 1 or 3.

A Simple Case Study, Part II

Imagine at point P = (r, θ) in $\mathcal{R}^2$ at a distance of r units from the origin O. Suppose further that there is *nothing* in the *tangible* space $\tau_{\mathcal{M}}$, so that relative to an *imaginary* observer [$\mathcal{O}$] at the *origin* in $\mathcal{R}^2$, the distance to P is simply r.

Now let's place our dark energy singularities at S = (±δ, 0) and ask ourselves how r changes, due to the presence of the *quantumlike* radial waves $\xi = 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)$, in the *intangible* space τ_ξ, which we see in the *coupled* equations below [pp 532-4] …

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)g_{u,v}(\delta\alpha) \quad (*)$$

Suppose that the *normalized* (σ = 1) solution [$\mathcal{N}$] to (*) is [$g_{u,v}$], so that the *general* solution is now $\sigma \cdot \mathcal{N}$ [pp 522-3], where σ > 0. To find out how r changes because of S, we perform the following integration along ℓ_θ [pp 395-7],

$$\int_0^r \sqrt{\sigma \cdot g_{1,1}}\, dr = \eta \pm i\zeta \quad (\dagger)$$

where $g_{1,1}$ is the *radial* component in $\mathcal{N}$ [we'll assume here that ζ ≥ 0].

Note that (†) could have an *imaginary* part, because the *radial* waves are both *positive* and *negative* in the range $0 < \theta \le \pi/2$, and *exponential* in nature if θ = 0 [see pages 345-9 for pictures of ξ, for different angles θ, in $\mathcal{R}^2$]. Thus, η is to be interpreted as *positive* distance and ζ as *negative* distance in (†), so that along ℓ_θ, r is now perceived by $\mathcal{O}$ to be $r + \eta - \zeta$.

As an example, we'll have a look at the following expression [labelling as (§)] …

$$\int_0^{2\pi} \sqrt{\sin(x)}\, dx = 4\, E(2) \approx 2.39628 + 2.39628\, i$$

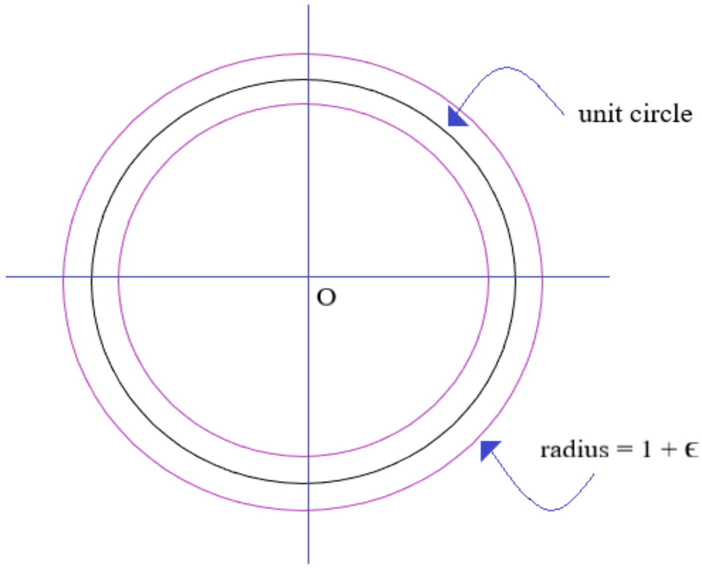
Here we see that the *real* and *imaginary* parts agree, so that if $g_{1,1}$ was $\sin(x)$, the change to r at 2π units from O would be *zero*.

Similar arguments can be put forth for the *angular* and *time* components in $\sigma \cdot \mathcal{N}$, using this particular interpretation of how dark energy in τ_ξ affects our perception of distance in $\mathcal{R}^2$.

A Simple Case Study, Part III

Let us bring back our *thin* annulus $\mathcal{A}$, as shown below, using our template $\mathrm{T}_{u,v}$ inside $\mathcal{A}$; that is to say, for $u,v = 1$, we have δ, $\Delta_{u,v} = 1$ and $h_{u,v}(\theta) \approx \sin(\theta)$, so that from our *potential* hypothesis,

$$g_{u,v} = 1 / (r \cdot h_{u,v}(\theta) + \Delta_{u,v}) .$$



We'll now situate our *imaginary* observer $\mathcal{O}$ at the origin O in $\mathcal{R}^2$, so that in the *absence* of any dark energy, and *nothing* in the tangible space $\tau_{\mathcal{M}}$, $\mathcal{O}$ will say the distance $\mathcal{D}$ *between* the inner and outer rings of $\mathcal{A}$ is always 2ϵ, for any choice of θ.

Now let's place our dark energy singularities at $S = (\pm\delta, 0)$ and ask ourselves how $\mathcal{D}$ changes, due to the presence of the *quantumlike* radial waves $\xi = 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)$, in the *intangible* space τ_ξ, which we see in the *coupled* equations below, where $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ …

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)g_{u,v}(\delta\alpha) \qquad (*)$$

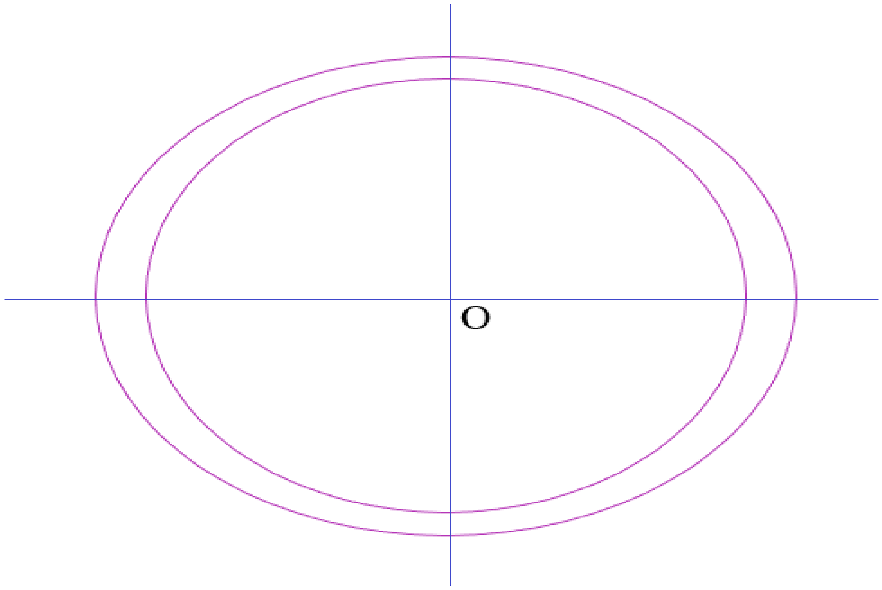
From our previous research note, the integral component in $\mathcal{A}$ is, for $\theta = 0$ [and thus $g_{1,1} = 1$] …

$$\int_{1-\epsilon}^{1+\epsilon} \sqrt{\sigma \cdot g_{1,1}}\, \mathrm{dr} = 2\epsilon \cdot \sqrt{\sigma} ,$$

so that to $\mathcal{O}$, it is the case that $\mathcal{D} = 2\epsilon + 2\epsilon \cdot \sqrt{\sigma}$. The distance, relative to $\mathcal{O}$, has *increased* because of the radial waves $\xi = 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)$, associated with dark energy.

Repeating for $\theta = \pi / 2$ [and thus $g_{1,1} = 1 / (r + 1)$, since $h_{1,1}(\theta) \approx \sin(\theta)$], we find that the integral component is now approximately $\epsilon \cdot \sqrt{2\sigma}$, so that $\mathcal{D} \approx 2\epsilon + \epsilon \cdot \sqrt{2\sigma}$, and thus has *increased* again because of ξ, but *less so* than the value when $\theta = 0$.

And this is because at $\theta = 0$, ξ is equal to $2\sigma \cdot \cosh(r)$ in $\mathcal{A}$, and *increasing*; whereas at the angle $\theta = \pi / 2$, ξ is equal to $2\sigma \cdot J_0(r)$ in $\mathcal{A}$, and *decreasing*. Thus, to $\mathcal{O}$ … the annulus appears to be more elliptical in shape, as shown in the picture below [not drawn to scale] …



OTHER CONSIDERATIONS

Whether ξ really has an influence in the *timelike* direction, on our imaginary observer $\mathcal{O}$ situated at O, is an *open* question, as far as I'm concerned. Certainly in the *tangible* space $\tau_{\mathcal{M}}$, with a perfect star centered at O, such an influence is apparent. But in the *intangible* space τ_{ξ}, what we really have here are *radial* waves [ξ], which are *quantumlike* in nature, emanating from O at different angles. And we see from the example above, how ξ can influence an observer's perception in $\mathcal{R}^2$, in the *radial* direction, which is *spacelike*.

If there is a justification for a *timelike* influence on $\mathcal{O}$, via ξ, then such an influence should be coded into the equations in (*), on the previous page [$u,v = 3$]. Otherwise, if no such influence really exists, then $u,v = 3$ should be omitted altogether in (*), in my view.

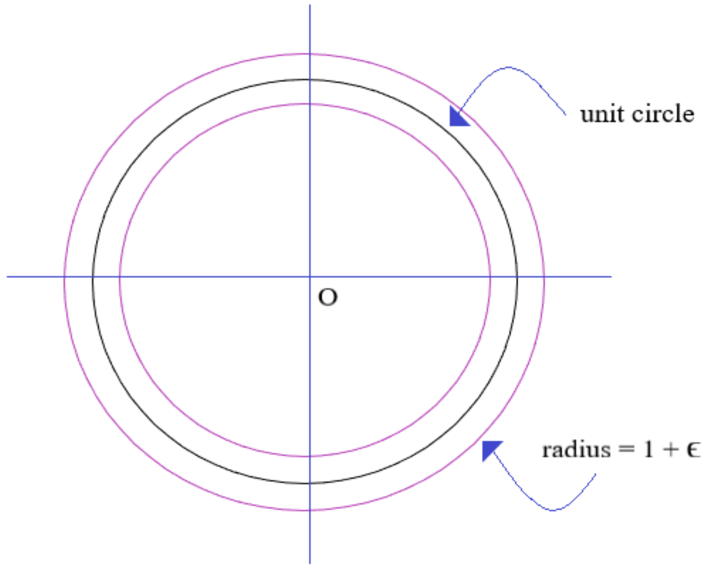
Similarly, we see from the picture above, that based *just* on an analysis of the radial component; that is to say $u,v = 1$, $\mathcal{O}$ can infer that the *unit* circle C_1 in $\mathcal{R}^2$ has become more elliptical. What then should we do with the *inner* block $\mathcal{B}$ [$u,v = 2$], especially since we already know that *angular* fluctuations along C_1 are really just *cross*-sectional views of ξ, for any given radius r.

In the end, this entire exercise of solving (*) may boil down to finding a solution for $g_{1,1}$ here, and nothing more. But at this point, it's too soon to say … ..

A Simple Case Study, Part IV

Let us bring back our *thin* annulus $\mathcal{A}$, as shown below, using our template $\mathrm{T}_{u,v}$ *inside* $\mathcal{A}$; that is to say, for $u,v = 1$, we have δ, $\Delta_{u,v} = 1$, so that from our *potential* hypothesis,

$$g_{u,v} \approx 1 / (r \cdot h_{u,v}(\theta) + \Delta_{u,v}) . \quad (\dagger)$$



The goal here is to develop a methodology for finding $h_{u,v}(\theta)$, so that we may use $g_{u,v}$ above to calculate *radial* distances inside $\mathcal{A}$, just as we did in the last research note. An improvement over the approximation $h_{u,v}(\theta) \approx \sin(\theta)$ in Part III, you might say.

The Riemann Curvature tensor must be at least *two*-dimensional, and from my reading of things, is proportional to R_{1212} [$1 = r$, $2 = \theta$] in an (r, θ) layout in $\mathcal{R}^2$, where we omit the time component altogether. From this, and using ($\dagger$), we should be able to calculate the Ricci tensor R_{11}, and the Ricci scalar R; so that in theory, at least, a solution to (*) below is possible for $u,v = 1$. Here, the *inner* block will default to r^2 [$u,v = 2$], and $\sigma = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$, so that we are looking for a *normalized* solution $\mathcal{N}$, for $u,v = 1$. The *general* solution will then be $\sigma \cdot \mathcal{N}$.

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha) \quad (*)$$

For $u,v = 1$, and using ($\dagger$), we see that (*) may be written as [σ, δ, $\Delta_{u,v} = 1$; where $(\pm\delta, 0)$ is the location of the dark energy singularities, $\mathcal{E} = 2\cosh(r\alpha)J_0(r\varepsilon)$, and $h = h_{u,v}(\theta)$] …

$$R_{1,1} - R / 2(r \cdot h + 1) \; = \; \mathcal{E} / (\alpha \cdot h + 1) .$$

And this results in the following quadratic

$$ah^2 + bh + c \; = \; 0 , \quad (\S)$$

where $a = 2r\alpha R_{1,1}$; $b = 2rR_{1,1} + 2\alpha R_{1,1} - \alpha R - 2r\mathcal{E}$; $c = 2R_{1,1} - R - 2\mathcal{E}$.

The *proper* solution to (§) will give us $h = h_{1,1}(r, \theta)$ in $\mathcal{A}$, but this will become $h_{1,1}(\theta)$ as r ---› 1. And this is the form for h in the *radial* solution, reproduced below, that we will use in $\mathcal{A}$ to calculate distances between the *inner* and *outer* rings of the annulus, just as we did in the previous research note [$u,v = 1$, $\Delta_{u,v} = 1$], where we let $h_{1,1}(\theta) \approx \sin(\theta)$.

$$g_{u,v} \approx 1 / (r \cdot h_{u,v}(\theta) + \Delta_{u,v}) \,. \qquad (\dagger)$$

Thus, (†) becomes an *approximation* in $\mathcal{A}$, for *small* ϵ, but from the perspective of integrating the expression below,

$$\int_{1-\epsilon}^{1+\epsilon} \sqrt{\sigma \cdot g_{1,1}} \ \mathrm{dr}$$

is much *easier* to deal with than $h = h_{1,1}(r, \theta)$ in (§) above. Indeed, it's unlikely we could even succeed at integrating, were we to use $h_{1,1}(r, \theta)$ in (†) …

OTHER CONSIDERATIONS

Strictly speaking, the quadratic (§) on the last page may be more complex, since terms like $R_{1,1}$ and R in the coefficients a, b, and c, could contain the function h, itself, and various derivatives. But regardless, in order to maintain consistency with the way in which (§) was derived; that is to say, via the template (†) for $u,v = 1$, and a *default* inner block $\mathcal{B} = r^2$ [$u,v = 2$]; it is best to let r ---› 1 *everywhere* in (§) first, and then solve for $h = h(\theta)$, accordingly.

For because (§) *is* derived via (†) and $\mathcal{B}$, necessarily h is a function of θ *only*, in (§). And so, to maintain this consistency when solving (§), we first let r ---› 1 *everywhere* here, and then proceed to solve (§) for h, in some suitable fashion.

A Simple Case Study, Part V

To show the reader just how complex the problem is, in terms of finding $h = h_{1,1}(\theta)$, I've calculated the *connection* coefficients for $R_{1,1}$ *only*, which arise by studying R_{1212}. They are listed below, where the template is $g_{u,v} \approx 1 / (r \cdot h_{u,v}(\theta) + \Delta_{u,v})$ [$u,v = 1$; δ, $\Delta_{u,v} = 1$], and the *inner* block $\mathcal{B} = r^2$.

$$\Gamma_1{}^1{}_1 = -h / 2(rh + 1) \;;\; \Gamma_1{}^2{}_2 = 1 / r \;;\; \Gamma_1{}^1{}_2 = -rh' / 2(rh + 1) \;;\; \Gamma_1{}^2{}_1 = h' / 2r(rh + 1)^2$$

$$\Gamma_2{}^2{}_2 = 0 \;;\; \Gamma_2{}^2{}_1 = 1 / r \;;\; \Gamma_1{}^2{}_{1,2} = d\Gamma_1{}^2{}_1/d\theta \;;\; \Gamma_1{}^2{}_{2,1} = -1 / r^2$$

Out of this, $R_{1,1}$ is formed, according to the formulas below, taken from Waner's book *Introduction to Differential Geometry and General Relativity* [6th printing, pages 88-89] …

$$\Gamma_b{}^a{}_c = \frac{1}{2} g^{ak}(g_{ck,b} + g_{kb,c} - g_{bc,k})$$

$$R_b{}^a{}_{cd} = [\Gamma_b{}^i{}_c \Gamma_i{}^a{}_d - \Gamma_b{}^i{}_d \Gamma_i{}^a{}_c + \Gamma_b{}^a{}_{c,d} - \Gamma_b{}^a{}_{d,c}].$$

$$R_{ab} = R_a{}^i{}_{bi} = g^{ij} R_{ajbi}$$

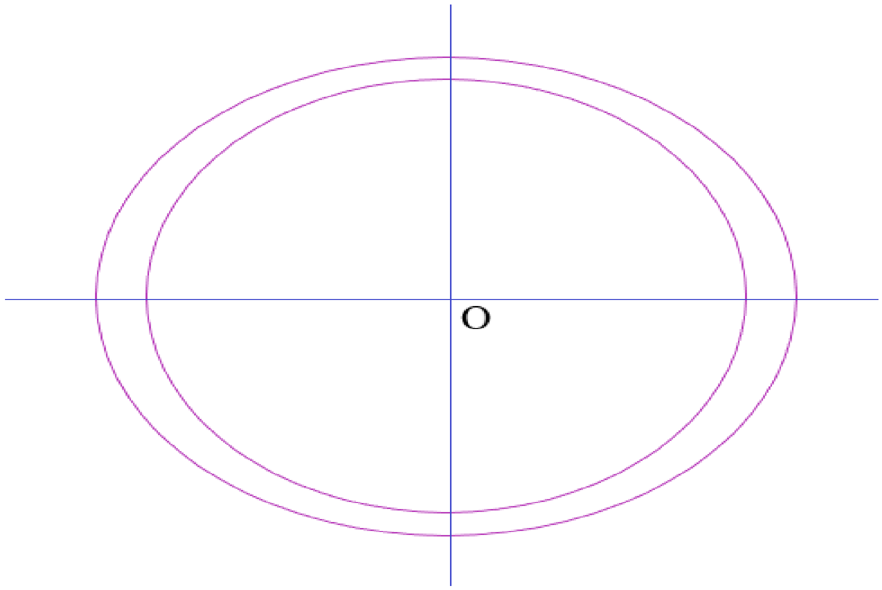
Note that $R_{1,1}$ is actually $R_1{}^2{}_{12}$, since $R_1{}^1{}_{11} = 0$, and you can see, *just* from calculating $R_{1,1}$, how difficult it is going to be to solve the equation below for $h(\theta)$, from Part IV [p 583], after first letting r ---› 1 here.

$$R_{1,1} - R / 2(r \cdot h + 1) = \mathcal{E} / (\alpha \cdot h + 1)\,.$$

But if a solution can be found, one of the conditions placed on $h(\theta)$ 'might' be $h(0) = 0$, and this stems from our analysis of Ⴇ $= 1 / (r \sin(\theta) + \Delta)$ [pp 559-62], where we saw that when $\theta = 0$, the match is *exact*, with respect to the *general* equivalency theorem $\mathcal{G}$ [see page 562, in particular].

A Simple Case Study, Part VI

In this note, we wish to calculate the *area* of the [distorted] elliptical annulus $\mathcal{E}$ below [pp 581-2], that our *thin* annulus $\mathcal{A}$ has *morphed* into, because of the dark energy *radial* waves generated from the singularities S at $(\pm\delta, 0)$; and emanating in *all* directions, from O.



Here, $g_{u,v} = 1 / (r \cdot h_{u,v}(\theta) + \Delta_{u,v})$, where $u,v = 1$, and $\delta, \Delta_{u,v} = 1$, with $h_{u,v}(\theta) \approx \sin(\theta)$, using our *potential* hypothesis (thus $g_{u,v}$ becomes a *simple* estimator). Now the integral $[\Delta_\epsilon]$ of interest to us, is shown below, and ultimately, because of *symmetry*, we'll multiply this integration by 4.

$$\int_0^{\pi/2} \int_{1-\epsilon}^{1+\epsilon} \sqrt{\sigma \cdot g_{1,1}}\ \sqrt{\sigma} r\, dr\, d\theta$$

We'll now situate our *imaginary* observer $\mathcal{O}$ at the origin O in $\mathcal{R}^2$, so that in the *absence* of any dark energy, and *nothing* in the tangible space $\tau_{\mathcal{M}}$, $\mathcal{O}$ will say that $\mathcal{E}$ really is $\mathcal{A}$, with *inner* and *outer* rings of radii $1 - \epsilon$ and $1 + \epsilon$, respectively; and hence $\mathcal{O}$ concludes that the area of $\mathcal{A}$ is just $4\pi\epsilon$.

Of interest to us is the following *indefinite* integral, where here $c = \sin(\theta)$ and $x = r$ …

$$\int x / \sqrt{cx + 1}\ dx \;=\; (2 / 3c^2)\sqrt{cx + 1}\,(cx - 2)\,,$$

so that Δ_ϵ becomes, after multiplying by 4,

.
.
.

$$(8\sigma / 3)\int_0^{\pi/2} (1 / \sin^2(\theta)) \cdot [\ \sqrt{(1+\ \epsilon)\sin(\theta)+1}\ ((1+\epsilon)\sin(\theta) - 2)\ -$$

$$\sqrt{(1-\ \epsilon)\sin(\theta)+1}\ ((1-\epsilon)\sin(\theta) - 2)\ ]\ d\theta$$

The first thing we want to check is that as θ ---› 0, the integral just above [Δ_ϵ] converges, since there is a *denominator* term of $\sin^2(\theta)$ here. So let $\sin(\theta) \approx \theta$, and expand each *square root* term using a Taylor series, for small θ ---› 0 and *fixed* ϵ; for example,

$$\sqrt{(1+\ \epsilon)\sin(\theta)+1}\ \approx\ \sqrt{(1+\ \epsilon)\theta+1}\ =\ \sqrt{1+(1+\ \epsilon)\theta}\ \approx\ 1 + ½ \cdot (1+\epsilon)\theta$$

Then after collecting terms, the *bracketed* expression in Δ_ϵ reduces to $2\epsilon \cdot \theta^2$, and since $\sin(\theta) \approx \theta$, we see that after dividing by the *denominator* term $\sin^2(\theta) \approx \theta^2$, we obtain 2ϵ. Thus, the integral converges as θ ---› 0.

Next, we want to evaluate Δ_ϵ as ϵ ---› 0, for *fixed* θ. Here we expand each *square root* term, using a Taylor series to *first* order, as per the following construction …

$$\sqrt{(1+\ \epsilon)\sin(\theta)+1}\ =\ \sqrt{(1+\ \sin(\theta))}\ \cdot\ \sqrt{(1+\ \epsilon\ \cdot \sin(\theta)\ /\ (1+\sin(\theta)))}$$

$$\approx\ \sqrt{(1+\ \sin(\theta))}\ \cdot\ \{1 + ½ \cdot \epsilon\sin(\theta)\ /\ (1+\sin(\theta))\}$$

If we now collect terms, the bracketed expression in Δ_ϵ … *divided* by the $\sin^2(\theta)$ term in the denominator, is equal to $3\epsilon / \sqrt{(1+\ \sin(\theta))}$, and so Δ_ϵ becomes …

$$(8\sigma\epsilon)\int_0^{\pi/2} 1 / \sqrt{(1+\ \sin(\theta))}\ d\theta\ =\ (8\sigma\epsilon)\{\log(3 + 2\sqrt{2})\} / \sqrt{2}$$

And this is the amount by which the *area* of the annulus $\mathcal{A}$ has *increased*, incrementally, due to the presence of dark energy, as it morphs into $\mathcal{E}$. That is to say, relative to $\mathcal{O}$, the area of $\mathcal{E}$ is the area of $\mathcal{A} + \Delta_\epsilon$.

As an example, if σ = 1, so we are dealing with a *normalized* solution for $g_{u,v}$, then $\Delta_\epsilon \approx 10\epsilon$, so that the area of $\mathcal{E}$ is nearly *double* that of $\mathcal{A}$, relative to $\mathcal{O}$. But in reality, σ is likely to be very small … perhaps on the order of the cosmological constant.

CORRECTION

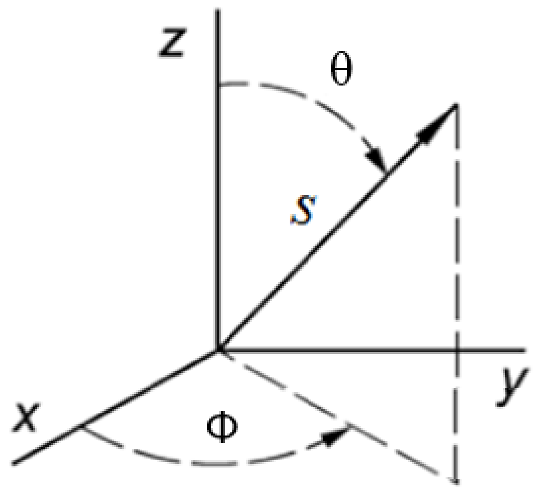
Since in the *normalized* solution $\mathcal{N}$ [σ = 1], the *inner* block is r^2 [pp 522-3], it becomes $\sigma \cdot r^2$ in the *general* solution, and so this research note has been updated in the integrals, to reflect the change. A correction was *also* made to the *right*-hand side of the integral just above, concerning the *log* argument.

A Simple Case Study, Part VII

In a *three*-dimensional model, the following *simple* estimator (*) for $g_{u,v}$ may be of interest to us, in our *thin* annulus $\mathcal{a}$ [spherical shell] containing the *unit* sphere, where $u,v = 1$, and $\delta, \Delta = 1$. Here, we are using a *physical* coordinate system, as shown below, where the dark energy singularities are located at $(\pm\delta, 0, 0)$, and $0 \leq \theta, \phi \leq \pi / 2$. Also, we should note that $\cos(\gamma) = \sin(\theta)\cos(\phi)$ in the *coupled* equations (†) for this setup.

$$g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta) \qquad (*)$$

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\cos(\gamma)) J_0(\delta r\sin(\gamma)) g_{u,v}(\delta\cos(\gamma)) \qquad (\dagger)$$



To see why (*) may be a *simple* estimator, let us start with $\phi = 0$. Then (*) reduces to the expression $1 / (r \cdot cos(\theta) + \Delta)$ in the *x*-*z* plane; and when $\theta = \pi / 2$, (*) becomes $1 / \Delta$ – its *maximum* value along the *x*-axis, where the dark energy $\xi = 2\sigma \cdot \cosh(\delta r\cos(\gamma)) J_0(\delta r\sin(\gamma))$ is also *strongest*.

On the other hand, when $\theta = 0$, ξ is at its *weakest* in the *x*-*z* plane; in line with (*), which takes on its *smallest* value of $1 / (r + \Delta)$. Now setting $\phi = \pi / 2$, we are in the *y*-*z* plane, and here (*) reduces to $1 / (r + \Delta)$, *no matter* the choice of θ, and that is because ξ is always at its *weakest* in this case. And finally, if $\theta = \pi / 2$, we are in the *x*-*y* plane, and (*) reduces to $1 / (r \cdot sin(\phi) + \Delta)$, which is what we expect.

For values of θ, ϕ in between 0 and $\pi / 2$, (*) is designed to interpolate *smoothly*, and so becomes an interesting candidate as a *simple* estimator for (†) in $\mathcal{a}$, *provided* it approximately satisfies the *three*-dimensional equivalency theorem, *on* the *unit* sphere [p 575; see also pp 559-562 for the analog in *two* dimensions, as well as the *potential* hypothesis on page 561, in particular].

NOTE TO THE READER

An earlier release put forth a *different* template in (*) than the one above; however, it is not fully *symmetric* in the *x*-*z* and and *x*-*y* planes, but the one above is …

OTHER CONSIDERATIONS

In the *two*-dimensional case, where there are singularities at $[(\pm\delta, 0), (0, \pm\delta)]$, our simple *normalized* $[\sigma = 1]$ estimator for the *thin* annulus $\mathcal{A}$ containing the *unit* circle, is simply $[u,v = 1 \,;\, \delta, \Delta = 1]$ …

$$g_{u,v} \;=\; 1 \,/\, (r \cdot \sin(\theta) + \Delta) \;+\; 1 \,/\, (r \cdot \cos(\theta) + \Delta) \,,$$

where $0 \leq \theta \leq \pi / 2$.

Since we're using a default *inner* block $\mathcal{B} = r^2$, it remains $\mathcal{B}$ when *adding* component solutions together [pp 522-3, 526-30]; so that the *normalized* solution $\mathcal{N}$ is $[g_{u,v} \,,\, \mathcal{B}]$, and the *general* solution is $\sigma \cdot \mathcal{N}$. The *time* component is omitted altogether, for now.

In the *three*-dimensional case, with singularities at $[(\pm\delta, 0, 0), (0, \pm\delta, 0), (0, 0, \pm\delta)]$, rotations are performed as per the notes on pages 532-4 and 536-7, where we define $[1 = x, 2 = y, 3 = z]$ …

$$\cos(\gamma)_1 = \sin(\theta)\cos(\phi) \;\;;\;\; \cos(\gamma)_2 = \sin(\theta)\sin(\phi) \;\;;\;\; \cos(\gamma)_3 = \cos(\theta)$$

Thus, $\cos(\gamma)_2$ is a 90 degree rotation of $\cos(\gamma)_1$ in the x-y plane, and $\cos(\gamma)_3$ is a 90 degree rotation of $\cos(\gamma)_1$ in the x-z plane, using *physical* coordinates.

The *normalized* estimator $[\sigma = 1]$ for our *thin* spherical shell, containing the *unit* sphere; where the default inner block $\mathcal{B} = [r^2 \,,\, r^2 sin^2(\theta)]$, is now $[u,v = 1 \,;\, \delta, \Delta = 1 \,;\, 0 \leq \theta, \phi \leq \pi / 2]$ …

$$g_{u,v} \;=\; 1 \,/\, (r \cdot \sin(\gamma)_1 + \Delta) \;+\; 1 \,/\, (r \cdot \sin(\gamma)_2 + \Delta) \;+\; 1 \,/\, (r \cdot \sin(\gamma)_3 + \Delta) \,.$$

A Simple Case Study, Part VIII

In this note, we are going to calculate the *increase* in volume of a spherical shell $\mathcal{A}$, with *inner* and *outer* spheres of radii $1 - \epsilon$ and $1 + \epsilon$, respectively; due to the presence of dark energy driven by the singularities at $(\pm\delta, 0, 0)$. We note that $\cos(\gamma) = \sin(\theta)\cos(\phi)$, in *physical* coordinates [p 588], and that the *inner* block defaults to $\mathcal{B} = [r^2\ , r^2 sin^2(\theta)]$ in a *normalized* $[\sigma = 1]$ solution.

We'll call this *ellipsoidal* annulus $\mathcal{E}$, that our *thin* shell $\mathcal{A}$ has *morphed* into, because of the dark energy *radial* waves generated from the singularities S at $(\pm\delta, 0, 0)$; and emanating in *all* directions, from the origin O.

Here, $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta_{u,v})$, where $u,v = 1$, and $\delta, \Delta_{u,v} = 1$, using our *potential* hypothesis (thus $g_{u,v}$ becomes a *simple* estimator). Now the integral $[\Delta_\epsilon]$ of interest to us, is shown below, and ultimately, because of *symmetry*, we'll multiply this integration by 8.

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_{1-\epsilon}^{1+\epsilon} \sqrt{\sigma \cdot g_{1,1}}\ \sqrt{\sigma} r \sqrt{\sigma} r\, sin(\theta)\ dr d\theta d\phi$$

We'll now situate our *imaginary* observer $\mathcal{O}$ at the origin O in $\mathcal{R}^3$, so that in the *absence* of any dark energy, and *nothing* in the tangible space $\tau_{\mathcal{M}}$, $\mathcal{O}$ will say that $\mathcal{E}$ really is $\mathcal{A}$, with *inner* and *outer* spheres of radii $1 - \epsilon$ and $1 + \epsilon$, respectively; and hence $\mathcal{O}$ concludes that the volume of $\mathcal{A}$ is just

$$(4\pi / 3)\{(1 + \epsilon)^3 - (1 - \epsilon)^3\}$$

Of interest to us is the following *indefinite* integral, where here $c = \sin(\gamma)$ and $x = r$ …

$$\int x^2 / \sqrt{cx + 1}\ dx = (2 / 15c^3) \sqrt{cx + 1}\ (3c^2x^2 - 4cx + 8)\ ,$$

so that Δ_ϵ becomes, after multiplying by 8,

$$(16/15)\ \sigma^{3/2} \int_0^{\pi/2} \int_0^{\pi/2} \Big[\sqrt{cr + 1}\ \{(3c^2r^2 - 4cr + 8) / c^3\}\Big]_{1-\epsilon}^{1+\epsilon} sin(\theta)\ d\theta d\phi$$

And here, the *bracketed* expression [B] is to be evaluated at its *upper* and *lower* limits of $r = 1 + \epsilon$ and $1 - \epsilon$, respectively; after which, we expand the result as a Taylor series to *first* order in ϵ, just as we did in Part VI [pp 586-7]. Almost magically, we find the *numerator* in B computes to $15c^3\epsilon$, and after dividing by the *denominator* c^3, yields $15\epsilon / \sqrt{c + 1}$. We then complete the balance of the integration, giving us the following expression for Δ_ϵ …

$$(16\epsilon)\sigma^{3/2} \int_0^{\pi/2} \int_0^{\pi/2} (sin(\theta) \,/\, \sqrt{c+1})\; d\theta d\phi$$

And this is the amount by which $\boldsymbol{a}$ expands into $\mathcal{E}$, incrementally, because of the dark energy *radial* waves. Note that $\mathcal{E}$ will be stretched *more* along the *x*-axis, than the *y*, *z* axes, because the dark energy singularities are at ($\pm\delta$, 0, 0).

CORRECTION

In a previous release, the denominator term $\sqrt{c+1}$, in the integral above, was omitted. Here, we have corrected that omission …

———

For the record, the integral above computes to ≈ 1.18367, as we see in the picture below, so that the *incremental* volume is $\Delta_\epsilon \approx 20\epsilon\sigma^{3/2}$.

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{1 + \sqrt{1 - \cos^2(y)\sin^2(x)}}}\, dy\, dx = 1.18367$$

A Simple Case Study, Part IX

Our *simple* estimator, for the *unit* circle C_1 in $\mathscr{R}^2$, is $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$ … where $u,v = 1$, and $\delta, \Delta = 1$; $0 \leq \theta \leq \pi / 2$ *and* the dark energy singularities are at $(\pm\delta, 0)$. What is important to observe here, is that $g_{u,v}$ is *consistent* with *spacelike* behavior [pp 580-91], and so it is natural to assume that the *potential* hypothesis, reproduced below, is referring to a template Ƭ $= 1 / (r \cdot h(\theta) + \Delta)$ that is *also spacelike* in nature.

> For some smooth function $h(\theta)$, which can be approximated by $\sin(\theta)$, where $0 < \theta < \pi / 2$, it is the case that $1 / (r \cdot h(\theta) + \Delta)$ is a suitable candidate for the *quantumlike* component in (*) below, on the unit circle …

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha) \qquad (*)$$

Thus, the *only* meaningful way to find $h(\theta)$, is by solving (*), and using Ƭ; where the *inner* block defaults to $\mathscr{B} = r^2$, in this case, *and* the initial condition $h(0) = 0$ holds. A tall order, given the complexities mentioned on pages 583-5, in particular. And since it is quite possible we will never actually find $h(\theta)$, we may have to be content with using *simple* estimators to calculate distance, area and volume, as we did in this *Simple Case Study* series. And that, in turn, is where the general equivalency theorem [$\mathscr{G}$] may be able to help us [pp 559-62, 575].

As to the *time* component, I am not sure, right now, if dark energy [ξ] really has a *temporal* influence on the *imaginary* observer $\mathscr{O}$, situated at the origin O in $\mathscr{R}^2$ [pp 581-2]; but if it does, then perhaps a *simple* estimator like $\mathsf{g}_{u,v} = 1 / (r \cdot \cos(\theta) + \Delta)$ might work here [$u,v = 3$] on C_1.

For with singularities at $(\pm\delta, 0)$, it behaves inversely to $g_{u,v}$ [$u,v = 1$], in so much as $\mathsf{g}_{u,v}$ *increases* as $g_{u,v}$ *decreases*, and conversely, as θ moves between 0 and $\pi / 2$ in $\mathscr{R}^2$. Said another way, the simple *timelike* estimator for singularities at $(\pm\delta, 0)$, is *equivalent* to the simple *spacelike* estimator for singularities at $(0, \pm\delta)$, where the latter approximately satisfies $\mathscr{G}$, as we know [pp 559-62].

And finally, similar remarks apply in $\mathscr{R}^3$, where here, *simple* estimators were discussed on pages 588-9.

When testing $\mathsf{g}_{u,v}$ as we did $g_{u,v}$ on pages 559-562, I saw that the agreement was *exact* if $\theta = \pi / 2$, and *exact* out to *six* decimal places if $\theta = 0$. I also noted that the result for $\theta = \pi / 4$ would *not* change; that is to say, it is .58164 versus .585902 for *both* $g_{u,v}$ and $\mathsf{g}_{u,v}$. This is probably enough evidence to support our guess that $\mathsf{g}_{u,v}$ may be used as a *simple* estimator, for the *time* component on the *unit* circle C_1, when there are singularities at $(\pm\delta, 0)$, *if* we choose to go down this path.

Hence, $\mathsf{g}_{u,v}$ becomes a *simple* estimator in the *thin* annulus $\mathscr{A}$ containing C_1, and by extension, a *simple* estimator in the *thin* shell containing a *unit* sphere; where here $\mathsf{g}_{u,v} = 1 / (r \cdot \cos(\gamma) + \Delta_{u,v})$, and $\cos(\gamma) = \sin(\theta)\cos(\phi)$ [providing, of course, $\mathsf{g}_{u,v}$ *approximately* satisfies the general equivalency theorem $\mathscr{G}$ in three dimensions (p 575), when there are singularities at $(\pm\delta, 0, 0)$].

Some additional testing was done for the *time* component $g_{u,v}$, on the *unit* circle, when $\theta = \pi / 6$ and $\delta, \Delta = 1$; with singularities at $(\pm\delta, 0)$. Here are the results …

$$\int_{0.5}^{300} \frac{\cos(y)\left(\frac{1}{1.75+0.866\, i\, y} - \frac{1}{1.75-0.866\, i\, y}\right) + i \sin(y)\left(\frac{1}{1.75+0.866\, i\, y} + \frac{1}{1.75-0.866\, i\, y}\right)}{(\pi\, i)\sqrt{y^2 - 0.25}}\, dy =$$

0.406792

$$2\sqrt{\frac{3}{13}}\; e^{-7/(2\sqrt{3})}$$

$$\frac{4 J_0\left(\frac{1}{2}\right)}{7}$$

Decimal approximation

0.127358759632430

value of R

Decimal approximation

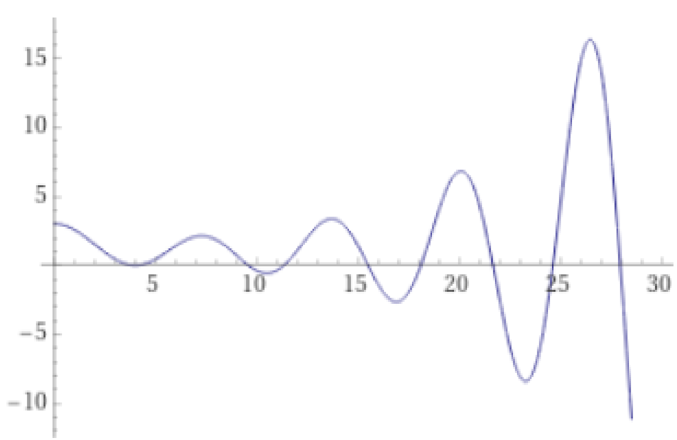
0.536268461280464

value of $\mathcal{E}$

From pages 559-62, we *add* the value of R to the value of the *harmonic* expression, just above; which gives us ~ .534151, and we can see here how *closely* it agrees with $\mathcal{E}$. Thus, we may conclude that $g_{u,v}$ is a *simple* estimator for the *time* component on the *unit* circle, and by extension, in the *thin* annulus $\mathcal{a}$ containing this circle.

Furthermore, we may conclude that in the presence of *stronger* dark energy [be it *positive* or *negative*], clock ticks *expand* and ruler ticks *contract*; and as the *stronger* dark energy becomes *weaker*, ruler ticks begin to *expand* and clock ticks begin to *contract*.

Note that *stronger* dark energy can be strongly *positive* or strongly *negative* [meaning further away from the *x*-axis], when looking at the quantumlike *radial* waves, as shown below; so that *clock* and *ruler* ticks can expand and contract in the *positive* or *negative* sense as well.



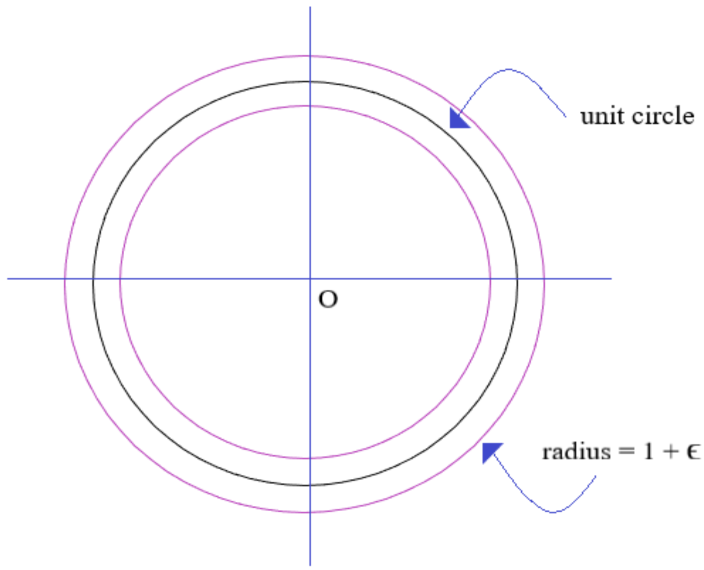
r = *x*-axis , $\theta = 80^o$, *radial*

A Simple Case Study, Part X

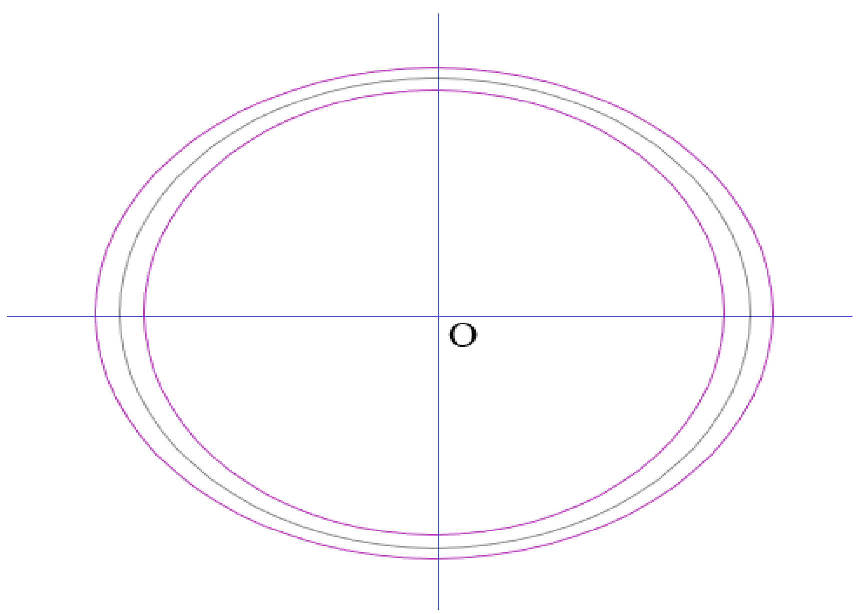
Let us bring back our *thin* annulus $\mathcal{A}$, as shown below, and note that for an *imaginary* observer $\mathcal{O}$ at the origin O in $\mathcal{R}^2$, where now there is *no* dark energy, and *nothing* in the tangible space $\tau_{\mathcal{M}}$; $\mathcal{O}$ will say the *area* between the *inner* and *outer* rings of $\mathcal{A}$ [with radii $1 - \epsilon$ and $1 + \epsilon$], is just $4\pi\epsilon$.

Now let $\epsilon \dashrightarrow 0$, so that we are actually *on* the unit circle C_1. Then the circumference of C_1 can be expressed as

$$\lim_{\epsilon \to 0} 4\pi\epsilon / 2\epsilon = 2\pi$$



Now let's place our dark energy singularities at S = $(\pm\delta, 0)$, and note that $\mathcal{A}$ morphs into the *elliptical* annulus $\mathcal{E}$, as shown in the diagram below [not drawn to scale]; where the lighter *green* ellipse E_1 is what C_1 has become, because of the dark energy *radial* waves ξ , emanating from O.



From part VI [pp 586-7], the *incremental* area in $\mathcal{E}$, that $\mathcal{A}$ has expanded into, is approximately $\Delta_\epsilon = 8\sigma\epsilon k$, where the constant $k = \{\log(3 + 2\sqrt{2})\} / \sqrt{2}$; and so to $\mathcal{O}$, the area of $\mathcal{E}$ is now the area of $\mathcal{A}$ plus Δ_ϵ, which is $\mathcal{A}' = 4\pi\epsilon + 8\sigma\epsilon k$.

If we assume that σ is *very* small, then we can divide $\mathcal{A}'$ by 2ϵ, using our lim ϵ ---› 0 argument above, and conclude that the *circumference* of E_1 in $\mathcal{E}$ is approximately $2\pi + 4\sigma k$.

———

In *three* dimensions, for a *thin* spherical shell $\mathcal{S}$ containing the *unit* sphere S_1, with inner and outer radii of $1 - \epsilon$ and $1 + \epsilon$, respectively; $\mathcal{O}$ will say in the absence of dark energy, and *nothing* in the tangible space, that the volume of $\mathcal{S}$ is, for small ϵ [pp 590-1] …

$$(4\pi / 3)\{(1 + \epsilon)^3 - (1 - \epsilon)^3\} \approx 8\pi\epsilon$$

Dividing this by 2ϵ, and using an ϵ ---› 0 argument, tells us that the *surface* area of S_1 is simply 4π. To find the *surface* area $\mathcal{S}'$ of S_1 as it morphs into an ellipsoid, due to the presence of dark energy, with singularities at $S = (\pm\delta, 0, 0)$; we add the *incremental* volume Δ_ϵ [pp 590-1] to $8\pi\epsilon$, and again, divide by 2ϵ, if σ is small. That is to say,

$$\mathcal{S}' \approx (8\pi\epsilon + 16\epsilon\sigma^{3/2}k) / 2\epsilon = 4\pi + 8\sigma^{3/2}k\ ,$$

where $\cos(\gamma) = \sin(\theta)\cos(\phi)$, $c = \sin(\gamma)$, and k is equal to

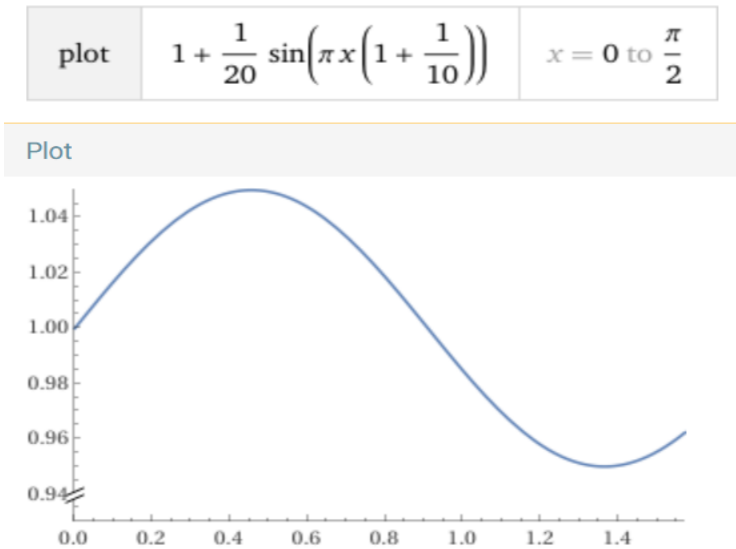
$$\int_0^{\pi/2}\int_0^{\pi/2} (\sin(\theta) / \sqrt{c + 1})\, d\theta d\phi$$

A Simple Case Study, Part XI

We can improve on our *simple spacelike* estimator for the *unit* circle C_1 in $\mathcal{R}^2$; that is to say, $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)^{(*)}$ … where u,v = 1, and $\delta, \Delta = 1$; $0 \le \theta \le \pi / 2$ *and* the dark energy singularities are at $(\pm\delta, 0)$. We do this by writing

$$g_{u,v} \approx 1 / (r \cdot \eta(\theta)\sin(\theta) + \Delta)\,, \qquad (\dagger)$$

where here $\eta(\theta) = 1 + (1/20)\sin(\pi\theta(1 + 1/10))$, as shown in the plot below.



Notice in the chart above, that we *over-compensate* if θ is approximately *less* than $\pi/4$, and do just the *opposite* if θ is approximately *greater* than $\pi/4$. Notice also that at $\theta = 0$, $\eta(\theta) = 1$, so that the general equivalency theorem [$\mathcal{G}$] is still *exact* for this choice of θ [pp 559-62]; and that at $\theta = \pi/2$, $\eta(\theta) \approx .96$.

We'll now calculate the various pieces that make up $\mathcal{G}$, using ($\dagger$), and see how things play out, when $\theta = \pi / 2$; the angle at which $\mathcal{H}_x$ + R deviated *most* from $\mathcal{E}$ [pp 559-62], using the *original* simple estimator, labelled (*) above. Here, $\mathcal{H}_x$ [shown below], is the *harmonic* expression itself, to which we add R, and then compare this result with $\mathcal{E}$. For the record, the *simple* pole is at $-\Delta / \eta(\theta)$, in this case.

$$\int_1^{300} -\frac{i\left(\left(-\frac{1}{\frac{24}{25}-iy}+\frac{1}{\frac{24}{25}+iy}\right)\cos(y)+i\left(\frac{1}{\frac{24}{25}-iy}+\frac{1}{\frac{24}{25}+iy}\right)\sin(y)\right)}{\pi\sqrt{-1+y^2}}\,dy = 0.30261$$

value of $\mathcal{H}_x$

$$\frac{2\exp\left(-\frac{1}{0.96}\right)}{\sqrt{\left(\frac{1}{0.96}\right)^2+1}}$$

Result

0.488742…

value of R

$$J_0(1)$$

Decimal approximation

0.765197686557966

value of $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$

Thus, the value of $\mathcal{H}_x$ + R is approximately .7913 here, versus $\mathcal{E} \approx .7652$. This is an improvement over the result on page 562 using (*), where $\mathcal{H}_x$ + R was ≈ .8281, and $\mathcal{E}$ remained the same at ≈ .7652.

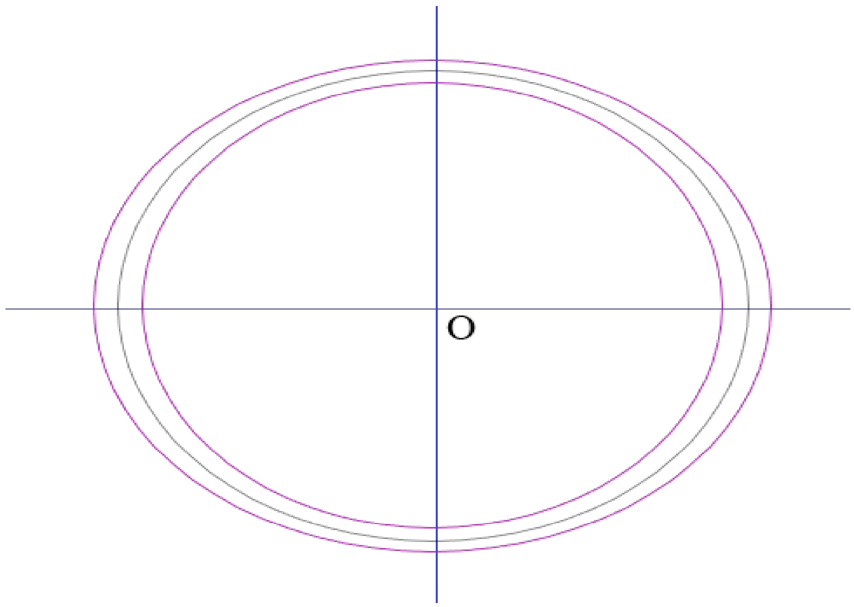
Furthermore, given the *oscillatory* nature of η(θ), we would expect to see improvement for *all* values of θ between 0 and π / 2; since in the original results [pp 559-62], $\mathcal{H}_x$ + R was *less* than $\mathcal{E}$ for θ approximately *less* than π / 4, and just the opposite for θ approximately *greater* than π / 4.

But finally, we have to recognize that by improving the *simple* estimator, as we did here; calculations for things like distance, area and volume, that were carried out in this *Simple Case Study* series, will inevitably become more complex.

There is always a trade-off in these matters, and so we can be satisfied that by using (*) instead of (†) for our calculations, we probably got fairly accurate results. In other words, would (†) yield anything much better ?

A Simple Case Study, Part XII

If we suppose the *simple* estimator for θ is *no* different than the *simple* estimator for $g_{1,1}$, on the *unit* circle C_1 in $\mathscr{R}^2$; then there is another way we can calculate the *circumference* of E_1, which is what C_1 becomes, when there are dark energy singularities at (±δ, 0) [E_1 is the *light* green ellipse as shown in the picture below, *not* drawn to scale; but see also pp 594-5].



For consider the following integral [Δ_ϵ], where $g_{u,v} = 1 / (r \cdot h_{u,v}(\theta) + \Delta_{u,v})$ when $u,v = 1$ *or* 2, and $\delta, \Delta_{u,v} = 1$, with $h_{u,v}(\theta) \approx \sin(\theta)$, using our *potential* hypothesis. Note that because of *symmetry*, we'll multiply this integration by 4, and note also that $u,v = 1$ *or* 2 means u and v are *both* 1 or 2.

$$\int_0^{\pi/2} \sqrt{\sigma \cdot g_{1,1}}\ \sqrt{\sigma \cdot g_{2,2}}\ d\theta$$

If we set $r = 1$ in *both* $g_{1,1}$ and $g_{2,2}$ above … then we are actually calculating the *incremental* circumference [Δ_ϵ] of E_1, due to dark energy; and this computes to 4σ, after multiplying the integral above by 4. Thus, to the imaginary observer $\mathscr{O}$ at the origin O in $\mathscr{R}^2$, the *total* circumference of E_1 is now $2\pi + 4\sigma$, which compares favorably with our estimate of $2\pi + 4\sigma k$ on pages 594-5, for *small* σ, where the constant $k = \{\log(3 + 2\sqrt{2})\} / \sqrt{2} \approx 1.25$. But again, this assumes $g_{1,1} = g_{2,2}$ on the *unit* circle.

For the *three* dimensional case, we suppose that on the *unit* sphere S_1, $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta_{u,v})$ when $u,v = 1, 2$ *or* 3, and $\delta, \Delta_{u,v} = 1$, with $\cos(\gamma) = \sin(\theta)\cos(\phi)$, using *physical* coordinates. Then our integral [Δ_ϵ] for the *incremental* surface area of E_1 becomes [where S_1 morphs into the distorted ellipsoid E_1, due to the dark energy singularities at (±δ, 0, 0)] …

$$\int_0^{\pi/2}\int_0^{\pi/2} \sqrt{\sigma \cdot g_{1,1}}\,\sqrt{\sigma \cdot g_{2,2}}\,\sqrt{\sigma \cdot g_{3,3}}\; d\theta d\phi$$

Note that because of *symmetry*, we'll multiply this integration by 8, so that after setting $r = 1$ in $g_{1,1}$, $g_{2,2}$, and $g_{3,3}$ above, Δ_ϵ computes to $8\sigma^{3/2}k$, where k is the constant below …

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\left(1+\sqrt{1-\cos^2(y)\sin^2(x)}\right)^{3/2}}\,dy\,dx = 1.01654$$

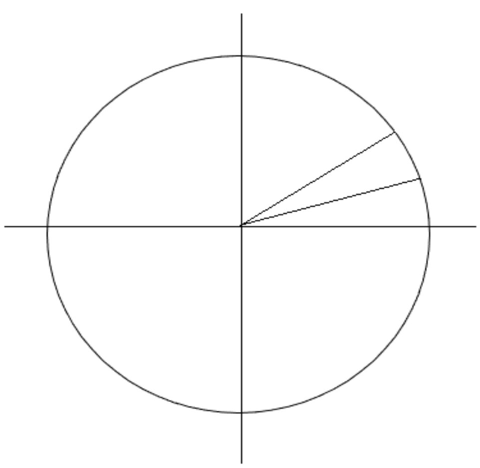
Thus, to the imaginary observer $\mathcal{O}$ at the origin O in $\mathcal{R}^3$ … the *total surface* area of E_1 now becomes $4\pi + 8\sigma^{3/2}k$; which compares favorably with our estimate of $4\pi + 8\sigma^{3/2}k$ on pages 594-5, for *small* σ, where k is equal to

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{1+\sqrt{1-\cos^2(y)\sin^2(x)}}}\,dy\,dx = 1.18367$$

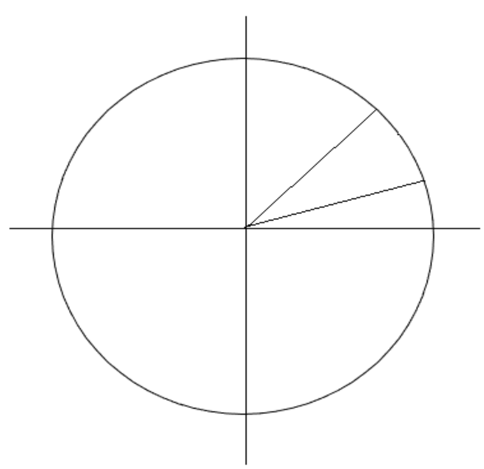
But again, this assumes $g_{1,1} = g_{2,2} = g_{3,3}$ on the *unit* sphere, and we should *also* note that *small* σ only applies in the arguments on pages 594-5; because there, we *only* use 2ϵ as our *radial* distance, when *area* or *volume* is divided by this quantity [see also pp 581-2]. And this requires that σ be small.

———

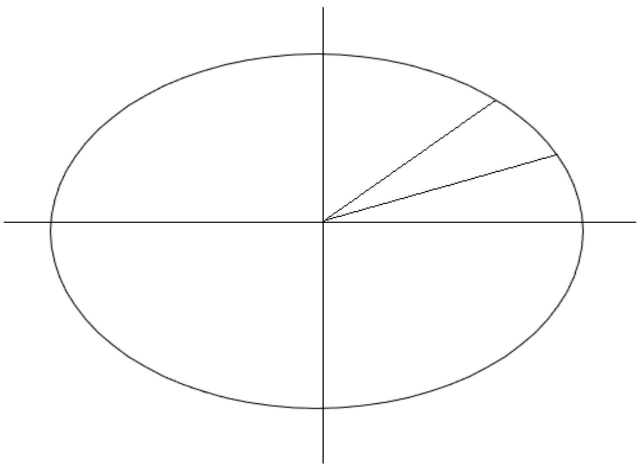
When calculating something like *circumference* or *surface* area, there are four cases to consider, as shown in the diagrams below; as θ moves from 0 to π / 2, within the context of general relativity and *dark* energy …

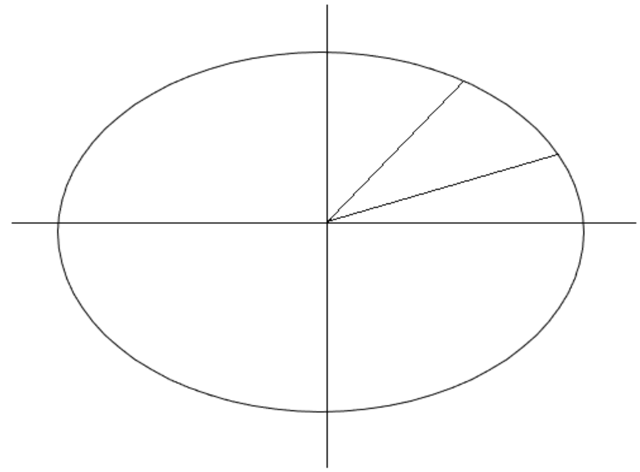


ruler tick and *angular* tick constant



ruler tick constant, *angular* tick changing





ruler tick changing, *angular* tick constant   *ruler* tick and *angular* tick changing

In calculating the *incremental* area or volume of a *thin* annulus containing the *unit* circle, or a *thin* shell containing the *unit* sphere [pp 586-7, 590-1], in the presence of dark energy [ξ]; we chose to go with the diagram on the *left*, just above, so that the *inner* block $\mathcal{B}$ was the default. And we did this to *simplify* the calculations.

But in this research note, to calculate the *circumference* or *surface area* of E_1, which is what the *unit* circle or *unit* sphere morphs into, because of ξ; we chose to go with the diagram on the *right*, just above, and hence the integrals on pages 598-9 are written the way they are.

Also, $g_{2,2}$ and $g_{3,3}$ are to be associated with the θ and ϕ components, respectively, using a *physical* coordinate system … ..

If, on the other hand, $g_{2,2}$ and $g_{3,3}$ behave more like the *timelike* estimator [pp 592-3], then the *normalized* integral on page 598 computes to

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1+\sin(x)}\,\sqrt{1+\cos(x)}}\,dx = \sqrt{2}\,\log(2) \approx 0.98026$$

so that $\Delta_\epsilon \approx 4\sigma$, after multiplying by 4; and this is about the same as using the *spacelike* estimator. And similarly for *three* dimensions: using the *timelike* estimator, the integral on pg 599 evaluates to

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\left(1+\cos(y)\sin(x)\right)\sqrt{1+\sqrt{1-\cos^2(y)\sin^2(x)}}}\,dy\,dx = 1.34305$$

and this *somewhat* agrees with the constant $k \approx 1.0165$, using *spacelike* estimators, though it is larger.

A Simple Case Study, Part XIII

To show that

$$\int_0^{\pi/2} \sqrt{\sigma \cdot g_{1,1}}\ \sqrt{\sigma \cdot g_{2,2}}\ d\theta \qquad (\dagger)$$

is the correct expression for calculating the *incremental* circumference [Δ_ϵ], as our *unit* circle C_1 morphs into the ellipse E_1, when there are dark energy singularities at ($\pm\delta$, 0); let us repeat the arguments from pages 586-7 and 594-5, where we now calculate Δ_ϵ as follows, presuming σ to be small [pp 581-2]. Thus, the *default* inner block $\mathcal{B}$ is replaced by the estimator $\sqrt{\sigma \cdot g_{2,2}}$ …

$$\Big[\lim_{\epsilon \to 0} \int_0^{\pi/2} \int_{1-\epsilon}^{1+\epsilon} \sqrt{\sigma \cdot g_{1,1}}\ \sqrt{\sigma \cdot g_{2,2}}\ dr\, d\theta\ \Big]\ /\ 2\epsilon \qquad (*)$$

We'll suppose $g_{u,v} = 1 / (r \cdot h_{u,v}(\theta) + \Delta_{u,v})$ when $u,v = 1$ *or* 2, and $\delta, \Delta_{u,v} = 1$, with $h_{u,v}(\theta) \approx \sin(\theta)$, using our *potential* hypothesis. Note that because of *symmetry*, we'll *also* multiply the integration, just above, by 4, after completing the calculation.

Of interest to us is the following *indefinite* integral, where $c = sin(\theta)$ and $x = r$ …

$$\int 1 / (cx + 1)\ dx\ =\ (1 / c) \log(cx + 1)$$

Using this, we evaluate the *inner* integral in (*), and then *expand* the result as a Taylor series for *small* ϵ. The expansion computes to

$$(1 / c)[\{\log(1 + c) + c\epsilon / (1+ c)\}\ -\ \{\log(1 + c) - c\epsilon / (1 + c)\}]\ =\ 2\epsilon / (1 + c)\,,$$

and we now complete the calculation, by computing the remainder of (*) and multiplying by 4 …

$$\Big[\lim_{\epsilon \to 0} \int_0^{\pi/2} 8\sigma\epsilon / (1 + c)\, d\theta\ \Big]\ /\ 2\epsilon\ =\ 4\sigma$$

In other words, there is *absolutely no* difference between this approach and *first* setting $r = 1$ in ($\dagger$), which is how these calculations were actually done in Part XII, to begin with. But here, by taking the $\epsilon \to 0$ approach, and using (*); it is easier to see the *interplay* between the *ruler* tick and the *angular* tick; whereas if I set $r = 1$ *initially*, in ($\dagger$), and do the calculations, it is harder to understand what $\sqrt{\sigma \cdot g_{1,1}}$ really is in ($\dagger$), in particular. Yet it is needed, in order that the two methods agree, and it should be mentioned that similar remarks apply in $\mathcal{R}^3$, but we won't do the calculations here.

Finally, when we look at the Wolfram integrals on pages 599-600, and reproduced here, where $x = \theta$ and $y = \phi$, it makes *no* difference if we map cos(y) to sin(y). For then, our dark energy singularities now lie along the y-axis at $(0, \pm\delta, 0)$, since $\cos(\gamma) = \sin(\theta)\sin(\phi)$ [pp 588-9]; but *no* such claim can be made along the z-axis, since here $\cos(\gamma) = \cos(\theta)$ … and the influence due to ϕ is lost altogether. A manifestation of the coordinate system we are working in, as we have said before.

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\left(1+\sqrt{1-\cos^2(y)\sin^2(x)}\right)^{3/2}}\,dy\,dx = 1.01654$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\left(1+\cos(y)\sin(x)\right)\sqrt{1+\sqrt{1-\cos^2(y)\sin^2(x)}}}\,dy\,dx = 1.34305$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\left(1+\sqrt{1-\sin^2(x)\sin^2(y)}\right)^{3/2}}\,dy\,dx = 1.01654$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\left(1+\sin(x)\sin(y)\right)\sqrt{1+\sqrt{1-\sin^2(x)\sin^2(y)}}}\,dy\,dx = 1.34305$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{1+\sqrt{1-\cos^2(y)\sin^2(x)}}}\,dy\,dx = 1.18367$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{1+\sqrt{1-\sin^2(x)\sin^2(y)}}}\,dy\,dx = 1.18367$$

A Simple Case Study, Part XIV

For the case where there are singularities at $(0, 0, \pm\delta)$ along the z-axis, using a *physical* coordinate system; we repeat the methods on pages 590-1, since here $\cos(\gamma) = \cos(\theta)$, when calculating the dark energy component [pp 588-9], as shown below …

$$\xi = 2\sigma \cdot \cosh(\delta r\cos(\gamma))J_0(\delta r\sin(\gamma))$$

Using the default *inner* block $\mathcal{B} = [r^2 , r^2 sin^2(\theta)]$, we find that Δ_ϵ computes to [where $c = \sin(\theta)$] …

$$(16\epsilon)\sigma^{3/2} \int_0^{\pi/2} \int_0^{\pi/2} (sin(\theta) / \sqrt{c+1})\, d\theta d\phi$$

And this is the [volume] amount by which $\mathcal{S}$ expands into $\mathcal{E}$, incrementally, because of the dark energy *radial* waves. Here, $\mathcal{S}$ is the *thin* spherical shell containing the *unit* sphere, with *inner* and *outer* radii of $1 - \epsilon$ and $1 + \epsilon$, respectively; and $\mathcal{E}$ is the *ellipsoidal* annulus, that our *thin* shell $\mathcal{S}$ has *morphed* into; because of the dark energy *radial* waves generated from the singularities S at $(0, 0, \pm\delta)$, and emanating in *all* directions, from the origin O.

Note that $\mathcal{E}$ will be stretched *more* along the z-axis, than the x, y axes, because the dark energy singularities are at $(0, 0, \pm\delta)$.

Now the integration, just above, computes to $8\pi\epsilon\sigma^{3/2}k$, where k evaluates to

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{1+\sin(x)}}\, dx = 2 + \frac{\log(3 - 2\sqrt{2})}{\sqrt{2}} \approx 0.75355$$

and thus, $\Delta_\epsilon \approx 18.939\,\epsilon\sigma^{3/2}$. Now let's compare this to Δ_ϵ on pages 590-1, where the dark energy singularities were at $(\pm\delta, 0, 0)$, along the x-axis. This value was $16\epsilon\sigma^{3/2}k$, where k evaluated to

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{1+\sqrt{1-\cos^2(y)\sin^2(x)}}}\, dy\, dx = 1.18367$$

so that here, $\Delta_\epsilon \approx 18.939\epsilon\sigma^{3/2}$. Remarkably, the two results are *identical* …

A Simple Case Study, Part XV

From pages 601-2, it is not hard to show that by *first* setting $r = 1$ in (†), the *correct* interpretation of this expression, for *small* σ, is actually (*), as shown below …

$$\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\sigma \cdot g_{1,1}}\ \sqrt{\sigma \cdot g_{2,2}}\ \sqrt{\sigma \cdot g_{3,3}}\ d\theta d\phi \qquad (\dagger)$$

$$\left[\lim_{\epsilon \to 0} \int_0^{\pi/2} \int_0^{\pi/2} \int_{1-\epsilon}^{1+\epsilon} \sqrt{\sigma \cdot g_{1,1}}\ \sqrt{\sigma \cdot g_{2,2}}\ \sqrt{\sigma \cdot g_{3,3}}\ dr d\theta d\phi \right] / 2\epsilon \qquad (*)$$

Here, (†) is defined to be the *incremental* surface area [Δ_ϵ], as our *unit* sphere S_1 morphs into the ellipsoid E_1, when there are dark energy singularities S at (±δ, 0, 0), *or* at (0, ±δ, 0), *or* at (0, 0, ±δ). Thus, the *default* inner block $\boldsymbol{\mathscr{B}}$ is replaced by the estimator $\sqrt{\sigma \cdot g_{2,2}}\ \sqrt{\sigma \cdot g_{3,3}}$; and we have already shown that for $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta_{u,v})$, where $u,v = 1$ *or* 2 *or* 3, and $\delta, \Delta_{u,v} = 1$ … this surface area Δ_ϵ does *not* change if S is either (±δ, 0, 0), *or* (0, ±δ, 0) [pp 601-2]. For definitions of sin(γ) using *physical* coordinates, see pages 588-9.

But since there can be *no* assurance that Δ_ϵ will *remain invariant* when S is equal to (0, 0, ±δ), we have a *choice* at this juncture: find estimators for $g_{2,2}$ and $g_{3,3}$ such that Δ_ϵ does *not* change, no matter the choice of S; *or* use the *default* inner block $\boldsymbol{\mathscr{B}} = [r^2\ ,\ r^2 sin^2(\theta)]$, where we showed in the last research note [p 603 and pp 601-2] that the incremental *volume* Δ_ϵ *remains invariant* (and thus Δ_ϵ too, after dividing Δ_ϵ by 2ϵ using a lim $\epsilon \to 0$ argument), *regardless* of what S is. A minor miracle, you might say, and the better choice, in my view; since we're not likely to find estimators for $g_{2,2}$ and $g_{3,3}$ that *approximately* satisfy the general equivalency theorem [$\boldsymbol{\mathscr{G}}$], and meet our *invariance* criteria, anytime soon [p 575].

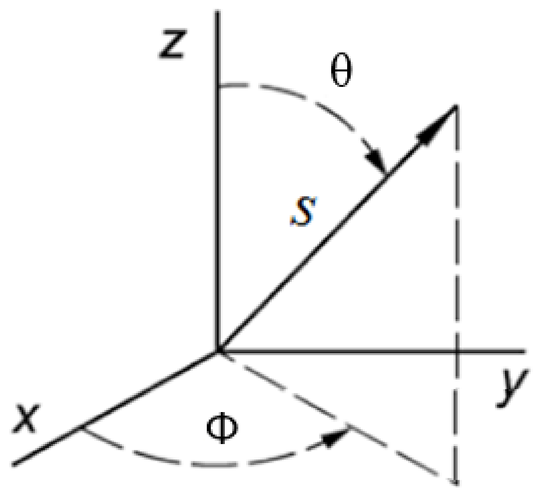
And so at this point in $\boldsymbol{\mathscr{R}}^3$, we can conclude that on the *unit* sphere S_1, and hence in a *thin* shell $\boldsymbol{\mathcal{S}}$ containing S_1, with *inner* and *outer* radii of $1 - \epsilon$ and $1 + \epsilon$, respectively; a suitable *normalized* solution $\boldsymbol{\mathcal{N}}$ (σ = 1) for the *coupled* equations is

$$g_{u,v} = [1 / (r \cdot \sin(\gamma) + \Delta_{u,v}),\ r^2,\ r^2 sin^2(\theta),\ 1 / (r \cdot \cos(\gamma) + \Delta_{u,v})]\ ,$$

where again, definitions of sin(γ) and cos(γ) using *physical* coordinates, can be found on pages 588-9. The *general* solution is then $\sigma \cdot \boldsymbol{\mathcal{N}}$, and of course, the presumption here is that $g_{1,1}$ and $g_{4,4}$ approximately satisfy $\boldsymbol{\mathscr{G}}$ on S_1 …

A Simple Case Study, Part XVI

Here, we are going to analyze the estimator $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta_{u,v})$ over the *unit* sphere [S_1], using our general equivalency theorem $\mathcal{G}$, in three dimensions [p 575]. The singularities are to be located at $(\pm\delta, 0, 0)$, where $u,v = 1$; $\delta, \Delta_{u,v} = 1$ and $0 \le \theta, \phi \le \pi / 2$; and again, we are using *physical* coordinates, as shown in the diagram below.



Since $\cos(\gamma) = \sin(\theta)\cos(\phi)$, we'll start by looking at the *x-z* plane. Here $\cos(\gamma) = \sin(\theta)$, so that $\sin(\gamma) = \cos(\theta) = \sin(\pi/2 - \theta)$, and we know from earlier testing that in this case, $g_{u,v}$ approximately satisfies $\mathcal{G}$ with a *reasonable* degree of precision [pp 559-62]; but weakens somewhat as θ approaches 0, heading *north*. We'll label this result *good*.

Now looking at the *x-y* plane, $\cos(\gamma) = \cos(\phi)$, so that $\sin(\gamma) = \sin(\phi)$, and we know from pages 559-62 that in this case, $g_{u,v}$ approximately satisfies $\mathcal{G}$ with a *reasonable* degree of precision. Indeed, as ϕ approaches $\pi / 2$ from *below*, the degree to which $g_{u,v}$ approximately satisfies $\mathcal{G}$ weakens somewhat, but is still acceptable to us. We'll also label this result *good*, as it is really *no* different than what is happening in the *x-z* plane.

Finally, let's look at the *y-z* plane. Here $\cos(\gamma) = 0$, so that $\sin(\gamma) = 1$, and this is the *same* as the case just above, where $\phi = \pi / 2$. Thus, when analyzing $g_{u,v}$ over the *whole* of S_1, we should expect the results to be *good* as well.

Let us now set $\theta, \phi = \pi / 4$ on S_1 and calculate the various components of $\mathcal{G}$, using the notes on page 575 as our guide. Here are the results, noting that $\alpha = \cos(\gamma) = ½$, and $\varepsilon = \sin(\gamma) = \sqrt{3} / 2$, and that the *simple* pole is at ≈ -1.6547, in the *generating* function $f(s)$. The value of $\mathcal{H}_x$ is shown below …

$$\int_{\frac{\sqrt{3}}{2}}^{500} \frac{\cos(y)\left(\frac{1}{1.433+0.866\,i\,y} - \frac{1}{1.433-0.866\,i\,y}\right) + i\sin(y)\left(\frac{1}{1.433+0.866\,i\,y} + \frac{1}{1.433-0.866\,i\,y}\right)}{(i\,\pi)\sqrt{y^2 - 0.75}}\, dy = 0.397824$$

$$2 \times \frac{\exp(-1.65469)}{\sqrt{1.65469^2 + 0.75}}$$

Result

0.204701…

value of R

$$\frac{1}{1.433}\, J_0\!\left(\frac{\sqrt{3}}{2}\right)$$

Result

0.572999…

value of $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$

From our theory concerning $\mathcal{G}$, we add the value of $\mathcal{H}_x$ to R, and compare to $\mathcal{E}$. In this case, we see that $\mathcal{H}_x$ + R ≈ .6025, and we note that it agrees reasonably well with $\mathcal{E}$. Indeed, it is *exactly* what we find on page 561, when $\theta = \pi / 3$ in an $\mathcal{R}^2$ layout.

And we expect the *same* to be true for the estimator $g_{u,v} = 1 / (r \cdot \cos(\gamma) + \Delta_{u,v})$, so that the *normalized* solution $\mathcal{N}$ ($\sigma = 1$) for the *coupled* equations; namely

$$g_{u,v} = [1 / (r \cdot \sin(\gamma) + \Delta_{u,v}),\ r^2,\ r^2 sin^2(\theta),\ 1 / (r \cdot \cos(\gamma) + \Delta_{u,v})]\ ,$$

is indeed a valid representation on the *unit* sphere S_1, and hence in a *thin* shell $\mathcal{S}$ containing S_1, with *inner* and *outer* radii of $1 - \epsilon$ and $1 + \epsilon$, respectively. The *general* solution is then $\sigma \cdot \mathcal{N}$ [p 604].

A Simple Case Study, Part XVII

In this note, we wish to offer up a *pair* of estimators for the *inner* block $\mathscr{B}$ in $\mathscr{R}^3$, which preserve *invariance* when looking at *incremental* surface area and volume, in the presence of dark energy [ξ]. In Part XV [p 604], I did not think such estimators could be found, which not only conserved the property of *invariance*, *but also*, approximately satisfied the general equivalency theorem $\mathcal{G}$ …

But consider the following timelike *normalized inner* block $\mathscr{B}$ on the *unit* sphere S_1 in $\mathscr{R}^3$, so that it *conforms* to this behavior and is also *similar* to the default *inner* block [r^2, $r^2 sin^2(\theta)$]; to wit, for singularities at $(\pm\delta, 0, 0)$ and $0 \le \theta, \phi \le \pi/2$ …

$$\mathscr{B} = [1 / (r \cdot \cos(\gamma) + \Delta_{u,v}) , \sin^2(\theta) / (r \cdot \cos(\gamma) + \Delta_{u,v})] ,$$

where $u,v = 2$ or 3; $\delta, \Delta_{u,v} = 1$ and $\cos(\gamma) = \sin(\theta)\cos(\phi)$, using *physical* coordinates.

We now calculate the *incremental* surface area [Δ_ϵ], by setting $r = 1$ in the expression below, as our *unit* sphere S_1 morphs into the ellipsoid E_1 [p 604], where $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta_{u,v})$ is the *radial* component when $u,v = 1$. And in (†), because of symmetry, we multiply by 8 to complete the calculation …

$$\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\sigma \cdot g_{1,1}} \sqrt{\sigma \cdot g_{2,2}} \sqrt{\sigma \cdot g_{3,3}}\, d\theta d\phi \qquad (\dagger)$$

Here are the *normalized* ($\sigma = 1$) snapshots from Wolfram, where $x = \theta$, and $y = \phi$ …

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{8 \sin(x)}{(1 + \cos(y)\sin(x)) \sqrt{1 + \sqrt{1 - \cos^2(y)\sin^2(x)}}}\, dy\, dx = 6.48798$$

$$\Delta_\epsilon \approx 6.488\sigma^{3/2} \text{ when } S = (\pm\delta, 0, 0)$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{8 \sin(x)}{(1 + \sin(x)\sin(y)) \sqrt{1 + \sqrt{1 - \sin^2(x)\sin^2(y)}}}\, dy\, dx = 6.48798$$

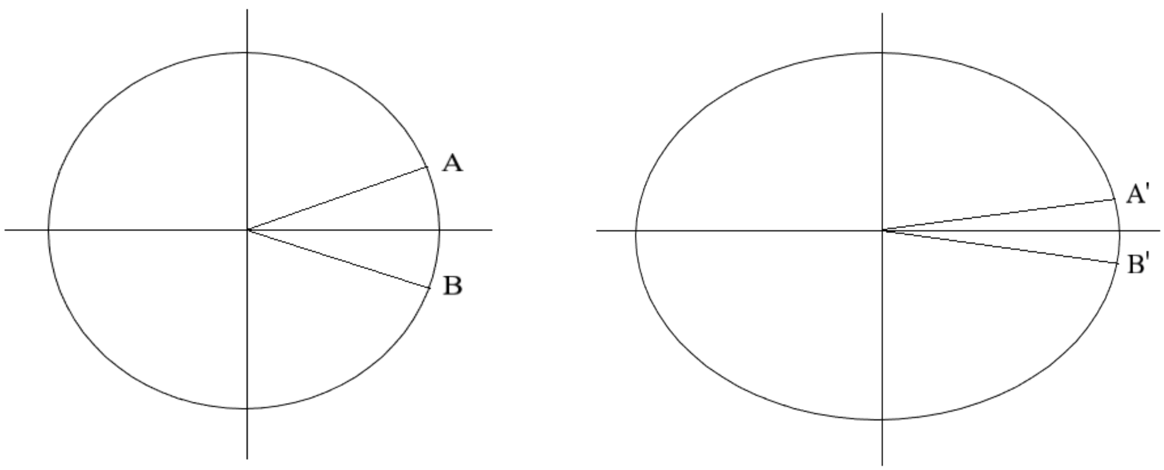
$$\Delta_\epsilon \approx 6.488\sigma^{3/2} \text{ when } S = (0, \pm\delta, 0)$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{8 \sin(x)}{\sqrt{1 + \sin(x)}\, (1 + \cos(x))}\, dy\, dx = \pi\left(8 \sinh^{-1}(1) - 2\sqrt{2} \log(3 + 2\sqrt{2})\right) \approx 6.4880$$

$$\Delta_\epsilon \approx 6.488\sigma^{3/2} \text{ when } S = (0, 0, \pm\delta)$$

Notice that *all three* values of Δ_ϵ are the *same*, so that for this *inner* block $\boldsymbol{\mathcal{B}}$, both *incremental* surface area and volume do *not* change, no matter our choice of S. Furthermore, when using the *default* inner block [r^2, $r^2 sin^2(\theta)$], *incremental* surface area was calculated to be $8\sigma^{3/2}k$ [no matter the choice of S], with the constant $k \approx 1.18367$, so that in this case $\Delta_\epsilon \approx 9.469\sigma^{3/2}$ [pp 594-5]. The two results differ, as they should.

As well, these *timelike* estimators for $\boldsymbol{\mathcal{B}}$ will satisfy $\boldsymbol{\mathcal{G}}$ on the *unit* sphere, and if we were to use *spacelike* estimators for $\boldsymbol{\mathcal{B}}$, I suspect these too would preserve *invariance*. But whether these estimators have any *real* meaning within the context of dark energy and general relativity, is another matter. Nonetheless, we note them here, for the properties they do have …



A POSSIBLE INTERPRETATION

In the left picture above, in the absence of ξ , we see a *unit* circle C, with an angle θ subtended by the arc AB. In the presence of dark energy singularities at ($\pm\delta$, 0), C will morph into the ellipse E, stretched *more* in the direction of *x* than *y*, and the angle θ that was subtended by AB now becomes the angle θ' subtended by the arc A′B′. Clearly $\theta' < \theta$, which means that for an *imaginary* observer $\boldsymbol{\mathcal{O}}$ at the origin O in $\boldsymbol{\mathcal{R}^2}$, angular ticks *expand* in the presence of *stronger* dark energy [*x*-axis], and just the opposite as the stronger dark energy *weakens* [*y*-axis]. And this is reminiscent of how the *clock* tick behaves, which is *inverse* to the *ruler* tick.

CORRECTIONS

A small change was made to the value of Δ_ϵ at the base of page 603. On pages 605-6, sin(γ) is actually $\sqrt{3}\,/\,2$, and *not* the value of ½ that was reported there in an earlier release. This changes the calculations for $\boldsymbol{\mathcal{H}_x}$, R, and $\boldsymbol{\mathcal{E}}$; and so … .. the pages have been updated, accordingly. The calculations can *also* be found on page 561, when $\theta = \pi\,/\,3$ in an $\boldsymbol{\mathcal{R}^2}$ layout. Again, the results are favorable when comparing $\boldsymbol{\mathcal{H}_x}$ + R to $\boldsymbol{\mathcal{E}}$.

A Simple Case Study, Part XVIII

We can save ourselves an enormous amount of calculating, by realizing that if $g' = f$, for some function f, then $g(1 + \epsilon) \approx g(1) + g'(1)\epsilon$, using basic calculus, if $\epsilon > 0$ is *small*. Similarly, we may write $g(1 - \epsilon) \approx g(1) - g'(1)\epsilon$, so that

$$g(1 + \epsilon) - g(1 - \epsilon) \approx 2\epsilon g'(1) = 2\epsilon f(1)$$

Thus,

$$\int_{1-\epsilon}^{1+\epsilon} f(x)\, dx \approx 2\epsilon f(1)$$

And so, when evaluating an expression like the following, from page 590, where $c = \sin(\gamma)$ and $x = r$ …

$$\int_{1-\epsilon}^{1+\epsilon} x^2 / \sqrt{cx+1}\, dx = \Big[(2 / 15c^3)\sqrt{cx+1}\,(3c^2x^2 - 4cx + 8)\Big]_{1-\epsilon}^{1+\epsilon}, \quad (*)$$

we actually *don't* need to perform a Taylor series on the *right*-hand side of (*). Instead, we conclude immediately that this must be $2\epsilon f(1)$, for *small* ϵ, where

$$f(x) = x^2 / \sqrt{cx+1}$$

A Simple Case Study, Part XIX

In this note, we are going to go through the calculations for the general equivalency theorem [$\mathcal{G}$], when $\theta, \phi = \pi / 4$ on the *unit* sphere S_1 in $\mathcal{R}^3$ [pp 605-6]. Here $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta_{u,v})$, with dark energy singularities at $(\pm\delta, 0, 0)$, where $u,v = 1$; $\delta, \Delta_{u,v} = 1$ and $0 \le \theta, \phi \le \pi / 2$, using *physical* coordinates. Thus, $\alpha = \cos(\gamma) = \sin(\theta)\cos(\phi)$ in this case, which computes to ½, and $\varepsilon = \sin(\gamma)$, which computes to $\sqrt{3} / 2$.

As to the *generating* function $f(s)$ [p 575], it becomes, where $r = s + \delta\alpha$ in $g_{u,v}$ …

$$f(s) \;=\; g_{u,v} \;\cdot\; 1 / \sqrt{s^2 + (\delta\varepsilon)^2} \;=\; 1 / ((s + \delta\alpha) \cdot \sin(\gamma) + \Delta_{u,v}) \;\cdot\; 1 / \sqrt{s^2 + (\delta\varepsilon)^2}$$

And so, the *simple* pole associated with the *first* term in $f(s)$ is $-p = -\Delta_{u,v} / \sin(\gamma) - \delta\alpha \approx -1.6547$; and we now calculate the *residue* associated with $-p$ from the Laplace inverse of $f(s)$ over some contour γ, as shown below, where the constant $\kappa = 1 / 2\pi \boldsymbol{i}$ [pp 559-62] …

$$\kappa \int e^{sr} f(s)\, ds\ .$$

This gives us $Res = exp(-p\mathrm{r}) \,/\, \sqrt{p^2 + (\delta\varepsilon)^2}$, and thus ... the value of R used in $\mathcal{G}$ is $2 \cdot Res \approx 0.2047$, since here r = 1.

We then go ahead and calculate $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta \mathrm{r} \varepsilon)$, where r = 1 in J_0, since we are on S_1 in $\mathcal{R}^3$; and here, $r = \delta\alpha$ in $g_{u,v}$, since $\delta\alpha$ is a *radial* measure. This gives us $\mathcal{E} \approx 0.5730$.

Finally, we calculate the harmonic expression $\mathcal{H}_x$, as shown below, where r maps to $\delta\alpha \pm \mathrm{i}y$ in $g_{u,v}$, so that $g_{u,v}(\delta\alpha \pm \mathrm{i}y, \theta, \phi) = 1 / ((\delta\alpha \pm \mathrm{i}y) \cdot \sin(\gamma) + \Delta_{u,v})$. And here, r = 1 in the *cos* and *sin* terms.

$$2\kappa \int_{\delta\varepsilon}^{\infty} \{\cos(y\mathrm{r})[g_{u,v}(\delta\alpha + \mathrm{i}y, \theta, \phi) \;-\; g_{u,v}(\delta\alpha - \mathrm{i}y, \theta, \phi)] \;+\; \boldsymbol{i}\sin(y\mathrm{r})\,[g_{u,v}(\delta\alpha + \mathrm{i}y, \theta, \phi) \;+\; g_{u,v}(\delta\alpha - \mathrm{i}y, \theta, \phi)]\}\; \mathrm{d}y \,/\, \sqrt{y^2 - (\delta\varepsilon)^2}$$

And this gives us a value of ≈ 0.3978, as shown below. We then compare $\mathcal{H}_x + \mathrm{R} \approx .6025$ to $\mathcal{E}$, and find that the agreement is reasonable. And indeed, this particular agreement actually holds for *any* $< \theta, \phi >$, such that $\cos(\gamma) = ½$, and $0 \le \theta, \phi \le \pi / 2$ on S_1 …

$$\int_{\frac{\sqrt{3}}{2}}^{500} \frac{\cos(y)\left(\frac{1}{1.433+0.866\,i\,y} - \frac{1}{1.433-0.866\,i\,y}\right) + i\sin(y)\left(\frac{1}{1.433+0.866\,i\,y} + \frac{1}{1.433-0.866\,i\,y}\right)}{(i\,\pi)\sqrt{y^2 - 0.75}}\, dy = 0.397824$$

A Simple Case Study, Part XX

In this note, we are going to provide *spacelike* and *timelike* estimators over the *whole* of $\mathcal{R}^2$, using the methodology that follows. Let us begin, then, by recalling the form for the dark energy itself, where there are singularities at (±δ, 0). That is to say,

$$\xi = 2\sigma \cdot \cosh(\delta r\cos(\theta))J_0(\delta r\sin(\theta))$$

For a *unit* circle C_1 of radius 1, with δ, σ = 1; we note that *on* C_1 at θ = 0, ξ = 2cosh(1), and *on* C_1 at θ = π / 2, ξ = $2J_0(1)$. Thus, the ratio $\mathcal{R}$ of these two values is simply cosh(1) / $J_0(1)$, and if we now define our *spacelike* estimator [$u,v = 1$] to be $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$, where Δ is some function of r; then it stands to reason that

$$\mathcal{R} = (r + \Delta) / \Delta\,, \quad (*)$$

since ruler ticks *contract* in the presence of *stronger* dark energy, and begin to *expand* as the dark energy begins to *weaken*.

In this case, $\mathcal{R} \approx 2.01$, so that from (*), Δ ≈ 1 if $r = 1$, and this is the value of Δ that was used originally when testing $g_{u,v}$ against the general equivalency theorem [$\mathcal{G}$] on pages 559-62.

Now let's look at the circle C_r of radius r = ½. In this case $\mathcal{R} \approx 1.20$, so that Δ ≈ 5 / 2, and the estimator is now $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$ on C_r. Thus, we can go ahead and calculate the various components of $\mathcal{G}$, when θ = 0 and π / 2, say, and see how things play out.

The generating function is

$$f(s) = g_{u,v} \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2} = 1 / ((s + \delta\alpha) \cdot \sin(\theta) + \Delta) \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2}\,,$$

and at θ = π / 2, the *simple* pole associated with the *first* term in $f(s)$ is $-p = -\Delta / \sin(\theta) - \delta\alpha = -\Delta$, since α = cos(θ), and ε = sin(θ). Thus, the residue is $Res = exp(-pr) / \sqrt{p^2 + (\delta\varepsilon)^2}$, so that the R value associated with $\mathcal{G}$ is R = 2 · *Res* ≈ 0.2128. Again, r = ½ in this calculation.

And since α = 0, $g_{u,v}(\delta\alpha) = 2/5$, and $J_0(\delta r\varepsilon) = J_0(½)$, because r = ½, and ε = 1. Thus, since it is the case that $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$, we see that $\mathcal{E} \approx 0.3754$.

Finally, we have the harmonic expression $\mathcal{H}_x$, and here $g_{u,v}(\delta\alpha \pm iy, \theta) = 1 / ((\delta\alpha \pm iy) \cdot \sin(\theta) + \Delta)$, since r maps to δα ± iy in this case. Here is the result, noting that r = ½ in the *cos* and *sin* terms …

$$\int_1^{300} -\frac{i\left(\left(-\frac{1}{\frac{5}{2}-iy}+\frac{1}{\frac{5}{2}+iy}\right)\cos\left(\frac{y}{2}\right)+i\left(\frac{1}{\frac{5}{2}-iy}+\frac{1}{\frac{5}{2}+iy}\right)\sin\left(\frac{y}{2}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy = 0.171251$$

And here are the snapshots for R and $\mathcal{E}$ …

$$2 \times \frac{\exp(-1.25)}{\sqrt{2.5^2 + 1}}$$

Result

0.212810…

value of R

$$\frac{2 J_0\left(\frac{1}{2}\right)}{5}$$

Decimal approximation

0.375387922896325

value of $\mathcal{E}$

Now we compare $\mathcal{H}_x$ + R to $\mathcal{E}$, and we find that $\mathcal{H}_x + \mathrm{R} \approx 0.3841$, which compares favorably with the value of $\mathcal{E}$.

When θ = 0, there are *no* poles associated with the first term in $f(s)$, and so R = 0, and the harmonic expression reduces to ($r = ½$, $\varepsilon = 0$, $\kappa = 1 / 2\pi \boldsymbol{i}$)

$$4\kappa \boldsymbol{i} / \Delta \cdot \int_{\varepsilon}^{\infty} \sin(yr)\, dy \,/\, y \;=\; 1 / \Delta$$

And $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r \varepsilon) = g_{u,v}(\delta\alpha)$, which *must* equal 1 / Δ in $\mathcal{R}^2$, for *any* choice of the argument to $g_{u,v}$, because $g_{u,v}(r, \theta) = 1 / \Delta$ now. Thus, the agreement is *exact* here, as expected [pp 559-62].

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In general, we may write, for *any* $r \geq 0$ ...

$$\mathcal{R} \;=\; \cosh(r) / J_0(r) \;=\; (r + \Delta) / \Delta \,, \qquad (*)$$

and solving this for Δ will give us the *spacelike* and *timelike* estimators, $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$ and $g_{u,v} = 1 / (r \cdot \cos(\theta) + \Delta)$ on C_r, respectively. And these estimators will approximately satisfy $\mathcal{G}$, because of the way in which Δ is calculated from (*). In $\mathcal{R}^3$, we simply replace θ with γ in $g_{u,v}$, noting that $\cos(\gamma) = \sin(\theta)\cos(\phi)$, with $0 \leq \theta, \phi \leq \pi / 2$; and that the singularities are at $(\pm\delta, 0, 0)$.

For large r, $\mathcal{R}$ becomes *very* large, so that Δ ---› 0, and thus in *very deep* space, the estimators reduce to $g_{u,v} = 1 / (r \cdot \sin(\gamma))$ and $g_{u,v} = 1 / (r \cdot \cos(\gamma))$ in $\mathcal{R}^3$. And as r ---› 0, $\cosh(r) / J_0(r)$ agrees with $(r + \Delta) / \Delta$, and it seems that Δ increases here as r decreases. If so, then $g_{u,v}$ ---› 0 in this case.

A Simple Case Study, Part XXI

We can now calculate the *incremental* volume [Δ_ϵ] of the ellipsoid $\mathcal{E}$, that our *unit* sphere has morphed into, using the results of the last research note. We're fairly confident that the *spatial* estimator $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta)$ [$u,v = 1$] will work for us in the range $0 \le r \le 1$, where here it is that case that $\Delta = r / (\mathcal{R} - 1)$, and $\mathcal{R} = \cosh(r) / J_0(r)$. Outside this range, we'd want to do more testing, using the general equivalency theorem [$\mathcal{G}$].

Now in this setup, using *physical* coordinates, the dark energy singularities are at ($\pm\delta$, 0, 0), where $\delta = 1$, and $\cos(\gamma) = \sin(\theta)\cos(\phi)$, with $0 \le \theta, \phi \le \pi/2$. As well, we'll use the default *inner* block $\mathcal{B} = [r^2, r^2 \sin^2(\theta)]$; so using *symmetry*, we multiply the *normalized* integration ($\sigma = 1$), as shown below, by $8\sigma^{3/2}$, where $x = r$, $y = \theta$ and $z = \phi$.

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{x^2 \sin(y)}{\sqrt{\frac{x}{-1+\frac{\cosh(x)}{J_0(x)}} + x\sqrt{1-\cos^2(z)\sin^2(y)}}} \, dz \, dy \, dx = 0.356058$$

Thus, $\Delta_\epsilon \approx 2.85\sigma^{3/2}$, or in round numbers, about $3\sigma^{3/2}$. Thus, to an imaginary observer $\mathcal{O}$ at the origin O in $\mathcal{R}^3$, with nothing in the *tangible* space, the *unit* sphere has morphed into a ellipsoid $\mathcal{E}$ whose volume is approximately $4\pi/3 + \Delta_\epsilon$. And again, $\mathcal{E}$ will be stretched more along the *x*-axis, than the *y*, *z* axes, due to the location of the singularities.

―――

For singularities at S = (0, $\pm\delta$, 0), the *normalized* integral becomes

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{x^2 \sin(y)}{\sqrt{\frac{x}{-1+\frac{\cosh(x)}{J_0(x)}} + x\sqrt{1-\sin^2(y)\sin^2(z)}}} \, dz \, dy \, dx = 0.356058$$

and for singularities at S = (0, 0, $\pm\delta$), the *normalized* integral is

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{x^2 \sin(y)}{\sqrt{\frac{x}{-1+\frac{\cosh(x)}{J_0(x)}} + x\sin(y)}} \, dz \, dy \, dx = 0.356058$$

Thus, in the range $0 \le r \le 1$, our estimator *preserves volume invariance*, no matter the choice of S, which is what we expect.

A Simple Case Study, Part XXII

Before we start, we'll mention that some *additional* commentary was added to page 613, regarding volume *invariance*. Now in this note, we'd like to study the circle C_r of radius $r = 2$, and see how things play out when calculating the various components of the general equivalency theorem [$\mathcal{G}$], for θ = 0 and θ = π / 2.

Now we already studied the circle of radius ½ on pages 611-12, so we needn't go through all the details again. Instead, we'll simply quote the results, for our estimator $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$, where

$$\mathcal{R} = \cosh(r) / J_0(r) = (r + \Delta) / \Delta$$

When θ = 0 the match is *exact*, so all we have to look at is θ = π / 2, where it is already known that the *deviation* between $\mathcal{H}_x$ + R and $\mathcal{E}$ is *greatest* [pp 559-62, 596-7]. Here are the results from Wolfram, assuming no mistakes were made in the calculations …

$$\int_1^{300} \frac{\cos(2y)\left(\frac{1}{0.12655+iy} - \frac{1}{0.12655-iy}\right) + i\sin(2y)\left(\frac{1}{0.12655+iy} + \frac{1}{0.12655-iy}\right)}{(\pi i)\sqrt{y^2-1}}\,dy = 0.45879$$

value of $\mathcal{H}_x$

$$2 \times \frac{\exp(-2 \times 0.12655)}{\sqrt{0.12655^2 + 1}}$$

Result

1.540494…

value of R

$$\frac{1}{0.12655} J_0(2)$$

Result

1.76919…

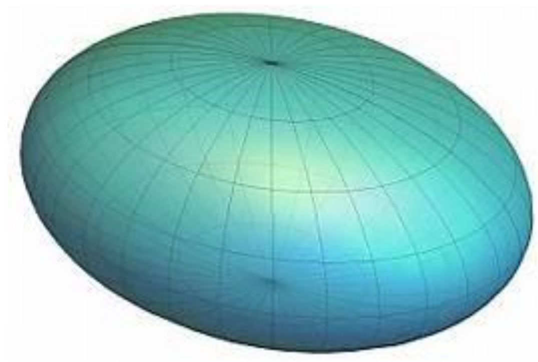
value of $\mathcal{E}$

Thus, $\mathcal{H}_x$ + R ≈ 1.99928, which is *somewhat* larger than $\mathcal{E}$, but all things considered, the two numbers are within about 10% of one another; which is the *maximum* deviation we should see, for θ between 0 and π / 2. As well, we're in *new* territory here, using $r / (\mathcal{R} - 1)$ as the estimator for Δ in $g_{u,v}$, and venturing out *beyond* the *unit* sphere, where $r = 1$. So we'll use this estimator in the range $0 \le r \le 2$ now; and in doing so, we find that the *incremental* volume [Δ_ϵ] for a sphere of radius $r = 2$ computes to $8\sigma^{3/2}k$, when the singularities are at (±δ, 0, 0), and k is

$$\int_0^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{x^2 \sin(y)}{\sqrt{\frac{x}{-1+\frac{\cosh(x)}{J_0(x)}} + x\sqrt{1-\cos^2(z)\sin^2(y)}}}\,dz\,dy\,dx = 3.3036$$

A Simple Case Study, Part XXIII

To calculate how much the *unit* sphere $\boldsymbol{S}$ has been stretched into an ellipsoid $\boldsymbol{\mathcal{E}}$ along the *x*-axis, when the singularities are at ($\pm\delta$, 0, 0) [with $\delta = 1$], we simply evaluate the expressions below, where $\sigma = 1$ …



$$\int_0^1 \frac{1}{\sqrt{\frac{x}{-1+\frac{\cosh(x)}{J_0(x)}}}}\,dx = 0.615881$$

$$\int_0^1 \frac{1}{\sqrt{x+\frac{x}{-1+\frac{\cosh(x)}{J_0(x)}}}}\,dx = 0.528654$$

incremental distance along *x*-axis

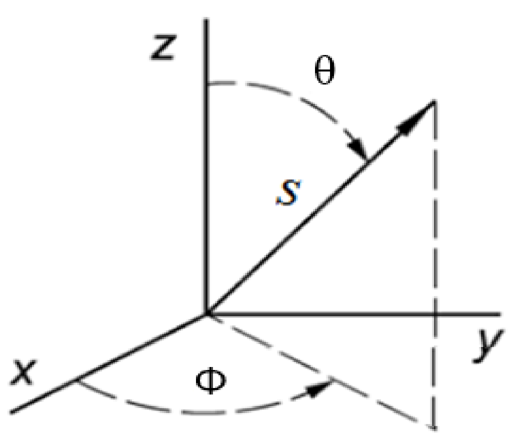
incremental distance along *y,z* axes

For in this case, our estimator is $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta)$, in the range $0 \le r \le 1$, where Δ is equal to $r / (\boldsymbol{\mathcal{R}} - 1)$ and

$$\boldsymbol{\mathcal{R}} = \cosh(r) / J_0(r) = (r + \Delta) / \Delta$$

And since *no* zeroes of the Bessel function $J_0(r)$ exist in the range $0 \le r \le 2$, we don't have to worry about that issue right now.

Now since the singularities are at ($\pm\delta$, 0, 0), $\cos(\gamma) = \sin(\theta)\cos(\phi)$, using *physical* coordinates, where $0 \le \theta, \phi \le \pi/2$.



And so, if $\theta = \pi/2$ and $\phi = 0$, we are on the *x*-axis, so that $\cos(\gamma) = 1$, and hence $g_{u,v} = 1 / \Delta$. On the other hand, if $\theta = \pi/2$ and $\phi = \pi/2$, we are on the *y*-axis, so that $\cos(\gamma) = 0$, and hence it is the case that $g_{u,v} = 1 / (r + \Delta)$. Finally, if $\theta = 0$, *irrespective* of what ϕ is, we are on the *z*-axis, and again, $\cos(\gamma) = 0$. Thus, the *incremental* distances shown on the last page are correct, when the singularities are at $(\pm\delta, 0, 0)$.

Hence, to the imaginary observer $\mathcal{O}$ at the origin O in $\mathcal{R}^3$, the ellipsoid $\mathcal{E}$ is stretched more in the direction of *x*, than *y* or *z*. And to $\mathcal{O}$, the distance to $\mathcal{E}$ along the *x*-axis is now $\approx 1 + 0.616\sqrt{\sigma}$; and the distance to $\mathcal{E}$ along the *y*,*z* axes is now $\approx 1 + 0.529\sqrt{\sigma}$.

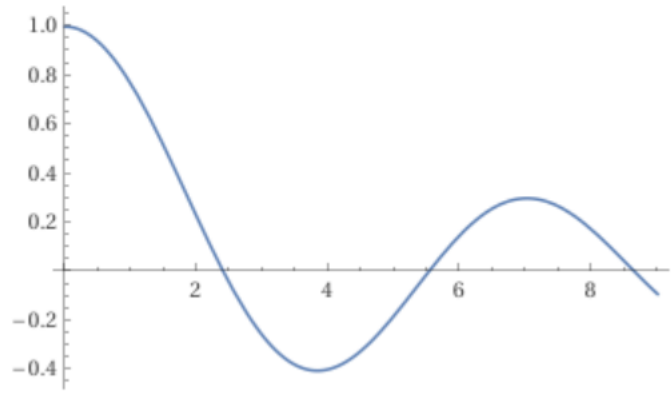
Finally, the *incremental* surface area $[\Delta_\epsilon]$ is $8\sigma^{3/2}k$, as $\mathcal{S}$ morphs into $\mathcal{E}$ [pp 598-600], and *k* is the integral below [$y = \theta$ and $z = \phi$]; which is *slightly* different than what we see on page 599 [there *k* is approximately 1.18367]. And that is because on page 599, $\Delta = 1$; whereas here, it is $r / (\mathcal{R} - 1)$, where $r = 1$ [approximately 0.9837]. And lastly, it should be said that Δ_ϵ is *invariant* with respect to the location of the singularities, as it should be.

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin(y)}{\sqrt{\frac{1}{-1+\frac{\cosh(1)}{J_0(1)}} + \sqrt{1-\cos^2(z)\sin^2(y)}}}\, dz\, dy = 1.18928$$

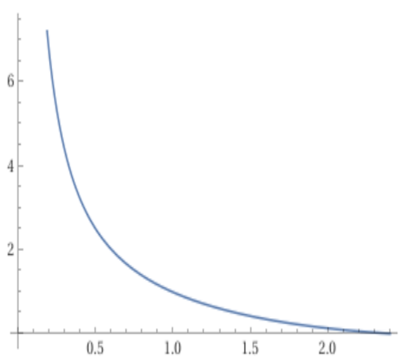
Now as to $\mathcal{R} = \cosh(r) / J_0(r)$, we should mention that it will become *infinite* if we encounter a *zero* of $J_0(r)$. In *very deep* space, that may not matter so much because Δ ---› 0 anyway, for large *r*, if we have $\Delta = r / (\mathcal{R} - 1)$; since here $J_0(r)$ ---› 0. And for $0 \le r \le 2$, we don't encounter any zeroes of $J_0(r)$; but otherwise, it is an issue that needs to be resolved, if we want a *usable* estimator for Δ, no matter the choice of *r*.

A Simple Case Study, Part XXIV

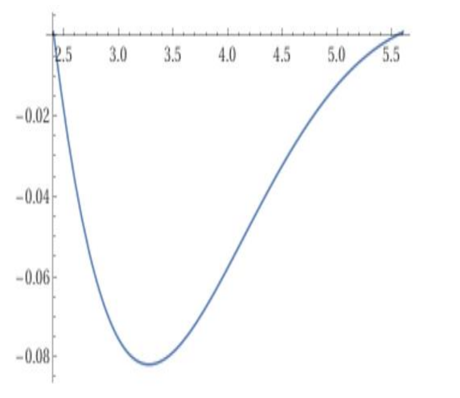
In this note, we are going to begin by looking at the intervals $R_1 = (r_1, r_2)$ and $R_2 = (r_2, r_3)$, where r_1, r_2 and r_3 are the *first*, *second* and *third* zeroes of $J_0(r)$, as shown in the plot below. These zeroes are at ≈ 2.4, 5.6 and 8.7, respectively, and here we'll define $R_0 = (0, r_1)$.



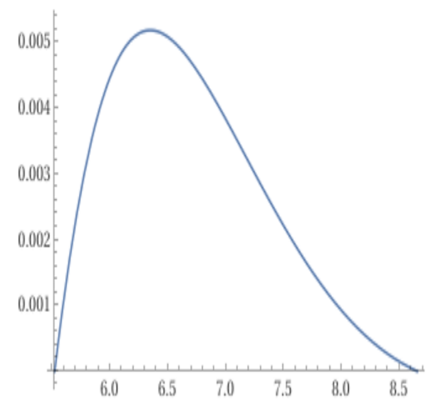
From the last research note, our estimator for Δ is $r\,/\,(\mathcal{R} - 1)$, where $\mathcal{R} = \cosh(r)\,/\,J_0(r)$, and the plots of Δ are shown below …



plot of Δ in R_0



plot of Δ in R_1



plot of Δ in R_2

Note that Δ is *greater* than or *equal* to zero in R_0 and R_2, but *less* than or *equal* to zero in R_1. Thus, when integrating our *spacelike* estimator $g_{u,v} = 1\,/\,(r \cdot \sin(\gamma) + \Delta)$ over these intervals, to find *incremental* distance; we have to be mindful of the *sign* of $\mathcal{D} = r \cdot \sin(\gamma) + \Delta$, since we are operating on $\sqrt{\sigma \cdot g_{u,v}}$. Here, we are in $\mathcal{R}^3$, with singularities at (±δ, 0, 0), and $\cos(\gamma) = \sin(\theta)\cos(\phi)$, using *physical* coordinates, where $0 \le \theta, \phi \le \pi/2$.

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$$\int_{0}^{2.4048} \frac{1}{\sqrt{\frac{x}{\frac{\cosh(x)}{J_0(x)}-1}}}\,dx = 4.79665$$

Δ_ϵ in R_0 along x-axis

$$\int_{2.4049}^{5.52} -\frac{1}{\sqrt{-\frac{x}{\frac{\cosh(x)}{J_0(x)}-1}}}\,dx = -23.9614$$

Δ_ϵ in R_1 along x-axis

$$\int_{5.5201}^{8.6537} \frac{1}{\sqrt{\frac{x}{\frac{\cosh(x)}{J_0(x)}-1}}}\,dx = 100.098$$

Δ_ϵ in R_2 along x-axis

Above we see the *normalized* integrals ($\sigma = 1$) for *incremental* distance [Δ_ϵ], covering off the range from 0 to r_3, supposing that they do indeed *converge*. Notice in R_1 that Δ is always *less* than or *equal* to zero, so we take the *negative* of the square root of Δ here, and then the *negative* of the integral, to give us a *negative* result; which is what we want [p 580].

Thus, the *net* effect of doing this results in a total *incremental* distance along the *x*-axis, due to dark energy [ξ], of about $81\sqrt{\sigma}$. And this is plausible, given that ξ grows exponentially in this particular direction [recall that $\xi = 2\sigma \cdot \cosh(\delta r\cos(\gamma))J_0(\delta r\sin(\gamma))$]. Now for the *y, z* axes …

$$\int_{0}^{2.4048} \frac{1}{\sqrt{x+\frac{x}{\frac{\cosh(x)}{J_0(x)}-1}}}\,dx = 1.5124$$

Δ_ϵ in R_0 along y-axis

$$\int_{2.4049}^{5.52} \frac{1}{\sqrt{x+\frac{x}{\frac{\cosh(x)}{J_0(x)}-1}}}\,dx = 1.60828$$

Δ_ϵ in R_1 along y-axis

$$\int_{5.5201}^{8.6537} \frac{1}{\sqrt{x+\frac{x}{\frac{\cosh(x)}{J_0(x)}-1}}}\,dx = 1.1842$$

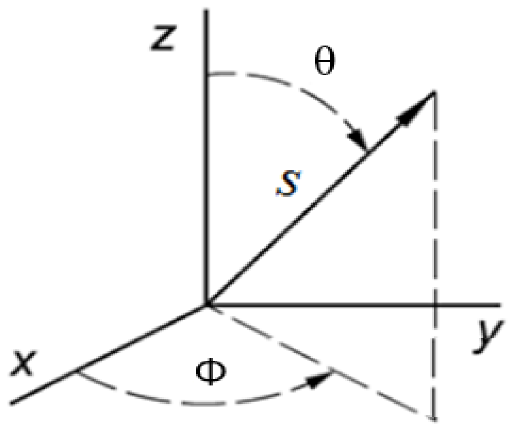
Δ_ϵ in R_2 along y-axis

Notice in this case, that in R_1, the function *inside* the square root is always *positive*, so we leave it as is, and calculate accordingly. The *net* effect is a total *incremental* distance along the *y*-axis, due to dark energy [ξ], of about $4\sqrt{\sigma}$. Again, a plausible result here, since ξ decreases like $J_0(r)$ in this direction. And this result is *equivalent* to the *increase* along the *z*-axis as well.

A Simple Case Study, Part XXV

Consider the following estimator for Δ, where $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta)$ in $\mathcal{R}^3$, with singularities at $(\pm\delta, 0, 0)$; where $\delta = 1$, and $\cos(\gamma) = \sin(\theta)\cos(\phi)$, using a *physical* coordinate system …

$$\mathcal{R} \;=\; \cosh(r\cos(\gamma))J_0(r\sin(\gamma)) \,/\, J_0(r) \;=\; (r + \Delta) \,/\, (r \cdot \sin(\gamma) + \Delta) \qquad (*)$$



Letting $\theta = \pi/2$, so that we are in the *x-y* plane, we see that (*) becomes

$$\mathcal{R} \;=\; \cosh(r\cos(\phi))J_0(r\sin(\phi)) \,/\, J_0(r) \;=\; (r + \Delta) \,/\, (r \cdot \sin(\phi) + \Delta)\,, \qquad (\dagger)$$

and at $\phi = 0$, reduces to

$$\mathcal{R} \;=\; \cosh(r) \,/\, J_0(r) \;=\; (r + \Delta) \,/\, \Delta\,;$$

which we are familiar with, from recent research notes. Now let $\phi \dashrightarrow \pi/2$, so that the expression (†) becomes

$$\mathcal{R} \;=\; J_0(r) \,/\, J_0(r) \;=\; (r + \Delta) \,/\, (r + \Delta) \;=\; 1$$

From a ‘ruler tick’ perspective, this is ideal, since these ticks *contract* if dark energy is *strong*, but begin to *expand* if dark energy *weakens*. So by incorporating the angle ϕ into $\mathcal{R}$, one might think we have actually *improved* the estimator for Δ, which is now

$$\Delta \;=\; r(1 - \mathcal{R} \cdot \sin(\phi)) \,/\, (\mathcal{R} - 1)$$

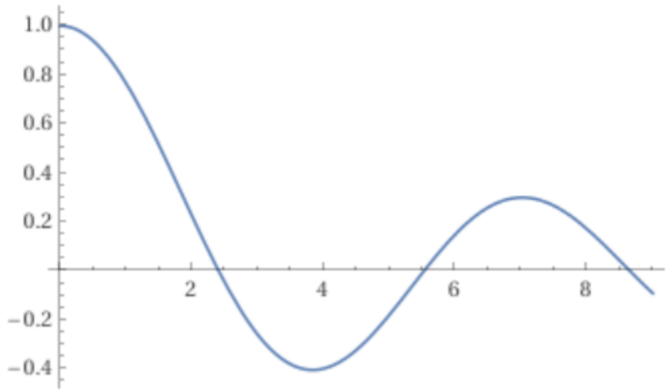
And indeed, at $\phi = 0$ we obtain the familiar result $\Delta \;=\; r \,/\, (\mathcal{R} - 1)$; but at $\phi = \pi/2$, the expression, just above, becomes (where $\mathcal{R} = 1$) …

$$\Delta \;=\; r(1 - \mathcal{R} \cdot \sin(\phi)) \,/\, (\mathcal{R} - 1) \;=\; r \cdot 0 \,/\, 0\,,$$

which is *indeterminate*. And so, the lesson learned here is that just because we think (*) is a better estimator for Δ, it actually isn’t. And so we’ll *stay* with what we have, for the time being, knowing that Δ is *just* a function of *r*, and nothing more …

A Simple Case Study, Part XXVI

Here, we'll calculate the *incremental* volume [Δ_ϵ], due to dark energy, as the sphere $\boldsymbol{S}$ of radius r_3 morphs into an ellipse $\boldsymbol{\mathcal{E}}$, where r_3 is the *third* zero of the Bessel function $J_0(r)$ [pp 617-18]. And here, we are in $\mathcal{R}^3$, where $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta)$, with singularities at $(\pm\delta, 0, 0)$; and $\cos(\gamma) = \sin(\theta)\cos(\phi)$, using *physical* coordinates, where $0 \leq \theta, \phi \leq \pi/2$. As well, $\Delta = r / (\mathcal{R} - 1)$, where it is the case that $\mathcal{R} = \cosh(r) / J_0(r)$.



plot of $J_0(r)$

What follows are the *normalized* ($\sigma = 1$) snapshots from Wolfram, where $x = r$, $y = \theta$ and $z = \phi$ …

$$\int_0^{2.4048}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x^2 \sin(y)}{\sqrt{x\sqrt{1-\sin^2(y)\cos^2(z)} + \frac{x}{\frac{\cosh(x)}{J_0(x)}-1}}}\, dz\, dy\, dx = 5.73955$$

Δ_ϵ in $[0, r_1]$

$$\int_{2.4049}^{5.52}\int_0^{\frac{31\pi}{64}}\int_0^{\frac{31\pi}{64}} \frac{x^2 \sin(y)}{\sqrt{x\sqrt{1-\sin^2(y)\cos^2(z)} + \frac{x}{\frac{\cosh(x)}{J_0(x)}-1}}}\, dz\, dy\, dx = 43.3693$$

Δ_ϵ in $[r_1, r_2]$

$$\int_{5.5201}^{8.6537}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x^2 \sin(y)}{\sqrt{x\sqrt{1-\sin^2(y)\cos^2(z)} + \frac{x}{\frac{\cosh(x)}{J_0(x)}-1}}}\, dz\, dy\, dx = 111.901$$

Δ_ϵ in $[r_2, r_3]$

Notice in the *second* integral associated with $R_1 = [r_1, r_2]$, we can only *approximate* Δ_ϵ, because the *denominator* of this expression is both *positive* and *negative*, inside the *outer* square root sign.

Thus, to an imaginary observer $\mathcal{O}$ situated at the origin O in $\mathcal{R}^3$, the *total incremental* volume Δ_ϵ, due to dark energy, is approximately $1288\sigma^{3/2}$ … after multiplying by 8, which we do, because of symmetry. And so $\mathcal{O}$ concludes that the *total* volume of the ellipse $\mathcal{E}$ that $\mathcal{S}$ has morphed into, is approximately $(4/3)\pi r_3^3 + \Delta_\epsilon$.

Finally, it should be said that we are using a default *inner* block $\mathcal{B} = [r^2, r^2 sin^2(\theta)]$, to do the calculations.

———

There seems to be *good* consistency between our findings on pages 617-18 and this research note, when the radius r is an *odd* zero of $J_0(r)$. That is to say, $r = r_1, r_3, r_5, \ldots$

For in such a case, *incremental* length, due to dark energy, *increases* along the x, y and z axes, and so does *incremental* volume. But if r is an *even* zero of $J_0(r)$, say r_2; then we see a *decrease* in *incremental* length along the x-axis [pp 617-18, also p 580], but an *increase* along the y, z axes. This is, in fact, directly related to the definition of $\Delta = r / (\mathcal{R} - 1)$, which has only been tested in the general equivalency theorem [$\mathcal{G}$], when $r \leq 2$. Furthermore, since $\mathcal{R} = \cosh(r) / J_0(r)$, Δ itself, is intimately tied to the *zeroes* of $J_0(r)$.

Nonetheless, we'll retain Δ in $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta)$ for the time being, and note that as r ---› 0, $g_{u,v}$ ---› 0, since here Δ ---› ∞. And for large r, $g_{u,v}$ ---› $1 / \Delta$ on the x-axis, since $\sin(\gamma) = 0$, and along the y, z axes, $g_{u,v}$ ---› $1 / r$, since $\sin(\gamma) = 1$ here, and Δ ---› 0.

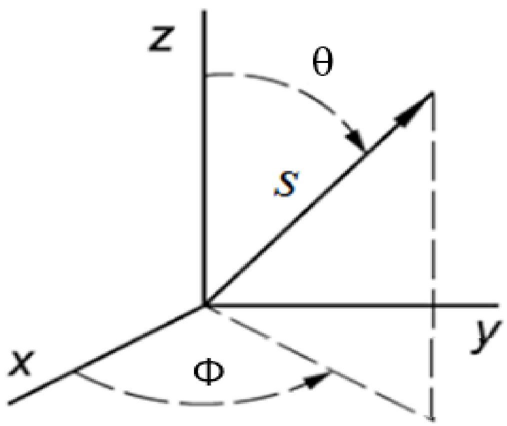
In general, though, where we choose r to be an *odd* zero of $J_0(r)$, I think we'll get reasonably good results, that demonstrate consistency, when looking at *incremental* length, area and volume, using the *spacelike* estimator $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta)$; and similarly, when using the *timelike* estimator $g_{u,v} = 1 / (r \cdot \cos(\gamma) + \Delta)$.

A Simple Case Study, Part XXVII

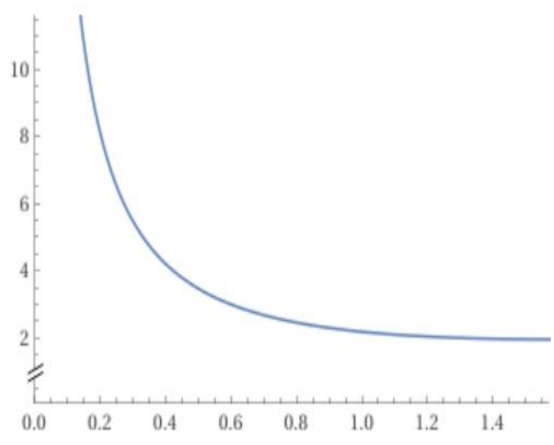
Continuing the discussion in Part XXV on page 619 … we'll now take a look at the following estimator for Δ. Let

$$\mathcal{R} \ = \ \cosh(r\cos(\phi))J_0(r\sin(\phi)) \, / \, \cosh(r) \ = \ \Delta \, / \, (r \cdot \sin(\phi) + \Delta) \ , \qquad (\dagger)$$

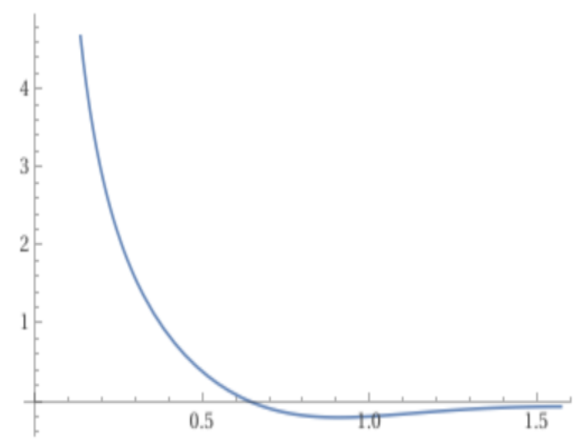
in the x-y plane, so that $\Delta = -r\mathcal{R} \cdot \sin(\phi) \, / \, (\mathcal{R} - 1)$, where ϕ is between 0 and $\pi/2$. One might think that $\mathcal{R}$ is well-defined, since it too follows the 'ruler tick' paradigm.



But here are some plots from Wolfram, that show this is not really the case …



plot of Δ, $0 \le \phi \le \pi/2$, $r = 1$



plot of Δ, $0 \le \phi \le \pi/2$, $r = 4$

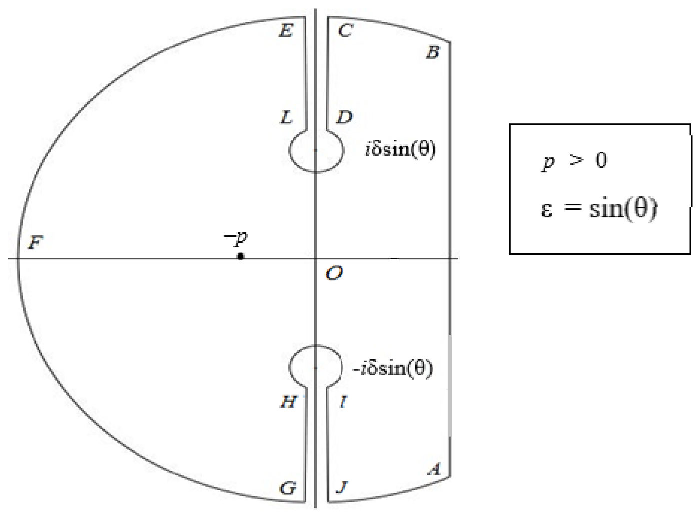
From these pictures, we see that $\Delta \dashrightarrow \infty$ as $\phi \dashrightarrow 0$, so that $g_{u,v} = 1 \, / \, (r \cdot \sin(\phi) + \Delta) \dashrightarrow 0$ along the x-axis, *no* matter the choice of r. Thus, integrating $\sqrt{\sigma \cdot g_{u,v}}$ along this axis, would produce nothing.

A Simple Case Study, Part XXVIII

In recent notes, we mentioned that our estimator $g_{u,v} = 1 / (r \cdot \sin(\phi) + \Delta)$, could be both *positive* and *negative* in the *x*-*y* plane, if Δ was *negative*. For Δ, the estimator is $\Delta = r / (\mathcal{R} - 1)$, where here $\mathcal{R} = \cosh(r) / J_0(r)$ and $0 \le \phi \le \pi/2$. Thus, Δ will turn negative if $J_0(r)$ is negative.

Now dark energy is defined to be $\xi = 2\sigma \cdot \cosh(\delta r\cos(\phi))J_0(\delta r\sin(\phi))$ in $\mathcal{R}^2$, and for singularities at $(\pm\delta, 0)$ with $\delta = 1$, will become *negative* if $J_0(r\sin(\phi)) < 0$. In such a case, we *impose* the condition that $g_{u,v} < 0$ as well. Thus, our only concern, in this case, is what to do with $\mathcal{D} = r \cdot \sin(\phi) + \Delta$, for it could be *positive* or *negative*, based on the current definition of Δ.

If we insist at $\phi = \pi / 2$, that the pole $-p = -\Delta$ be *negative*, when traversing the contour below, in order to calculate the *residue* associated with the generating function $f(s)$ [pp 559-62, 575]; then necessarily Δ must be *positive*. And we can arrange for that by letting $\mathcal{R} = \cosh(r) / |J_0(r)|$, since $\cosh(r)$ is always greater than or equal to $|J_0(r)|$, and hence $\Delta = r / (\mathcal{R} - 1) \ge 0$ [pp 617-18].



We would then write $g_{u,v} = sgn(J_0(r\sin(\phi))) / (r \cdot \sin(\phi) + \Delta)$, where *this* version of *sgn* is the *sign* function, being +1 if $J_0(r\sin(\phi)) \ge 0$ and –1 if $J_0(r\sin(\phi)) < 0$. As an example, the *first* integral on page 618, in the range $R_1 = (r_1, r_2)$ along the *x*-axis; where r_1 and r_2 are the *first* and *second* zeros of $J_0(r)$, now becomes [since $\phi = 0$ and thus $sgn(J_0(r\sin(\phi))) = 1$] …

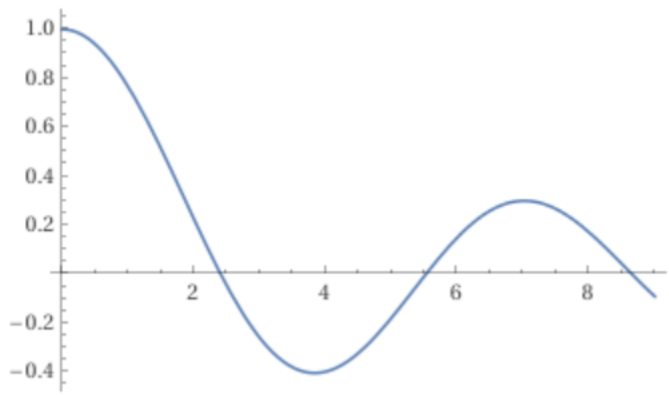
$$\int_{2.4049}^{5.52} \frac{1}{\sqrt{\frac{x}{\frac{\cosh(x)}{|J_0(x)|} - 1}}}\, dx = 23.7895$$

And this is *very close* to the number –23.9614 quoted there, *but* the *sign* has changed, as it should, since along the *x*-axis, dark energy is wholly *positive* with *exponential* growth. In $\mathcal{R}^3$ we would write $g_{u,v} = sgn(J_0(r\sin(\gamma))) / (r \cdot \sin(\gamma) + \Delta)$; where $\cos(\gamma) = \sin(\theta)\cos(\phi)$ in *physical* coordinates and the singularities are at $(\pm\delta, 0, 0)$, with $0 \le \theta, \phi \le \pi/2$, and $\delta = 1$. And in ξ, ϕ would map to γ ...

As another example, the *second* integral on page 618, in the range $R_1 = (r_1, r_2)$ along the *y*-axis; where r_1 and r_2 are the *first* and *second* zeros of $J_0(r)$, now becomes [since the angle $\phi = \pi/2$ and thus $sgn(J_0(r\sin(\phi))) = -1$] …

$$\int_{2.4049}^{5.52} -\frac{1}{\sqrt{x + \frac{x}{\frac{\cosh(x)}{|J_0(x)|} - 1}}}\, dx = -1.58639$$

And this is *very close* to the number 1.60828 quoted there, *but* the *sign* has changed, as it should, since along the *y*-axis, dark energy is *negative* in R_1, as you can see from the picture below …



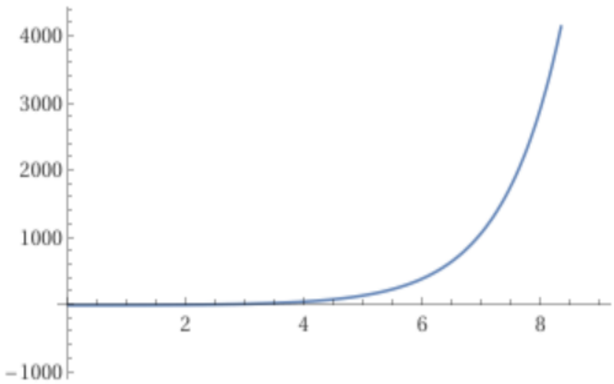
Plot of $J_0(r)$ from 0 to r_3 at $\phi = \pi/2$

Thus, with these revisions, our imaginary observer $\mathcal{O}$ at the origin O, will say the *incremental* distance along the *x*-axis, due to dark energy, and out to a radius of r_3, is approximately $128\sqrt{\sigma}$. And similarly, the *incremental* distance along the *y*, *z* axes, out to r_3, is approximately $\sqrt{\sigma}$.

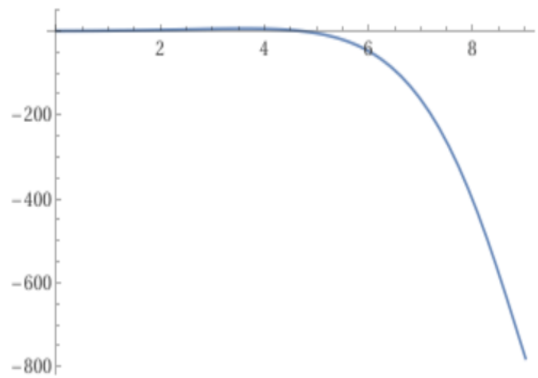
For the *second* and *third* integrals on page 620, where we are calculating *incremental* volume, we need to know what $sgn(J_0(r\sin(\gamma)))$ is, in this triple integration, since $\sin(\gamma)$ can vary between 0 and 1.

Thus, $r\sin(\gamma)$ can vary between 0 and r_3, so that $sgn(J_0(r\sin(\gamma)))$ oscillates between –1 and +1, along any *radial* line $\ell_{\theta\phi}$, emanating from the origin O. Unfortunately, the Wolfram site is not able to calculate the integration in this case; but we may be able to approximate it by knowing the *incremental* distances quoted above, and knowing too, that the *volume* of the *original* sphere computes to $(4/3)\pi r_3^3$ [in reality, the sphere of radius r_3 doesn't morph into an ellipsoid $\mathcal{E}$; rather, an object with an *undulating* surface on either side of the surface of $\mathcal{E}$].

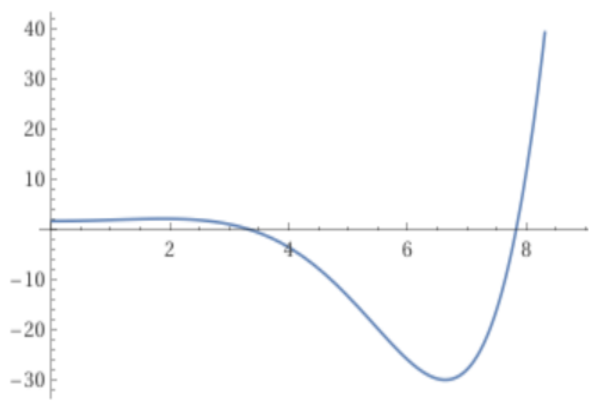
Below are some plots of the dark energy ξ ($\sigma = 1$) in $\mathcal{R}^2$, at different angles ϕ to the *x*-axis, and you can see just how different the *radial* patterns actually are …



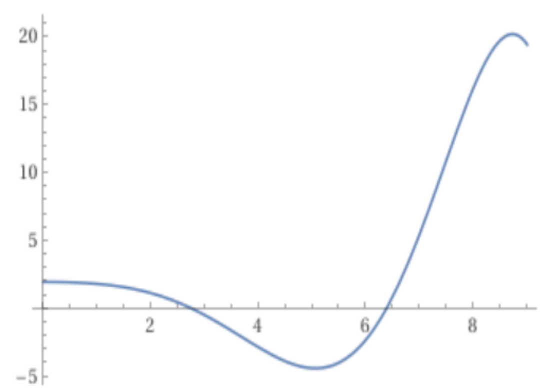
ξ from 0 to r_3 at $\phi = 0$



ξ from 0 to r_3 at $\phi = \pi/6$



ξ from 0 to r_3 at $\phi = \pi/4$



ξ from 0 to r_3 at $\phi = \pi/3$

So if, for example, using our estimator $g_{u,v} = sgn(J_0(r\sin(\phi))) / (r \cdot \sin(\gamma) + \Delta)$, we were calculating *incremental* distance out to r_3 when $\phi = \pi / 4$, we'd want to know where $J_0(r\sin(\phi))$ changes *sign* in this picture. And that occurs at about $r \approx 3.5$ and 7.8. Again, r_3 is the *third* zero of $J_0(r)$.

———

In my opinion, this approach is better than what we find in the previous research notes, for it allows us to align $g_{u,v}$ with the *sign* of dark energy [ξ] on the one hand; yet on the other, provides an estimator for Δ that seems reasonable. And, it allows us to calculate *incremental* distance, due to dark energy, out to *any* arbitrary radius *r*, along any *radial* line $\ell_{\theta\phi}$. However, if we want to calculate things like *incremental* area or volume, we will probably have to write custom routines to carry out these integrations.

Finally, for $g_{u,v} = sgn(J_0(r\sin(\gamma))) / (r \cdot \sin(\gamma) + \Delta)$ we note that as $r \dashrightarrow 0$, $g_{u,v} \dashrightarrow 0$, since here it is the case that $\Delta \dashrightarrow \infty$. And for large *r*, $g_{u,v} \dashrightarrow 1 / \Delta$ on the *x*-axis, since $\sin(\gamma) = 0$; and along the *y*, *z* axes, $g_{u,v} \dashrightarrow sgn(J_0(r)) / r$, since $\sin(\gamma) = 1$ here, and $\Delta \dashrightarrow 0$.

The following table shows *incremental* distances [Δ_ϵ] out to r_3 *and* r_4 in $\mathcal{R}^2$, due to dark energy [ξ], where σ is equal to 1 and the singularities are at S = ($\pm\delta$, 0), with δ = 1.

angle ϕ	Δ_ϵ , r_3	Δ_ϵ , r_4
0	128.96	559.79
π / 80	16.15	21.11
π / 40	12.33	15.85
π / 20	9.34	11.82
π / 6	1.52	0.76
π / 4	0.47	1.64
π / 3	0.89	0.78
π / 2	1.11	0.13

You can see from the table that Δ_ϵ is *greatest* near the *x*-axis, but falls away rapidly, even as ϕ increases, ever so slightly. And this is because S is located on the *x*-axis, where ξ is *strongest*.

These distances, due to symmetry, would be the *same* in all four quadrants of $\mathcal{R}^2$, relative to the origin O. And so, for the imaginary observer $\mathcal{O}$ at O, the circle of radius r_3 or r_4 has increased by $\sqrt{\sigma} \cdot \Delta_\epsilon$ along the *radial* lines ℓ_x, where $x = \pm\phi$ or $x = \pm(\pi - \phi)$.

Note that $g_{u,v} = sgn(J_0(r\sin(\phi))) \,/\, (r \cdot \sin(\phi) + \Delta)$, and that $\Delta = r \,/\, (\mathcal{R} - 1)$; where here it is the case that $\mathcal{R} = \cosh(r) \,/\, |J_0(r)|$. And in $\mathcal{R}^3$ we would write $g_{u,v} = sgn(J_0(r\sin(\gamma))) \,/\, (r \cdot \sin(\gamma) + \Delta)$; where $\cos(\gamma) = \sin(\theta)\cos(\phi)$ in *physical* coordinates and the singularities are at ($\pm\delta$, 0, 0), with the usual constraints; namely $0 \le \theta, \phi \le \pi/2$, and $\delta = 1$.

So if along $\ell_{\theta\phi}$, where $\theta, \phi = \pi/4$, we wanted to know what Δ_ϵ was ($\sigma = 1$), when $r = r_4 \approx 11.79$ – the *fourth* zero of $J_0(r)$; we note that $\cos(\gamma) = ½$, so that $\sin(\gamma) = \sqrt{3}/2$, and hence the result is …

$$\int_0^{11.79} \frac{\operatorname{sgn}\left(J_0\left(\frac{x\sqrt{3}}{2}\right)\right)}{\sqrt{\frac{x\sqrt{3}}{2} + \frac{x}{\frac{\cosh(x)}{|J_0(x)|} - 1}}}\, dx = 0.775982$$

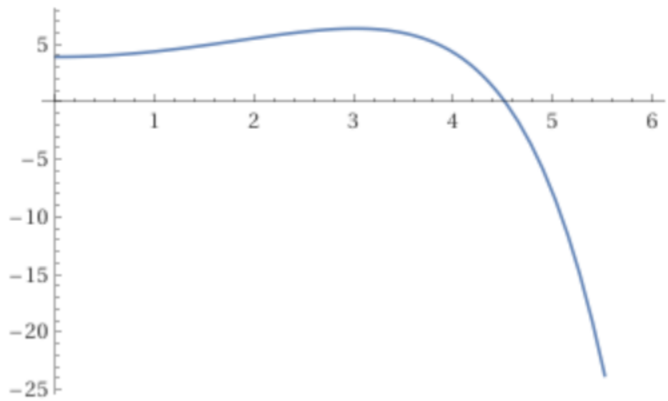
And in this case, Wolfram allows us to use the *sgn* function for *single* integrals. Thus, to an imaginary observer $\mathcal{O}$ at O, the sphere of *radius* r_4 has *increased* by $\sqrt{\sigma} \cdot \Delta_\epsilon$ along the *radial* line $\ell_{\theta\phi}$, due to dark energy, where $\Delta_\epsilon \approx 0.776$. And so, we can map out these *incremental* distances for a sphere of *arbitrary* radius $r = r_k$, for *any* $\ell_{\theta\phi}$, where r_k is the *k-th* zero of $J_0(r)$.

A Simple Case Study, Part XXIX

In this note, we wish to calculate *incremental* distance [Δ_ϵ] in $\mathcal{R}^2$, due to dark energy, when there are singularities at ($\pm\delta$, 0) *and* (0, $\pm\delta$), where $\delta = 1$. In this case, the *total* dark energy function computes to

$$\xi = 2\sigma \cdot \{\cosh(\delta r\cos(\phi))J_0(\delta r\sin(\phi)) \; + \; \cosh(\delta r\sin(\phi))J_0(\delta r\cos(\phi))\} \, ,$$

and our *two* gravitational components are $g_{u,v} = 1 / (r \cdot \sin(\phi) + \Delta)$ and $h_{u,v} = 1 / (r \cdot \cos(\phi) + \Delta)$; where the *first* one is associated with ($\pm\delta$, 0), and the *second* with (0, $\pm\delta$) [see also pp 623-6].



plot of ξ when $\phi = \pi / 6$

We *define* the *incremental* distance [Δ_ϵ] to be, where r_k is the *k-th* zero of $J_0(r)$ …

$$\int_0^{r_k} sgn(\xi) \cdot (\sqrt{\sigma \cdot g_{u,v}} \; + \; \sqrt{\sigma \cdot h_{u,v}}\,) \, dr \qquad (\dagger)$$

As an example, if $\phi = \pi / 6$ and $k = 3$, then with $\sigma = 1$ and $x = r$, Δ_ϵ computes to …

$$\int_0^{8.6536} \operatorname{sgn}\left(\cosh\left(x\cos\left(\frac{\pi}{6}\right)\right)J_0\left(x\sin\left(\frac{\pi}{6}\right)\right) + \cosh\left(x\sin\left(\frac{\pi}{6}\right)\right)J_0\left(x\cos\left(\frac{\pi}{6}\right)\right)\right) \left(\frac{1}{\sqrt{x\sin\left(\frac{\pi}{6}\right) + \frac{x}{\frac{\cosh(x)}{|J_0(x)|} - 1}}} + \frac{1}{\sqrt{x\cos\left(\frac{\pi}{6}\right) + \frac{x}{\frac{\cosh(x)}{|J_0(x)|} - 1}}}\right) dx = 2.19476$$

And this compares favorably with the *sum* of 1.52 + .89 = 2.41 in the table on page 626, where these are the values of Δ_ϵ, when $\phi = \pi / 6$ and $\pi / 3$, respectively. We could have just added these

two numbers together, and defined Δ_ϵ to be the *sum*; but I think the method above is more accurate, because it takes into account the *sign* of *total* dark energy, for any given radius r, and angle ϕ.

———

In $\mathcal{R}^3$, the *total* dark energy function is, with singularities at $[(\pm\delta, 0, 0), (0, \pm\delta, 0), (0, 0, \pm\delta)]$ …

$$\xi = 2\sigma \cdot \{\cosh(\delta r\cos(\gamma)_1)J_0(\delta r\sin(\gamma)_1) + \cosh(\delta r\cos(\gamma)_2)J_0(\delta r\sin(\gamma)_2) + \cosh(\delta r\cos(\gamma)_3)J_0(\delta r\sin(\gamma)_3)\},$$

where $1 = x$, $2 = y$, $3 = z$, using *physical* coordinates; and from pages 588-9,

$$\cos(\gamma)_1 = \sin(\theta)\cos(\phi)\ \ ;\ \ \cos(\gamma)_2 = \sin(\theta)\sin(\phi)\ \ ;\ \ \cos(\gamma)_3 = \cos(\theta)\ .$$

We *define* the incremental distance $[\Delta_\epsilon]$ to be, where r_k is the *k-th* zero of $J_0(r)$ …

$$\int_0^{r_k} sgn(\xi) \cdot (\sqrt{\sigma \cdot f_{u,v}}\ +\ \sqrt{\sigma \cdot g_{u,v}}\ +\ \sqrt{\sigma \cdot h_{u,v}})\ dr \qquad (\dagger)$$

and here,

$$f_{u,v}\ =\ 1 / (r \cdot \sin(\gamma)_1 + \Delta)\ \ ;\ \ g_{u,v}\ =\ 1 / (r \cdot \sin(\gamma)_2 + \Delta)\ \ ;\ \ h_{u,v}\ =\ 1 / (r \cdot \sin(\gamma)_3 + \Delta)\ .$$

———

In $\mathcal{R}^2$, with singularities at $S = [(\pm\delta, 0), \pm(\sqrt{2} \cdot \delta / 2, \sqrt{2} \cdot \delta / 2), (0, \pm\delta), \pm(\sqrt{2} \cdot \delta / 2, -\sqrt{2} \cdot \delta / 2)]$, so that they *all* lie on a circle of radius C_δ; the *total* dark energy function is $[0 \le \phi \le \pi / 2, \delta = 1$; and noting that *cosh* and J_0 are *even* functions] …

$$\xi = 2\sigma \cdot \{\cosh(\delta r\cos(\phi))J_0(\delta r\sin(\phi))\ +\ \cosh(\delta r\cos(\phi - \pi / 4))J_0(\delta r\sin(\phi - \pi /4))\ +$$
$$\cosh(\delta r\sin(\phi))J_0(\delta r\cos(\phi))\ +\ \cosh(\delta r\cos(\phi + \pi / 4))J_0(\delta r\sin(\phi + \pi /4))\}\ .$$

The terms in ξ correspond to the terms in S, and here,

$$f_{u,v}\ =\ 1 / (r \cdot |\sin(\phi)|\ +\ \Delta)\ \ ;\ \ g_{u,v}\ =\ 1 / (r \cdot |\sin(\phi - \pi /4)| + \Delta)\ \ ;$$

$$h_{u,v}\ =\ 1 / (r \cdot |\cos(\phi)| + \Delta)\ \ ;\ \ k_{u,v}\ =\ 1 / (r \cdot |\sin(\phi + \pi /4)| + \Delta)\ \ .$$

We then calculate *incremental* distance $[\Delta_\epsilon]$ along any line ℓ_ϕ, using the expression below; where again, the estimator $\Delta = r / (\mathcal{R} - 1)$, and $\mathcal{R}\ =\ \cosh(r) / |J_0(r)|$.

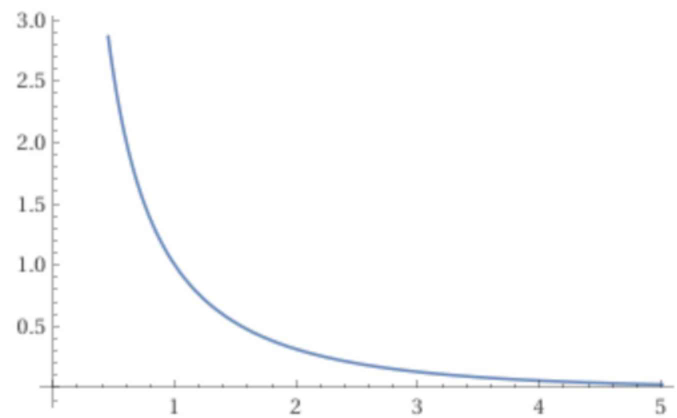
$$\int_0^{r_k} sgn(\xi) \cdot (\sqrt{\sigma \cdot f_{u,v}}\ +\ \sqrt{\sigma \cdot g_{u,v}}\ +\ \sqrt{\sigma \cdot h_{u,v}})\ +\ \sqrt{\sigma \cdot k_{u,v}})\ \ dr \qquad (\dagger)$$

A Simple Case Study, Part XXX

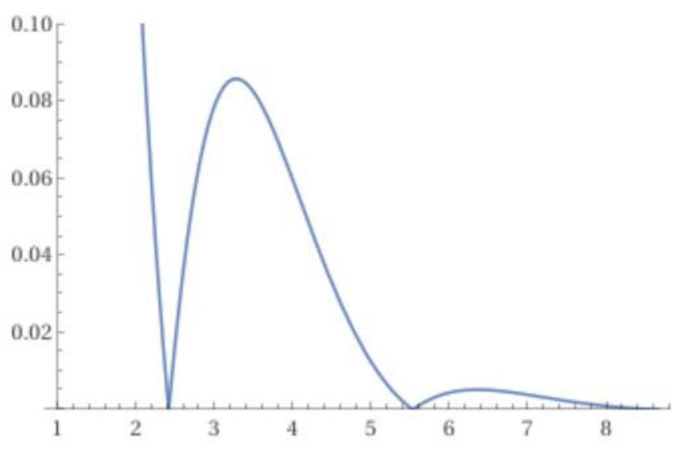
We can also consider estimators for Δ of type $\Delta' = \beta^{r-1} / r$, when $\beta > 0$ [in this case $\beta = 16/25$]. These estimators distinguish themselves from $\Delta = r / (\mathcal{R} - 1)$, where $\mathcal{R} = \cosh(r) / |J_0(r)|$, in that they are *continuously* differentiable, and do *not* become *zero* if r is a *zero* of $J_0(r)$. Thus, the estimator $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta')$ would *not* become *infinite* for *any* choice of r, if $\sin(\gamma) = 0$; which it is, along the x-axis in $\mathcal{R}^3$, using *physical* coordinates [$0 \leq \theta, \phi \leq \pi/2$, and $\delta = 1$].

As $r \dashrightarrow 0$, both Δ and $\Delta' \dashrightarrow \infty$, so that $g_{u,v} \dashrightarrow 0$. For large r, the ratio $\Delta / \Delta' \sim r^2 |J_0(r)| 2^{-r}$, if we have $\beta = 16/25$, so that $\Delta \dashrightarrow 0$ much more quickly than Δ'. If, therefore, we believe Δ is decaying *too* quickly as $r \dashrightarrow \infty$, then we might consider estimators of type Δ' as an alternative.

In the case, where $r = ½$ or $r = 1$, $g_{u,v} = 1 / (r \cdot \sin(\gamma) + \Delta')$ approximately satisfies the general equivalency theorem [$\mathcal{G}$], *if* $\phi = \pi/2$, since here $\Delta' = 5/2$ or 1 [pp 559-562, 611-12]. However, for a radius of $r = 2$, there is no match for Δ' in $\mathcal{G}$, using $g_{u,v}$; but there is for Δ [p 614]. Nonetheless, we want to retain Δ' for the properties that it does have; and in particular, for its *slower* decay rate as $r \dashrightarrow \infty$ …



plot of Δ′ when β = 16/25



plot of Δ

A Simple Case Study, Part XXXI

As an alternative to defining *incremental* distance the way we did on pages 627-8, we can opt for the following definition, where there are singularities at $(\pm\delta, 0)$ *and* $(0, \pm\delta)$, and $\delta = 1$. In this case, the *total* dark energy function computes to

$$\xi = 2\sigma \cdot \{\cosh(\delta r\cos(\phi))J_0(\delta r\sin(\phi)) + \cosh(\delta r\sin(\phi))J_0(\delta r\cos(\phi))\} ,$$

and our *two* gravitational components are $g_{u,v} = 1 / (r \cdot \sin(\phi) + \Delta)$ and $h_{u,v} = 1 / (r \cdot \cos(\phi) + \Delta)$; where the *first* one is associated with $(\pm\delta, 0)$, and the *second* with $(0, \pm\delta)$ [see also pp 623-6]. Here, $\Delta = r / (\mathcal{R} - 1)$, where $\mathcal{R} = \cosh(r) / |J_0(r)|$.

We *define* the *incremental* distance $[\Delta_\epsilon]$ to be, where r_k is the *k-th* zero of $J_0(r)$ …

$$\int_0^{r_k} sgn(\xi) \cdot (\sqrt{\sigma \cdot g_{u,v} + \sigma \cdot h_{u,v}}\,)\, dr \qquad (\dagger)$$

As an example, if $\phi = \pi/6$ and $k = 3$, then with $\sigma = 1$ and $x = r$, Δ_ϵ computes to …

$$\int_0^{8.6536} \operatorname{sgn}\left(\cosh\left(x\cos\left(\frac{\pi}{6}\right)\right)J_0\left(x\sin\left(\frac{\pi}{6}\right)\right) + \cosh\left(x\sin\left(\frac{\pi}{6}\right)\right)J_0\left(x\cos\left(\frac{\pi}{6}\right)\right)\right) \sqrt{\frac{1}{x\sin\left(\frac{\pi}{6}\right) + \frac{x}{\frac{\cosh(x)}{|J_0(x)|}-1}} + \frac{1}{x\cos\left(\frac{\pi}{6}\right) + \frac{x}{\frac{\cosh(x)}{|J_0(x)|}-1}}}\; dx =$$

$$1.55278$$

This number is certainly *smaller* than 2.19476, which is the value we see on page 627, using the definition for Δ_ϵ put forth there; however, for the sake of completeness we include this approach, and will let the reader decide which is more philosophically appealing.

Notice too, that the number above; namely 1.55278, is only *marginally* larger than the value of 1.52 found in the table on page 626 for $\phi = \pi/6$ and $k = 3$, when the singularities are at $(\pm\delta, 0)$. In other words, using this methodology, adding the singularities at $(0, \pm\delta)$ *only* contributes about 0.03 to Δ_ϵ, in this case. Personally, I find this contribution to be too small …

A Simple Case Study, Part XXXII

To calculate *incremental* volume [Δ_ϵ] in $\mathcal{R}^3$, with singularities at S = [($\pm\delta$, 0, 0), (0, $\pm\delta$, 0), (0, 0, δ)], we use the *same* setup on pages 627-8, using the *default* inner block $\mathcal{B} = [r^2 , r^2 sin^2(\theta)]$. The result is to be multiplied by 8, using symmetry … ..

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{r_k} sgn(\xi)\cdot\left(\sqrt{\sigma\cdot f_{u,v}} + \sqrt{\sigma\cdot g_{u,v}} + \sqrt{\sigma\cdot h_{u,v}}\right)\sqrt{\sigma}r\sqrt{\sigma}r sin(\theta)\, dr d\theta d\phi \quad (\dagger)$$

To calculate *incremental* volume [Δ_ϵ] in $\mathcal{R}^3$, with singularities at S = [($\pm\delta$, 0, 0), (0, $\pm\delta$, 0), (0, 0, δ)], we use the *same* setup on pages 627-8; but now we'll use a *non-standard inner* block $\mathcal{B}'$, such as the one shown on pages 607-8, and reproduced here [note that it mimics $\mathcal{B}$] …

$$\mathcal{B}' = [1 / (r\cdot \cos(\gamma) + \Delta_{u,v}) , \sin^2(\theta) / (r\cdot\cos(\gamma) + \Delta_{u,v})]$$

Since *inner* blocks are to be *added* together if they are *uniquely* different from one another, let us write [pp 627-8]

$$\beta = 1 / (r\cdot\cos(\gamma)_1 + \Delta) + 1 / (r\cdot\cos(\gamma)_2 + \Delta) + 1 / (r\cdot\cos(\gamma)_3 + \Delta)$$

Then Δ_ϵ becomes [which is to be multiplied by 8] … ..

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{r_k} sgn(\xi)\cdot\left(\sqrt{\sigma\cdot f_{u,v}} + \sqrt{\sigma\cdot g_{u,v}} + \sqrt{\sigma\cdot h_{u,v}}\right)\cdot\sigma\beta sin(\theta)\, dr d\theta d\phi \quad (*)$$

And from previous research, we *also* know that *both* (†) and (*) *preserve* volume *invariance*, so that if S were to be rotated about the origin O, in some fashion, Δ_ϵ would *not* change [pp 607-8].

We can also define (*) as a sum over 'volume elements', so that if we let

$$\psi = \sqrt{\sigma\cdot f_{u,v}}\cdot 1 / (r\cdot\cos(\gamma)_1 + \Delta) + \sqrt{\sigma\cdot g_{u,v}}\cdot 1 / (r\cdot\cos(\gamma)_2 + \Delta) + \sqrt{\sigma\cdot h_{u,v}}\cdot 1 / (r\cdot\cos(\gamma)_3 + \Delta),$$

then (*) becomes

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{r_k} sgn(\xi)\cdot\sigma\psi sin(\theta)\, dr d\theta d\phi \quad (\S)$$

and this may be an improvement over (*) above.

And finally, were we to adopt the methods on page 630, then (*) becomes

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{r_k} sgn(\xi)\cdot(\sqrt{\sigma\cdot f_{u,v} + \sigma\cdot g_{u,v} + \sigma\cdot h_{u,v}}\cdot\sigma\beta sin(\theta)\, dr d\theta d\phi \quad (\ddagger)$$

since now, the gravitational components are *first* added together, before forming ds^2; whether we are looking at dr^2, $d\theta^2$ or $d\phi^2$.

Such is *not* the case in (§) above, for here each volume element is created *individually*, and then all *three* are added together. And in (*) above, it's a *hybrid* approach – the dr^2 elements are created *individually*, but the $d\theta^2$ and $d\phi^2$ elements are *first* added together.

In my opinion, from a consistency viewpoint, we either choose (§) or (‡); and of these two, I would say (§) is the preferred option.

A Simple Case Study, Part XXXIII

We can *also* consider estimators of type $\Delta' = \{r / (\mathcal{R} - 1)\} + \kappa$, where $\mathcal{R} = \cosh(r) / |J_0(r)|$ and κ is some small *positive* number, such that $g_{u,v} = 1 / (r \cdot \sin(\phi) + \Delta')$ *still* satisfies the general equivalency theorem [$\mathcal{G}$] in $\mathcal{R}^2$, *if* $\phi = \pi / 2$ and $r = ½$, 1 or 2 [pp 559-562, 611-12, 614]. For example, κ might be 10^{-2} or 10^{-3} [here, the singularities are at $(\pm\delta, 0)$, where $\delta = 1$ and $0 \le \phi \le \pi/2$].

For in such a case, R is equal to $2 \cdot Res$, where *Res* is approximately $exp(-\kappa r) / \sqrt{\kappa^2 + (\delta\varepsilon)^2}$, for large r [here $\alpha = \cos(\phi) = 0$, and $\varepsilon = \sin(\phi) = 1$]. Thus R ---› 0, in this case, as r ---› ∞. As to the harmonic expression $\mathcal{H}_x$, it becomes [where $\boldsymbol{\kappa} = 1 / 2\pi \boldsymbol{i}$]

$$2\boldsymbol{\kappa}\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\kappa + iy) - 1/(\kappa - iy)] + \boldsymbol{i}\sin(yr)\,[1/(\kappa + iy) + 1/(\kappa - iy)]\}\,dy \,/\sqrt{y^2 - \varepsilon^2}$$

Now since κ is small, we can *omit* it in the integration … and so for large r, $\mathcal{H}_x$ becomes, approximately ($\varepsilon = 1$) …

$$-2/\pi \int_{\varepsilon}^{\infty} \cos(yr)\, dy \,/ y\sqrt{y^2 - \varepsilon^2}\,,$$

which will tend to *zero* as r ---› ∞, since the integral *in kind*; namely

$$-2/\pi \int_{\varepsilon}^{\infty} \cos(yr)\, dy \,/\sqrt{y^2 - \varepsilon^2} \;=\; Y_0(r\varepsilon)\,,$$

is a Bessel function of the *second* kind; which is known to tend to *zero* as r ---› ∞ [see page 170 in G. N. Watson's book *A Treatise On The Theory Of Bessel Functions*, Second Edition, for this representation of $Y_0(r\varepsilon)$, as well as page 327 in these notes].

Finally, we have $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$ in $\mathcal{G}$, and since $\alpha = 0$, it is the case that $g_{u,v} = 1 / \Delta'$, which for large r, becomes $1 / \kappa$; and hence $\mathcal{E} = (1 / \kappa) \cdot J_0(\delta r\varepsilon)$, which will *also* tend to *zero* as r ---› ∞. Thus, $\mathcal{G}$ is satisfied for large r, since $\mathcal{H}_x$ + R *and* $\mathcal{E}$ both tend to *zero* here.

No such claim can be made if $\kappa = 0$, since if, for example, $r = r_k$ is the *k-th* zero of $J_0(r)$, where k is *large*, then we have $\Delta' = r / (\mathcal{R} - 1) = 0$, and $J_0(\delta r\varepsilon) = 0$, since *both* δ and ε are equal to 1. Thus, $\mathcal{E}$ is not *properly* defined in this case, because $1 / \Delta' = \infty$. And while we might be able to calculate an 'in the limit' value for $\mathcal{E}$ as r ---› r_k, there can be *no* assurance that $\mathcal{G}$ would be satisfied here.

As well, by introducing the constant κ, we see that along the x-axis, where dark energy [ξ] grows *exponentially*; the gravitational component is approximately $g_{u,v} = 1 / \kappa$, for large r, giving us more control on just how fast *incremental* distance [Δ_ϵ] is increasing as r ---› ∞. Without κ, we are constrained by the behavior of $\Delta' = r / (\mathcal{R} - 1)$, which tends to *zero* very quickly for large r …

A Simple Case Study, Part XXXIV

When r ---› 0, $\Delta' = \{r / (\mathcal{R} - 1)\} + \kappa$ can be written as $r / (\mathcal{R} - 1)$, since κ is *small* and $\mathcal{R} - 1$ approaches *zero*. In fact, since $\mathcal{R} = \cosh(r) / |J_0(r)|$, and $J_0(r)$ ---› 1 as r ---› 0; we find that $\mathcal{R}$ behaves as $1 + ½ \cdot r^2$ as r ---› 0, and so, $\Delta' \approx 2 / r$. Let us now set $\Delta = \Delta'$, so that $\Delta r = 2$, and note again that $g_{u,v} = 1 / (r \cdot \sin(\phi) + \Delta')$, where $\phi = \pi/2$. Essentially, the *same* setup from the previous research note.

Then *Res* is approximately $exp(-\Delta r) / \sqrt{\Delta^2 + (\delta\varepsilon)^2}$, which will tend to *zero* as r ---› 0, since here it is the case that Δ ---› ∞. And so, R will *also* ---› 0, where R = 2 · *Res*. As to the harmonic expression $\mathcal{H}_x$, it becomes [noting that $\Delta r = 2$, $\boldsymbol{\kappa} = 1 / 2\pi\boldsymbol{i}$, and $\varepsilon = 1$] …

$$2\boldsymbol{\kappa}\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\Delta + iy) - 1/(\Delta - iy)] + \boldsymbol{i}\sin(yr)\,[1/(\Delta + iy) + 1/(\Delta - iy)]\}\,dy \,/\sqrt{y^2 - \varepsilon^2}$$

And this too will tend to *zero* as r ---› 0, as we see in the example below; where $r = 1 / 1000$, and thus $\Delta = 2000$.

$$\int_1^{300} -\frac{i\left(\left(-\frac{1}{2000-iy} + \frac{1}{2000+iy}\right)\cos\left(\frac{y}{1000}\right) + i\left(\frac{1}{2000-iy} + \frac{1}{2000+iy}\right)\sin\left(\frac{y}{1000}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy =$$
$$0.0000476265$$

Finally, we have $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$ in $\mathcal{G}$, the general equivalency theorem … and this now computes to $\mathcal{E} = (1 / \Delta) \cdot J_0(\delta r\varepsilon)$ [$\alpha = 0$, $\varepsilon = 1$ and $\delta = 1$]; and so $\mathcal{E}$ tends to *zero* as r ---› 0. Thus, $\mathcal{G}$ is satisfied for r ---› 0, since $\mathcal{H}_x$ + R *and* $\mathcal{E}$ both tend to *zero* here.

Indeed, the match between $\mathcal{H}_x$ + R *and* $\mathcal{E}$ is always *exact* if $\phi = 0$, and as r ---› 0, we expect that $\mathcal{H}_x$ + R *and* $\mathcal{E}$ will *both* tend to *zero*, no matter the choice of ϕ, such that $0 < \phi \leq \pi/2$. Thus, we have shown from the last research note and this one, that $\mathcal{G}$ is *approximately* satisfied by $g_{u,v}$, for small r and for large r.

A Simple Case Study, Part XXXV

We didn't do too much testing on the *time* component in $\mathcal{R}^2$ [see pp 592-3], so here we'll do a little more on the circle of radius $r = ½$. And here, $g_{u,v} = 1 / (r \cdot \cos(\theta) + \Delta)$ where $0 \le \theta \le \pi / 2$, *and* the dark energy singularities are at $(\pm\delta, 0)$, where $\delta = 1$. As well, we'll let $\Delta = r / (\mathcal{R} - 1)$, where from past research notes, $\mathcal{R} = \cosh(r) / |J_0(r)|$, and thus $\Delta \approx 5/2$ in this case [pp 611-12].

The *generating* function is

$$f(s) = g_{u,v} \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2} = 1 / ((s + \delta\alpha) \cdot \cos(\theta) + \Delta) \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2}\,,$$

so at $\theta = 0$, the *simple* pole associated with the *first* term in $f(s)$ is $-p = -\Delta / \cos(\theta) - \delta\alpha = -(\Delta + 1)$, since $\alpha = \cos(\theta)$, and $\varepsilon = \sin(\theta)$. Thus, $-p \approx -7/2$ if $\theta = 0$, and so the residue is $Res = exp(-pr) / \sqrt{p^2 + (\delta\varepsilon)^2}$; so that the R value associated with the general equivalency theorem $\mathcal{G}$ is $R = 2 \cdot Res$, which is ≈ 0.0993, as shown below. Again, $r = ½$ in this calculation.

$$\frac{4}{7\,e^{7/4}}$$

Decimal approximation

$$0.099299396257397700223020371\ldots$$

And since $\alpha = 1$, $g_{u,v}(\delta\alpha) = 2/7$; and $J_0(\delta r\varepsilon) = 1$ because $\varepsilon = 0$. Thus, since it is the case that $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$, we see that $\mathcal{E} = 2/7$.

Finally, for $r = ½$, the harmonic expression $\mathcal{H}_x$ computes to

$$\int_0^{300} \frac{\cos\left(\frac{y}{2}\right)\left(\frac{1}{3.5+i y} - \frac{1}{3.5-i y}\right) + i \sin\left(\frac{y}{2}\right)\left(\frac{1}{3.5+i y} + \frac{1}{3.5-i y}\right)}{(\pi\, i)\, y}\, dy = 0.186425$$

Now $\mathcal{H}_x + R$ evaluates to ≈ 0.285724, while $\mathcal{E} = 2/7 \approx 0.285714$. The two are *very* close, as we can see here; and from previous notes [pp 559-562] we know the agreement will be *exact* if the angle θ is equal to $\pi / 2$, since here there is *no* pole associated with the *first* term in $f(s)$, and hence $R = 0$.

Thus, $\Delta = r / (\mathcal{R} - 1)$ is not only the right choice for the *spatial* estimator, but also the *timelike* estimator as well; and can be enhanced by writing $\Delta' = \{r / (\mathcal{R} - 1)\} + \kappa$, where κ is small [see pages 633-4].

———

Indeed, if $\theta = \pi / 2$, then $g_{u,v} = 1 / \Delta$, so that $\boldsymbol{\mathcal{E}} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon) = (1 / \Delta) \cdot J_0(r\varepsilon)$, where $r = ½$, and $\alpha = 0$; and $\delta, \varepsilon = 1$.

Now the harmonic expression $\boldsymbol{\mathcal{H}_x}$ computes to [$\kappa = 1 / 2\pi\boldsymbol{i}$] …

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[1/\Delta - 1/\Delta] + \boldsymbol{i}\sin(yr)\,[1/\Delta + 1/\Delta]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

and from page 327, this is equal to

$$(1 / \Delta) \cdot 2/\pi \int_{\varepsilon}^{\infty} \sin(yr)\, dy \,/\sqrt{y^2 - \varepsilon^2} \;=\; (1 / \Delta) \cdot J_0(r\varepsilon)\ .$$

Thus, the match is *exact*, in so much as $\boldsymbol{\mathcal{H}_x}$ is equal to $\boldsymbol{\mathcal{E}}$.

———

On page 634, we offered some evidence that the harmonic expression $\boldsymbol{\mathcal{H}_x}$, as shown below, does indeed tend to *zero* as the radius r ---› 0 [$\Delta r = 2$, $\kappa = 1 / 2\pi\boldsymbol{i}$, and $\varepsilon = 1$] …

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\Delta + iy) \;- 1/(\Delta - iy)] + \boldsymbol{i}\sin(yr)\,[1/(\Delta + iy) \;+\; 1/(\Delta - iy)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

Here are some more snapshots from Wolfram that illustrate this idea …

$$\int_1^{1\,000\,000} -\frac{\boldsymbol{i}\left(\left(-\frac{1}{20\,000-iy}+\frac{1}{20\,000+iy}\right)\cos\left(\frac{y}{10\,000}\right)+\boldsymbol{i}\left(\frac{1}{20\,000-iy}+\frac{1}{20\,000+iy}\right)\sin\left(\frac{y}{10\,000}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy =$$
$$0.0000364697$$

$r = 1 / 10{,}000$; $\Delta = 20{,}000$

$$\int_1^{1\,000\,000} -\frac{\boldsymbol{i}\left(\left(-\frac{1}{200\,000-iy}+\frac{1}{200\,000+iy}\right)\cos\left(\frac{y}{100\,000}\right)+\boldsymbol{i}\left(\frac{1}{200\,000-iy}+\frac{1}{200\,000+iy}\right)\sin\left(\frac{y}{100\,000}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy = 3.68034\times 10^{-6}$$

$r = 1 / 100{,}000$; $\Delta = 200{,}000$

On page 633, we stated the following …

———

Now since κ is small, we can *omit* it in the integration … and so for large r, $\mathcal{H}_x$ becomes, approximately (ε = 1) …

$$-2/\pi \int_{\varepsilon}^{\infty} \cos(yr)\, dy \,/ y\sqrt{y^2 - \varepsilon^2}\ ,$$

which will tend to *zero* as r ---› ∞, since the integral *in kind*; namely

$$-2/\pi \int_{\varepsilon}^{\infty} \cos(yr)\, dy \,/ \sqrt{y^2 - \varepsilon^2} \;=\; Y_0(r\varepsilon)\ ,$$

is a Bessel function of the *second* kind; which is known to tend to *zero* as r ---› ∞.

———

We can offer up some evidence of this, by looking at the following snapshots from Wolfram …

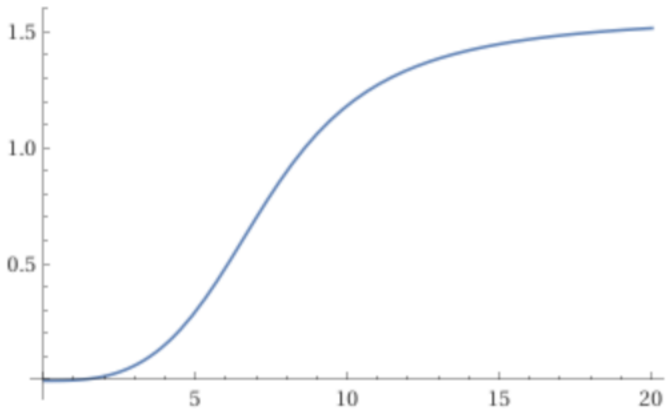
$$\int_1^{32\pi} -\frac{2\cos(48\,y)}{\pi\, y\,\sqrt{-1+y^2}}\,dy = -0.00893725$$

$$\int_1^{32\pi} -\frac{2\cos(48\,y)}{\pi\,\sqrt{-1+y^2}}\,dy = -0.0101336$$

Here, r = 48 and we *don't* require a large *upper* limit on the integrations to confirm things; for in this case $Y_0(r) \approx -0.0101335868$, and you can see how well it agrees with the *second* integral. Now notice too, that the *first* integral is small; and indeed, if our assertion is correct, this integral will *also* tend to zero as r ---› ∞, just like the second …

A Simple Case Study, Part XXXVI

We can also look at estimators for Δ of type $\Delta = \{r / (\mathcal{R} - 1)\} + \kappa(r)$, where $\mathcal{R} = \cosh(r) / |J_0(r)|$ and $\kappa(r) = \arctan(r^3 / \eta)$, where η is suitably large. In this case we'll let $\eta = 400$, and a plot is shown below …



plot of $\kappa(r)$ when $\eta = 400$

Notice that $\kappa(r)$ tends to $\pi / 2$ asymptotically, and that even for smaller r, it is already pretty close to this value. On the other hand, when r is *very* small, $\kappa(r)$ is virtually negligible, and so won't affect our calculations for the various components of the general equivalency theorem [$\mathcal{G}$] in this range [recall that our testing to date was in the range $0 \le r \le 2$, and $r \longrightarrow \infty$].

Now let's *arbitrarily* choose $r = 24$. Here, for our *spacelike* component $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$ in $\mathcal{R}^2$, with $\theta = \pi / 2$; *Res* computes to $exp(-\Delta r) / \sqrt{\Delta^2 + (\delta\varepsilon)^2}$, and since $\Delta \approx \pi / 2$, *Res* is negligible. And so $R = 2 \cdot Res$, associated with $\mathcal{G}$, is *also* negligible [recall too that $\alpha = \cos(\theta)$, and $\varepsilon = \sin(\theta)$, with $\delta = 1$, and singularities at $(\pm\delta, 0)$].

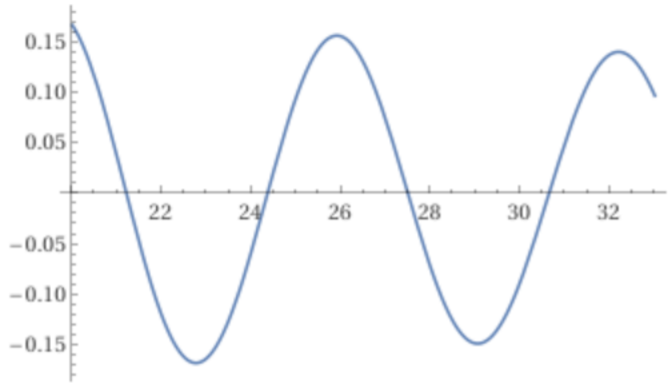
Now $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$, and since $\alpha = 0$ and $\varepsilon = 1$; $\mathcal{E} = (1 / \Delta) \cdot J_0(r\varepsilon) \approx -0.0358$. That only leaves us with the harmonic expression $\mathcal{H}_x$, and this computes to …

$$\int_1^{120} -\frac{i\left(\left(-\frac{1}{\frac{\pi}{2}-iy} + \frac{1}{\frac{\pi}{2}+iy}\right)\cos(24y) + i\left(\frac{1}{\frac{\pi}{2}-iy} + \frac{1}{\frac{\pi}{2}+iy}\right)\sin(24y)\right)}{\pi\sqrt{-1+y^2}}\,dy = -0.0705571 + 0\,i$$

Thus, $\mathcal{H}_x$ and $\mathcal{E}$ are *both* small, and compare favorably; but the *most* important thing to observe here, is that they are *both* small, and will tend to *zero* as *r increases*. But even for values of r that are *not* large, we see that we can get reasonable agreement in $\mathcal{G}$ for this choice of Δ.

In turn, this gives us an estimator for $g_{u,v}$ that just might work for all $r \ge 0$, and similarly for the *timelike* component; namely, $g_{u,v} = 1 / (r \cdot \cos(\theta) + \Delta)$. And, of course, similar remarks apply in $\mathcal{R}^3$ …

Here is a plot of $J_0(r)$ in the range r = 20 to 33, and below that, a table which compares $\mathcal{H}_x$ and $\mathcal{E}$ for various values of r in this range. Note that for these values of r, the *first* term in the expression $\Delta = \{r / (\mathcal{R} - 1)\} + \kappa(r)$ is *very* small, and can be omitted. The *second* term in Δ; namely $\kappa(r) = \arctan(r^3 / \eta)$, where $\eta = 400$, thus implies $\Delta \approx \pi / 2$, for the setup described on the previous page …



$\mathcal{H}_x$	$\mathcal{E}$	r
–0.0846	–0.1034	23
0.0746	0.0993	26
–0.0645	–0.0941	29
0.0544	0.0879	32

Notice in the table above, that the sign of $\mathcal{H}_x$ seems to track the sign of $\mathcal{E}$, which is somewhat amazing to me. As well, if r is a *root* of $J_0(r)$ then $\mathcal{E}$ will be *zero*, so the deviation reduces to $\mathcal{H}_x$ in the range above and beyond. And indeed, $\mathcal{H}_x$ becomes *smaller* as r increases, so that this particular deviation *tends* to *zero*, as it should …

A Simple Case Study, Part XXXVII

The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$ and here, $\delta = 1$ [similar results hold in $\mathcal{R}^3$].

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	12	.079	.078
1	12	.067	.064
2	12	.022	.019
3	12	–.019	–.022
4	12	–.033	–.033
5	12	–.017	–.015
6	12	.011	.012
7	12	.024	.025
8	12	.015	.014
9	12	–.006	–.007
10	12	–.020	–.020
20	12	.014	.014
30	12	–.007	–.007

Now since *Res* computes to $exp(-\Delta r) / \sqrt{\Delta^2 + (\delta\varepsilon)^2}$, and $R = 2 \cdot Res$, we only need to include R in the calculations if r becomes very small. If we do, we get the agreement we expect when comparing $\mathcal{H}_x$ + R with $\mathcal{E}$. But for the table above, and indeed, for all $r \geq ½$, we exclude R, and you can see just how well $\mathcal{H}_x$ and $\mathcal{E}$ track each other, in this case. And similar results hold for *any* θ between 0 and $\pi / 2$, and also hold true for the *time* component $g_{u,v} = 1 / (r \cdot \cos(\theta) + \Delta)$.

Thus, as an alternative to estimators developed in our previous research, I would suggest we use the ones here. And here are a few snapshots from the table above, for $\mathcal{H}_x$ and $\mathcal{E}$, when $r = 4$ …

$$\int_1^{1000} -\frac{i\left(\left(-\frac{1}{12-iy}+\frac{1}{12+iy}\right)\cos(4y)+i\left(\frac{1}{12-iy}+\frac{1}{12+iy}\right)\sin(4y)\right)}{\pi\sqrt{-1+y^2}}\,dy = -0.0333277 + 0\,i$$

$$\frac{J_0(4)}{12}$$

Decimal approximation

-0.03309581748865

A Simple Case Study, Part XXXVIII

The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$ and here, $\delta = 2$ [similar results hold in $\mathcal{R}^3$]. We haven't done any analysis except at $\delta = 1$, and so when $\delta > 0$ is *arbitrary*, we note that the transformation $r' = r / \delta$ implies $\delta' = 1$ if $r = \delta$. Thus, in this *transformed* space, the estimator $g_{u,v}$ becomes $1 / (r' \cdot \sin(\theta) + \Delta)$, where $\Delta \approx 12$; so that in the *original* space, the estimator is now $g_{u,v} = 1 / \{(r / \delta) \cdot \sin(\theta) + \Delta\}$. And this is what we use to calculate the various components of the general equivalency theorem $\mathcal{G}$. Here is the table, where the R value is omitted because *Res* is negligible …

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	12	.067	.064
1	12	.022	.019
2	12	–.033	–.033
3	12	.011	.012
4	12	.016	.014
5	12	–.020	–.020
6	12	.002	.004
7	12	.015	.014
8	12	–.014	–.014
9	12	–.002	–.001
10	12	.014	.014
20	12	.001	.001
30	12	–.007	–.008

And here are a few snapshots from the table above, for $\mathcal{H}_x$ and $\mathcal{E}$, when $r = 2$, $\theta = \pi / 2$, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$; and $\delta = 2$. Note again that $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r \varepsilon)$ here [see also page 511] …

$$\int_2^{300} -\frac{2\, i \left(\left(-\frac{1}{24-i y} + \frac{1}{24+i y}\right) \cos(2 y) + i \left(\frac{1}{24-i y} + \frac{1}{24+i y}\right) \sin(2 y)\right)}{\pi \sqrt{-4+y^2}}\, dy = -0.0333276 + 0\, i$$

$$\frac{J_0(4)}{12}$$

Decimal approximation

-0.03309581748865

A Simple Case Study, Part XXXIX

The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$. The angle is $\theta = \pi / 4$, with singularities at $(\pm\delta, 0)$ and here, $\delta = 1$ [similar results hold in $\mathcal{R}^3$]. A few minor corrections were made to the table below, in a previous release.

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	12	.078	.078
1	12	.071	.070
2	12	.047	.045
3	12	.014	.012
4	12	–.014	–.016
5	12	–.030	–.030
6	12	–.030	–.030
7	12	–.017	–.016
8	12	.002	.003
9	12	.018	.019
10	12	.024	.024
20	12	.013	.012
29	12	.009	.009

Now since *Res* computes to $exp\,(-pr) \;/\; \sqrt{p^2 + (\delta\varepsilon)^2}$; where $-p$ is the pole obtained from the generating function $f(s)$, and R = 2 · *Res*; we only need to include R in the calculations if r becomes very small. If we do, we get the agreement we expect when comparing $\mathcal{H}_x$ + R with $\mathcal{E}$. But for the table above, and indeed, for all $r \geq ½$, we exclude R, and you can see just how well $\mathcal{H}_x$ and $\mathcal{E}$ track each other, in this case. And similar results hold for *any* θ between 0 and $\pi / 2$, and also hold true for the *time* component $g_{u,v} = 1 / (r \cdot \cos(\theta) + \Delta)$.

Thus, we seem to have a *pair* of estimators here that are valid for *any* choice of $r, \delta > 0$; and for *any* choice of θ between 0 and $\pi / 2$ in $\mathcal{R}^2$ [similar results hold in $\mathcal{R}^3$]. As such, we'll call these *ideal* estimators, since it is unlikely we'll find another pair that has all of these properties.

———

If $\delta \cdot r = k$, for any $k > 0$, so that $\delta \cdot r$ is an *invariant*, then $\mathcal{H}_x$, R and $\mathcal{E}$ are *also* invariants; so that we only need to test at $\delta = 1$. For example, if $k = 4$, then testing at $r, \delta = 2$ is *no* different than testing at $r = 4$ and $\delta = 1$. The *potential* hypothesis [$\mathcal{P}$], mentioned on pages 592-3, still applies for our *ideal* spacelike and timelike estimators; but now for all $r, \delta > 0$ and $0 \leq \theta \leq \pi / 2$. Thus, if $\mathcal{P}$ holds, the *ideal* estimators approximate solutions to the *coupled* equations in $\mathcal{R}^2$; since these spacelike and timelike solutions will always satisfy $\mathcal{H}_x$ + R = $\mathcal{E}$ *exactly*. Similar remarks also hold in $\mathcal{R}^3$.

To show that $\mathcal{H}_x$, R and $\mathcal{E}$ are *also* invariants if $\delta \cdot r = k$ is invariant, let's start with the generating function below, where $g_{u,v} = \delta / (r \cdot \sin(\theta) + \delta\Delta)$ and the singularities are at $(\pm\delta, 0)$ in $\mathcal{R}^2$; and r maps to $s + \delta\alpha$ …

$$f(s) = g_{u,v} \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2} = \delta / ((s + \delta\alpha) \cdot \sin(\theta) + \delta\Delta) \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2}$$

The *simple* pole associated with the *first* term in $f(s)$ is $-p = -\delta\Delta / \sin(\theta) - \delta\alpha$, where $\alpha = \cos(\theta)$, and $\varepsilon = \sin(\theta)$; and $0 < \theta \leq \pi / 2$. If we define $q = \Delta / \sin(\theta) + \alpha$, then it is not hard to show that *Res* computes to $exp(-kq) / \sqrt{q^2 + \varepsilon^2}$. Thus, $R = 2 \cdot Res$ is invariant, in this case, if $\delta \cdot r = k$.

Since $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r\varepsilon)$, this computes to $\{1 / (\alpha \cdot \sin(\theta) + \Delta)\} \cdot J_0(k\varepsilon)$, which again, is invariant. And finally, if we define $\gamma = (\alpha \cdot \sin(\theta) + \Delta)$, then $\mathcal{H}_x$ computes to $[\kappa = 1/2\pi \boldsymbol{i}]$ …

$$2\kappa\delta \int_{\delta\varepsilon}^{\infty} \{\cos(yr)[1/(\delta\gamma + iy\varepsilon) - 1/(\delta\gamma - iy\varepsilon)] + \boldsymbol{i}\sin(yr)[1/(\delta\gamma + iy\varepsilon) + 1/(\delta\gamma - iy\varepsilon)]\}\, dy \Big/ \sqrt{y^2 - (\delta\varepsilon)^2}$$

If we now let $y = \delta u$, then the expression above becomes the following, which is *also* invariant …

$$2\kappa \int_{\varepsilon}^{\infty} \{\cos(ku)[1/(\gamma + iu\varepsilon) - 1/(\gamma - iu\varepsilon)] + \boldsymbol{i}\sin(ku)[1/(\gamma + iu\varepsilon) + 1/(\gamma - iu\varepsilon)]\}\, du \Big/ \sqrt{u^2 - \varepsilon^2}$$

If $\theta = 0$, there is *no* pole associated with the *first* term in $f(s)$. Thus $R = 0$, and $\mathcal{E} = 1 / \Delta$; while the harmonic expression, just above, with $\gamma = \Delta$ and $\varepsilon = 0$, computes to

$$4\kappa\boldsymbol{i} / \Delta \cdot \int_{\varepsilon}^{\infty} \sin(ku)\, du \,/\, u = 1 / \Delta$$

The match is, therefore, *exact* in this case …

———

We'll now test $\mathcal{H}_x$ + R versus $\mathcal{E}$, using the calculations above for the *spatial* component, when the radius $r = 1/16$ and $\delta = 1$; and the angle $\theta = \pi / 2$. Since r is *small*, we'll need to include R in the computations. Here, $q = \Delta$ (≈ 12) and $k = 1/16$, and the snapshots from Wolfram follow on the next page … ..

.
.
.

$$\int_1^{300} -\frac{i\left(\left(-\frac{1}{12-iy}+\frac{1}{12+iy}\right)\cos\left(\frac{y}{16}\right)+i\left(\frac{1}{12-iy}+\frac{1}{12+iy}\right)\sin\left(\frac{y}{16}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy = 0.00474287$$

value of $\mathcal{H}_x$

$$2\times\frac{\exp(-0.75)}{\sqrt{145}}$$

Result

0.0784558…

value of R

$$\frac{J_0\left(\frac{1}{16}\right)}{12}$$

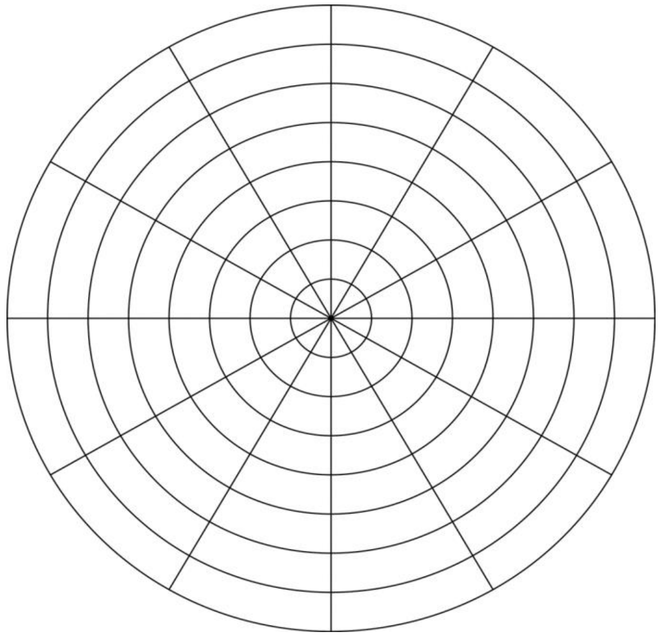
Decimal approximation

0.083251972991059

value of $\mathcal{E}$

Here $\mathcal{H}_x$ + R ≈ 0.08320, which compares very favorably with $\mathcal{E}$, so that our *ideal* estimators *also* apply for very small r. And here is an *extended* table from page 640, where δ = 1 and θ = π / 2 … ..

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	12	.079	.078
1	12	.067	.064
2	12	.022	.019
3	12	–.019	–.022
4	12	–.033	–.033
5	12	–.017	–.015
6	12	.011	.012
7	12	.024	.025
8	12	.015	.014
9	12	–.006	–.007
10	12	–.020	–.020
20	12	.014	.014
30	12	–.007	–.007
40	12	.001	.001
50	12	.004	.005
60	12	–.007	–.008
70	12	.008	.008
80	12	–.006	–.006
90	12	.003	.002



The general equivalency theorem [$\mathcal{G}$] in the plane $\mathcal{R}^2$, with singularities at (±δ, 0), where α = cos(θ) and ε = sin(θ), states that [p 575]

$$\mathcal{H}_x + \mathrm{R} = \mathcal{E}^{\,(1)} \quad \textit{if and only if} \quad \mathrm{C}_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha)\mathrm{J}_0(\delta r\varepsilon)\mathrm{g}_{u,v}(\delta\alpha)^{\,(2)}$$

Since we now know that our *ideal* spacelike and timelike estimators satisfy (1) approximately; yet *everywhere* in $\mathcal{R}^2$, and that spacelike and timelike *solutions* to (2) satisfy (1) *exactly* in $\mathcal{R}^2$; one can only conclude that the *ideal* estimators are very likely *approximators* for these solutions.

For if we go to any point in the picture above, where a *radial* line intersects a circle, and calculate each of $\mathcal{H}_x$, R and $\mathcal{E}$; we will find that (1) is approximately true for the *ideal* estimators, and *exactly* true for the solutions. What are the odds that this could be so, and yet *no* connection existed between the *ideal* estimators and the solutions ? In my view, the odds are *zero*.

Similar remarks apply for the corresponding *ideal* estimators in $\mathcal{R}^3$, which have been discussed in several places throughout this *Simple Case Study* series, for different choices of Δ that were *not* ideal. As such, we should feel comfortable with using these *ideal* estimators in our calculations, for the time being, until something better comes along ... ..

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As to the *potential* hypothesis [pp 592-3], associated with our *ideal* estimators, it now reads as follows, for the *spacelike* component …

> For some smooth function $h(\theta)$, which can be approximated by $\sin(\theta)$, where $0 \le \theta \le \pi / 2$, it is the case that $1 / (r \cdot h(\theta) + \Delta)$ is a suitable candidate for the *quantumlike* component in (2) above, for all r, $\delta > 0$, and Δ some constant

And for the *timelike* component, it reads thusly …

> For some smooth function $\boldsymbol{h}(\theta)$, which can be approximated by $cos(\theta)$, where $0 \le \theta \le \pi / 2$, it is the case that $1 / (r \cdot \boldsymbol{h}(\theta) + \Delta)$ is a suitable candidate for the *quantumlike* component in (2) above, for all r, $\delta > 0$, and Δ some constant

Thus, we should interpret Δ as the *same* constant for both the *spacelike* and *timelike* elements here, which [hopefully] can be found by solving (2) for $h(\theta)$ and $\boldsymbol{h}(\theta)$, according to the following template in $\mathscr{R}^2$ (σ, $\delta = 1$) …

$$g_{u,v} = [1 / (r \cdot h(\theta) + \Delta) , r^2 , 1 / (r \cdot \boldsymbol{h}(\theta) + \Delta)]$$

Our expectation is that a solution, *if* it exists, will show that $\Delta \approx 12$. Yet again, I would say that finding such a solution on paper is highly unlikely, but do believe the *potential* hypotheses above are actually valid, because they are the *natural* extensions of the *ideal* estimators, themselves.

But *if* a solution can be found, one of the conditions placed on $h(\theta)$ might be $h(\theta) = 0$ if $\theta = 0$, and this is because when $\theta = 0$, (1) is true *exactly* for the *ideal* spacelike estimator. Similarly, we might require for the *timelike* element that $\boldsymbol{h}(\theta) = 0$ if $\theta = \pi / 2$, for the same reason.

Finally, it should be said that solutions to (2), using $g_{u,v}$, must be *fully* covariant, so that $\nabla_u g^{u,v} = 0$ holds, in particular, for the spacelike and timelike elements. This could help in determining the true value of Δ in $g_{u,v}$.

Note as well, that these are *normalized* solutions ($\sigma = 1$), which we call $\mathcal{N}$, so that the *general* solutions are $\sigma \cdot \mathcal{N}$ [pp 522-3]. If the singularities are at ($\pm\delta$, 0), then the *normalized* spacelike and timelike solutions would be $\delta / (r \cdot h(\theta) + \delta\Delta)$ and $\delta / (r \cdot \boldsymbol{h}(\theta) + \delta\Delta)$, respectively [p 641].

A Simple Case Study, Part XL

The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$. The angle is $\theta = \pi / 6$, with singularities at $(\pm\delta, 0)$ and here, $\delta = 1$ [similar results hold in $\mathcal{R}^3$].

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	12	.079	.079
1	12	.076	.075
2	12	.062	.062
3	12	.042	.041
4	12	.019	.018
5	12	–.003	–.004
6	12	–.020	–.021
7	12	–.030	–.030
8	12	–.032	–.032
9	12	–.026	–.026
10	12	–.015	–.014
20	12	–.020	–.020
30	12	–.001	–.001
40	12	.013	.013
50	12	.008	.008
60	12	–.007	–.007
70	12	–.010	–.010
80	12	.001	.001
90	12	.009	.009

Now since *Res* computes to $exp(-pr) / \sqrt{p^2 + (\delta\varepsilon)^2}$, and R = 2 · *Res*, we only need to include R in the calculations if r becomes very small. If we do, we get the agreement we expect when comparing $\mathcal{H}_x$ + R with $\mathcal{E}$. But for the table above, and indeed, for all $r \geq$ ½, we exclude R, and you can see just how well $\mathcal{H}_x$ and $\mathcal{E}$ track each other, in this case. And similar results hold for *any* θ between 0 and $\pi / 2$, and also hold true for the *time* component $g_{u,v} = 1 / (r \cdot \cos(\theta) + \Delta)$.

And here are a few snapshots from the table above, for $\mathcal{H}_x$ and $\mathcal{E}$, when $r = 90$ …

$$\int_{0.5}^{300} \frac{\cos(90\,y)\left(\frac{1}{12.433+0.5\,i\,y} - \frac{1}{12.433-0.5\,i\,y}\right) + i\sin(90\,y)\left(\frac{1}{12.433+0.5\,i\,y} + \frac{1}{12.433-0.5\,i\,y}\right)}{(\pi\,i)\sqrt{y^2 - 0.25}}\,dy =$$

0.00935736

$$\frac{1}{12.433} J_0\left(\frac{90}{2}\right)$$

Result

0.00931542…

A Simple Case Study, Part XLI

The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is $g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta)$. The angle is $\theta = \pi / 3$, with singularities at $(\pm\delta, 0)$ and here, $\delta = 1$ [similar results hold in $\mathcal{R}^3$]. In some cases, such as r = 10, 50, 90; Wolfram would not report a value for $\mathcal{H}_x$, so here we used the *next* number instead.

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	12	.077	.077
1	12	.068	.066
2	12	.033	.031
3	12	–.005	–.007
4	12	–.029	–.030
5	12	–.029	–.029
6	12	–.010	–.009
7	12	.012	.013
8	12	.024	.024
9	12	.018	.017
11	12	–.015	–.016
20	12	–.011	–.011
30	12	.013	.013
40	12	–.008	–.008
51	12	.008	.008
60	12	.006	.006
70	12	–.008	–.008
80	12	.006	.006
91	12	–.006	–.006

Now since *Res* computes to $exp(-pr) / \sqrt{p^2 + (\delta\varepsilon)^2}$, and R = 2 · *Res*, we only need to include R in the calculations if r becomes very small. If we do, we get the agreement we expect when comparing $\mathcal{H}_x$ + R with $\mathcal{E}$. But for the table above, and indeed, for all $r \geq ½$, we exclude R, and you can see just how well $\mathcal{H}_x$ and $\mathcal{E}$ track each other, in this case. And similar results hold for *any* θ between 0 and $\pi / 2$, and also hold true for the *time* component $g_{u,v} = 1 / (r \cdot \cos(\theta) + \Delta)$.

And here are a few snapshots from the table above, for $\mathcal{H}_x$ and $\mathcal{E}$, when r = 9 …

$$\int_{\frac{\sqrt{3}}{2}}^{300} \frac{\cos(9y)\left(\frac{1}{12.433+0.866\,i\,y} - \frac{1}{12.433-0.866\,i\,y}\right) + i\sin(9y)\left(\frac{1}{12.433+0.866\,i\,y} + \frac{1}{12.433-0.866\,i\,y}\right)}{(\pi\, i)\sqrt{y^2 - 0.75}}\, dy = 0.0183316$$

$$\frac{1}{12.433}\, J_0\left(9 \times \frac{\sqrt{3}}{2}\right)$$

Result

0.0174187…

A Simple Case Study, Part XLII

In this note, we'll demonstrate briefly that the elements of the following *diagonal* matrix are always fully covariant.

$$g_{u,v} = [1 / (r \cdot \sin(\theta) + \Delta) \ , \ r^2 \ , \ 1 / (r \cdot \cos(\theta) + \Delta)]$$

If we define T to be $g^{u,v}$, then from Waner's book *An Introduction To Differential Geometry and General Relativity* [6th printing], the following hold true, where (1) is the first expression and (2) is the second expression, as shown below …

$$T^{ab}{}_{|b} = \frac{\partial T^{ab}}{\partial x^b} + \Gamma_k{}^a{}_b T^{kb} + \Gamma_b{}^b{}_k T^{ak}$$

$$\Gamma_h{}^p{}_k = \frac{1}{2} g^{lp} \left(\frac{\partial g_{kl}}{\partial x^h} + \frac{\partial g_{lh}}{\partial x^k} - \frac{\partial g_{hk}}{\partial x^l} \right)$$

Setting *a*, *b* = 1, we calculate the *first* covariant partial derivative $T^{ab}{}_{|b}$ associated with the *first* row, and this computes to

$$\sin(\theta) - 2 \cdot \{½ \cdot \sin(\theta) / (r \cdot \sin(\theta) + \Delta)\} \cdot (r \cdot \sin(\theta) + \Delta) = 0 \ .$$

For *a* = 1, *b* = 2 we obtain

$$-(r \cdot \sin(\theta) + \Delta) / r \ + \ (r \cdot \sin(\theta) + \Delta) / r = 0 \ ,$$

and for *a* = 1, *b* = 3 the result is

$$-½ \cdot (r \cdot \sin(\theta) + \Delta) \cdot \cos(\theta) / (r \cdot \cos(\theta) + \Delta) \ + \ ½ \cdot (r \cdot \sin(\theta) + \Delta) \cdot \cos(\theta) / (r \cdot \cos(\theta) + \Delta) = 0$$

Thus, *all three* covariant partial derivatives are *zero*, so that $\nabla_v g^{u,v} = 0$ holds in this case, for the *first* row, where *u* = 1, and *v* = 1, 2, 3. In a similar way, we expect this to be true for the *second* and *third* rows, so that $\nabla_v g^{u,v} = 0$ holds in general.

And similarly, from our *potential* hypothesis [pp 645-6], we expect $\nabla_v g^{u,v} = 0$ holds true as well, for the elements of $g_{u,v}$ below …

$$g_{u,v} = [1 / (r \cdot h(\theta) + \Delta) \ , \ r^2 \ , \ 1 / (r \cdot \boldsymbol{h}(\theta) + \Delta)]$$

A Simple Case Study, Part XLIII

In $\mathcal{R}^3$, with singularities at S = (±δ, 0, 0), for δ > 0, and cos(γ) = sin(θ)cos(ϕ) in *physical* coordinates, where 0 ≤ θ, ϕ ≤ π / 2; the *normalized* (σ = 1) *ideal spacelike* and *timelike* estimators are now δ / (*r* · sin(γ) + δΔ) and δ / (*r* · cos(γ) + δΔ), respectively. Thus the normalized *diagonal* matrix becomes

$$\mathcal{N} = [\delta / (r \cdot \sin(\gamma) + \delta\Delta) \text{ , } r^2 \text{ , } r^2 sin^2(\theta) \text{ , } \delta / (r \cdot \cos(\gamma) + \delta\Delta)] \text{ ,}$$

so that the *general* matrix is σ · $\mathcal{N}$ [pp 522-3]. Note that the *default* inner block $\mathcal{B} = [r^2 \text{ , } r^2 sin^2(\theta)]$ has *no* dependency on δ, and that Δ ≈ 12, since these are *ideal* estimators.

If we are dealing with an *ideal non-standard* inner block, such as the one mentioned on pages 607-8, and reproduced below (but now recalculated to factor in δ) …

$$\mathcal{B} \;=\; [\delta / (r \cdot \cos(\gamma) + \delta\Delta) \text{ , } \delta \cdot \sin^2(\theta) / (r \cdot \cos(\gamma) + \delta\Delta)] \text{ ;}$$

then the normalized *diagonal* matrix becomes

$$\mathcal{N} = [\delta / (r \cdot \sin(\gamma) + \delta\Delta) \text{ , } \mathcal{B} \text{ , } \delta / (r \cdot \cos(\gamma) + \delta\Delta)] \text{ ,}$$

and the *general* matrix is again, σ · $\mathcal{N}$. Note that these inner blocks, be it the standard [default] or non-standard, both preserve volume *invariance* [Δ_ϵ], induced by dark energy, itself. Thus, if the singularities [S] were rotated in some fashion about the origin O, Δ_ϵ would not change [and similar remarks apply in $\mathcal{R}^2$] …

———

To show that the *default* inner block $\mathcal{B}$ has no dependency on δ, let's start by examining things in *contravariant* mode, for $\mathcal{R}^2$. Here $\mathcal{B} = 1 / r^2$, so that if we look at the *square root* of $\mathcal{B}$, this is 1 / *r*. If there was a dependency on δ, then 1 / *r* would transform as δ / *r* [p 641], so that in *covariant* mode, $\mathcal{B}$ would now become $(r / \delta)^2$.

From pages 586-7, we now calculate the incremental *area* [Δ_ϵ] of the [distorted] elliptical annulus $\mathcal{E}$ that our *thin* annulus $\mathcal{A}$ [containing the *unit* circle] has *morphed* into, because of the dark energy *radial* waves generated from the singularities S at (±δ, 0); and emanating in *all* directions, from O. If we assume that $\mathcal{B} = r^2$; this is, in the *first* quadrant, where $g_{u,v} \;=\; \delta / (r \cdot \sin(\theta) + \delta\Delta)$ [σ = 1] …

$$\int_0^{\pi/2} \int_{1-\epsilon}^{1+\epsilon} \sqrt{g_{u,v}}\; r\,dr\,d\theta$$

and from page 609; becomes, approximately …

$$\int_0^{\pi/2} 2\epsilon f(1)\, d\theta$$

where $f(r) = r \cdot \sqrt{g_{u,v}}$. Now if we were to let $\mathcal{B} = (r / \delta)^2$, so that $\mathcal{B}$ *has* a *dependency* on δ; then Δ_ϵ would compute to, in the *first* quadrant …

$$\int_0^{\pi/2} (2\epsilon / \delta) f(1)\, d\theta$$

and this integral *diverges* to ∞ as δ ---› 0, because of the term $1 / \sqrt{\delta}$ that now emerges in the calculation. Thus, *no* such dependency on δ exists if $\mathcal{B}$ is the *default* inner block [and similarly for $\mathcal{R}^3$].

For the record, when testing $\mathcal{H}_x$ + R against $\mathcal{E}$, using the setup in this note, with δ = 1; the pole is at $-p = -\Delta / \sin(\gamma) - \alpha$, so that $Res = exp(-pr) / \sqrt{p^2 + \varepsilon^2}$, and thus R is $2 \cdot Res$. Here, $\alpha = \cos(\gamma)$, and $\varepsilon = \sin(\gamma)$. The harmonic expression $\mathcal{H}_x$, where $\eta = \alpha \cdot \sin(\gamma) + \Delta$, computes to …

$$2\kappa \int_\varepsilon^\infty \{\cos(yr)[1/(\eta + iy\varepsilon) - 1/(\eta - iy\varepsilon)] + \boldsymbol{i}\sin(yr)[1/(\eta + iy\varepsilon) + 1/(\eta - iy\varepsilon)]\}\, dy / \sqrt{y^2 - \varepsilon^2}$$

And finally, $\mathcal{E} = (1 / \eta) \cdot J_0(r\varepsilon)$. If sin(γ) = 0, then there is *no* pole associated with the generating function $f(s)$, so that R = 0, and the match is *exact*. That is to say, $\mathcal{H}_x = \mathcal{E}$, and again, these remarks pertain to the *spacelike* component $1 / (r \cdot \sin(\gamma) + \Delta)$ in $\mathcal{R}^3$. But they *also* apply to the *timelike* component $1/ (r \cdot \cos(\gamma) + \Delta)$ in $\mathcal{R}^3$, after making the obvious changes [see pages 635-7 for examples of how this is done in $\mathcal{R}^2$].

Note that testing in $\mathcal{R}^3$ is really *no* different than testing in $\mathcal{R}^2$, so that it is always *sufficient* to test in $\mathcal{R}^2$ only [pp 605-6, 610]. And finally, from our *invariance* principle [pp 642-6], we only need to test at δ = 1 …

A Simple Case Study, Part XLIV

In this note, we wish to revisit the *associativity* principle under addition [APA], relative to σ; which was discussed on pages 480-91, among other places. But here, we will *avoid* some of the arguments made previously, which relied on a lim δ ---› 0 argument, when studying the equation below in $\mathscr{R}^2$; where there are singularities at (±δ, 0), and α = cos(θ), ε = sin(θ).

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)g_{u,v}(\delta\alpha) \quad (2)$$

Let us begin by stating that *n* singularities at O, *each* with strength σ, relative to the dark energy density function λ(s) ≈ σ / s, is *no* different than *one* singularity at O with strength *n*σ. And furthermore, we shall note the following *equivalency*, which we'll label (†): S is a singularity at O of strength σ *if and only if* it is the case that $C_{u,v} \approx \sigma \cdot g_{u,v}(0)$; with a solution $g_{u,v}(r, \sigma)$, in the *radial* and *time* components, in particular [the *inner* block $\mathscr{B}$ can be the default, if we like].

Now let's consider the expression

$$g_{u,v}(r, 2\sigma) + g_{u,v}(r, -\sigma)$$

From (†), $g_{u,v}(r, 2\sigma)$ represents a singularity of strength 2σ at O [which is equivalent to *two* singularities of strength σ at O], and $g_{u,v}(r, -\sigma)$ represents a singularity of strength –σ at O. Thus the *sum* above must represent a singularity of strength σ at O, and so we may write, using (†) again …

$$g_{u,v}(r, 2\sigma) + g_{u,v}(r, -\sigma) = g_{u,v}(r, \sigma) \qquad (*)$$

But from page 491, we also know that

$$g_{u,v}(r, \sigma) + g_{u,v}(r, -\sigma) = g_{u,v}(r, 0) = 0 ,$$

so that (*) becomes $g_{u,v}(r, 2\sigma) = 2g_{u,v}(r, \sigma)$. And this result can now be used on pages 522-3 to form the more *general* principle, concerning APA.

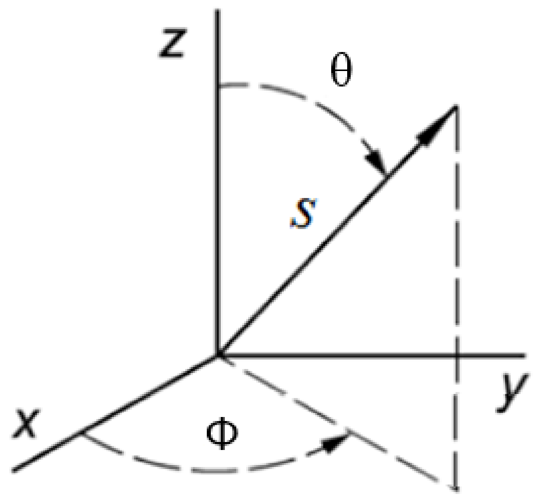
What is important to see here, is that APA does *not* depend on a lim δ ---› 0 argument in (2) above, so that when considering possible *normalized* (σ = 1) solutions to (2) and shown below, from our *potential* hypotheses [pp 645-6]; that is to say,

$$\mathcal{N} = [\delta / (r \cdot h(\theta) + \delta\Delta) , \mathscr{B} , \delta / (r \cdot \boldsymbol{h}(\theta) + \delta\Delta)]$$

in the *radial* and *time* elements; we needn't worry about what happens as δ ---› 0 in $\mathcal{N}$. The *general* solution will always be σ · $\mathcal{N}$, in accordance with our earlier findings …

A Simple Case Study, Part XLV

In $\mathcal{R}^3$, with singularities at S = (±δ, 0, 0), for δ > 0, and cos(γ) = sin(θ)cos(ϕ) in *physical* coordinates, where 0 ≤ θ, ϕ ≤ π / 2; the *potential* hypotheses on pages 645-6 [$\mathcal{P}$] still hold, but now the *argument* to the functions *h* and ***h*** becomes *h*(*η*) and ***h***(*η*), respectively. And in this case, we have *η* = π / 2 – *arccos*(sin(γ)). Thus, when scaling up from $\mathcal{R}^2$ to $\mathcal{R}^3$, all we need to do is *remap* θ in $\mathcal{P}$, accordingly [recall that the ideal *spacelike* and *timelike* estimators in $\mathcal{R}^3$ are 1 / (*r* · sin(γ) + Δ) and 1/ (*r* · cos(γ) + Δ), respectively, if δ = 1 and Δ ≈ 12 (p 650)].



Let's do a few examples using our *physical* coordinates diagram, as shown above. When ϕ = 0 and θ = 0, cos(γ) = 0 and thus sin(γ) = 1. Hence, *η* = π / 2 so that *h*(*η*) agrees with what we see in $\mathcal{R}^2$, at 90 degrees *from* the *x*-axis.

When ϕ = 0 and θ = π / 2, cos(γ) = 1 and thus sin(γ) = 0. Hence, *η* = 0 so that *h*(*η*) agrees with what we see in $\mathcal{R}^2$, *along* the *x*-axis, where our singularities are.

Now if ϕ = π / 2, then cos(γ) = 0, *no* matter our choice of θ, so that again sin(γ) = 1. And hence, we have *η* = π / 2 so that again, *h*(*η*) agrees with what we see in $\mathcal{R}^2$, at 90 degrees *from* the *x*-axis. And similar remarks apply to ***h***(*η*).

———

When testing *spacelike* and *timelike* estimators in $\mathcal{R}^2$, where Δ was a function of *r* [p 611 *ff*.], we chose *not* to map *r* in Δ(*r*) to its *complex* equivalent in the generating function *f* (*s*), Nor did we map *r* in Δ(*r*) to its *complex* equivalent in the harmonic expression $\mathcal{H}_x$. For to do so would have made the calculations impossible to carry out. Indeed, Δ(*r*) should be seen as a *real*-*valued* function *only*; that is now *exempt* from any mapping associated with $\mathcal{H}_x$, R and $\mathcal{E}$ in the general equivalency theorem $\mathcal{G}$; in so far as any testing is concerned, in these research notes. For the case where Δ is a constant, such as our *ideal* estimators, we needn't worry about this distinction …

———

The *potential* hypotheses [$\mathscr{P}$] in $\mathscr{R}^2$, on pages 645-6, read as follows, for the *spacelike* and *timelike* elements, respectively; where $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, with singularities at $(\pm\delta, 0)$ …

$$C_{u,v} \approx 2\sigma \cdot \cosh(\delta r\alpha) J_0(\delta r\varepsilon) g_{u,v}(\delta\alpha) \ ^{(2)}$$

> For some smooth function $h(\theta)$, which can be approximated by $\sin(\theta)$, where $0 \leq \theta \leq \pi / 2$, it is the case that $1 / (r \cdot h(\theta) + \Delta)$ is a suitable candidate for the *quantumlike* component in (2) above, for all r, $\delta > 0$, and Δ some constant
>
> For some smooth function $\boldsymbol{h}(\theta)$, which can be approximated by $\cos(\theta)$, where $0 \leq \theta \leq \pi / 2$, it is the case that $1 / (r \cdot \boldsymbol{h}(\theta) + \Delta)$ is a suitable candidate for the *quantumlike* component in (2) above, for all r, $\delta > 0$, and Δ some constant

Initially, σ and δ are set to 1, and we see if we can find solutions to (2), subject to certain *initial* conditions [pp 645-6], using the template below; where $\mathscr{B} = r^2$ is the *default* inner block in $\mathscr{R}^2$ …

$$g_{u,v} = [1 / (r \cdot h(\theta) + \Delta) , \mathscr{B} , 1 / (r \cdot \boldsymbol{h}(\theta) + \Delta)] \quad (\dagger)$$

By mapping θ to γ, where $\cos(\gamma) = \sin(\theta)\cos(\phi)$ in *physical* coordinates, we arrive at the *equivalent* hypotheses in $\mathscr{R}^3$, where $0 \leq \theta, \phi \leq \pi / 2$; so that the *argument* to the functions h and $\boldsymbol{h}$ becomes $h(\gamma)$ and $\boldsymbol{h}(\gamma)$, respectively. Note that we map θ to γ in (2) as well.

Now we said on the last page that $\eta = \pi / 2 - \arccos(\sin(\gamma))$ was, in fact, this argument; and indeed, since $\arccos(\sin(\gamma)) = \pi / 2 - \gamma$, it must be the case that $\eta = \gamma$. And if we can find solutions to (2) for the *radial* and *time* elements in (†), then it is likely that they are *also* solutions to (2) in $\mathscr{R}^3$, where $\mathscr{B} = [r^2, r^2 \sin^2(\theta)]$ is the *default* inner block, and the matrix is now $[\sigma, \delta = 1]$

$$g_{u,v} = [1 / (r \cdot h(\gamma) + \Delta) , \mathscr{B} , 1 / (r \cdot \boldsymbol{h}(\gamma) + \Delta)]$$

For in this case, just above, the *radial* and *time* components in $g_{u,v}$ will satisfy $\mathscr{H}_x + R = \mathcal{E}\ ^{(1)}$ *exactly*. For more on *ideal* estimators in $\mathscr{R}^3$, see pages 650-1.

A Simple Case Study, Part XLVI

In this short note, we outline a *heuristical* proof for the *existence* of $h(\theta)$ and $\boldsymbol{h}(\theta)$, in the *potential* hypotheses [$\mathscr{P}$] for $\mathscr{R}^2$ on pages 645-6, and mentioned again on page 654. We'll also see how Δ surfaces in the *spacelike* component in $\mathscr{P}$; namely, $1 / (r \cdot h(\theta) + \Delta)$, and note that similar remarks will apply for the *timelike* element; namely $1 / (r \cdot \boldsymbol{h}(\theta) + \Delta)$, as well.

To start, the *ideal* spacelike estimator [σ, $\delta = 1$] is $\boldsymbol{S}_0 = 1 / (r \cdot \sin(\theta) + \Delta)$, where $r > 0$, $0 \leq \theta \leq \pi / 2$, and $\Delta \approx 12$. We'll refer to this region as Q1; and we see using $\boldsymbol{S}_0$ in Q1, how well $\boldsymbol{\mathscr{H}_x}$ + R tracks $\boldsymbol{\mathcal{E}}$ in the tables; starting on page 640, and going forward from there. Let us define $f_0(\theta) = sin(\theta)$, and let us define $\Delta_0 = 12$.

Now this is not the 'end of the road' for our *ideal* estimator $\boldsymbol{S}_0$; for even though the tracking is quite good, we can always (in the *existence* sense) find *another ideal* estimator $\boldsymbol{S}_1 = 1 / (r \cdot f_1(\theta) + \Delta_1)$, where the tracking is even *better* in Q1 than it was for $\boldsymbol{S}_0$. We expect $f_1(\theta)$ to approximate $f_0(\theta)$ and Δ_1 to approximate Δ_0, as they should.

Now we continue the process out to ∞, building (in the *existence* sense) our *infinite* set of *ideal* estimators $\{\boldsymbol{S}_n \mid n = 0, 1, 2, 3, \ldots\}$, where at each stage … the tracking becomes more and more precise. In the limit, we *define* $1 / (r \cdot h(\theta) + \Delta) = \lim n \dashrightarrow \infty \{\boldsymbol{S}_n\}$, and I would submit to the reader that this *is* the function referred to in $\mathscr{P}$. A similar construction can be done for *timelike* estimators, leading to a definition for $1 / (r \cdot \boldsymbol{h}(\theta) + \Delta)$. Finally, we scale up to $\mathscr{R}^3$, using the last research note on pages 653-4 … ..

A Simple Case Study, Part XLVII

The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here $\eta = 1/10$, $\Delta = 11$ and $\gamma = 1 / \eta$. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$].

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	11	.0853	.0855
1	11	.0699	.0695
2	11	.0208	.0204
3	11	–.0234	–.0236
4	11	–.0362	–.0361
5	11	–.0164	–.0161
6	11	.0135	.0137
7	11	.0273	.0273
8	11	.0158	.0156
9	11	–.0080	–.0082
10	11	–.0223	–.0223
11	11	–.0157	–.0156
12	11	.0041	.0043
13	11	.0188	.0188
20	11	.0152	.0152
30	11	–.0079	–.0079

Now since *Res* computes to $exp(-\gamma\Delta r) / \sqrt{(\gamma\Delta)^2 + (\delta\varepsilon)^2}$, and $R = 2 \cdot Res$, we only need to include R in the calculations if r becomes *very* small. If we do, we should get the agreement we expect when comparing $\mathcal{H}_x$ + R with $\mathcal{E}$. But for the table above, and indeed, for all $r \geq ½$, we exclude R, and you can see just how well $\mathcal{H}_x$ and $\mathcal{E}$ track each other, in this case. And similar results hold for *any* θ between 0 and $\pi / 2$, and also hold true for the *time* component $g_{u,v} = 1 / (r \cdot \eta\cos(\theta) + \Delta)$.

And here are a few snapshots from the table above, for $\mathcal{H}_x$ and $\mathcal{E}$, when $r = 3$ …

$$\int_1^{300} -\frac{i\left(\left(-\frac{1}{11-\frac{iy}{10}}+\frac{1}{11+\frac{iy}{10}}\right)\cos(3y)+i\left(\frac{1}{11-\frac{iy}{10}}+\frac{1}{11+\frac{iy}{10}}\right)\sin(3y)\right)}{\pi\sqrt{-1+y^2}}\,dy =$$

$$-0.0233793 + 0\,i$$

$$\frac{J_0(3)}{11}$$

Decimal approximation

$$-0.02364108680926$$

The *potential* hypotheses [$\mathscr{P}$] still apply, but now our starting point is the *ideal* estimators mentioned above. As well, the *heuristical* argument put forth on page 655 still holds; and if anything, is made *stronger* by virtue of the table above ! A few minor corrections were made to the table below, in a previous release …

The following table compares the harmonic expression $\mathscr{H}_x$ with $\mathscr{E}$, for the *spacelike* component in $\mathscr{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here $\eta = 1/10$ and $\Delta = 11$. The angle is $\theta = \pi/4$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathscr{R}^3$]. Wolfram would not report a number for $\mathscr{H}_x$, when $r = 12$ or 30. In this case, that number was skipped …

r	Δ	$\mathscr{H}_x$	$\mathscr{E}$
1/2	11	.0878	.0877
1	11	.0797	.0795
2	11	.0508	.0506
3	11	.0142	.0140
4	11	–.0176	–.0178
5	11	–.0347	–.0348
6	11	–.0336	–.0335
7	11	–.0177	–.0175
8	11	.0040	.0041
9	11	.0213	.0214
10	11	.0271	.0271
11	11	.0199	.0199
13	11	–.0122	–.0122
14	11	–.0217	–.0217
20	11	.0137	.0136
31	11	–.0100	–.0100

You can see again, just how *accurate* the tracking actually is, and so I would say with a *high* degree of confidence, that the *ideal* spacelike and timelike estimators mentioned in this research note, can be used in our calculations to measure *dark energy* effects in our *space-time* fabric; such as *incremental* length, area and volume. And here are a few snapshots from the table above, for $\mathscr{H}_x$ and $\mathscr{E}$, when $r = 14$ …

$$\int_{\frac{1}{\sqrt{2}}}^{300} \frac{\cos(14\,y)\left(\frac{1}{11.05+0.0707\,i\,y} - \frac{1}{11.05-0.0707\,i\,y}\right) + i\sin(14\,y)\left(\frac{1}{11.05+0.0707\,i\,y} + \frac{1}{11.05-0.0707\,i\,y}\right)}{(\pi\, i)\sqrt{y^2 - 0.5}}\,dy = -0.0217176$$

$$\frac{J_0\left(14 \times \frac{\sqrt{2}}{2}\right)}{11.05}$$

Result

$-0.0217472\ldots$

The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is *now* $g_{u,v}$ = 1 / ($r \cdot \eta\sin(\theta) + \Delta$). Here $\eta = 1/20$, $\Delta = 10$ and $\gamma = 1 / \eta$. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$].

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	10	.0940	.0938
1	10	.0768	.0765
2	10	.0227	.0224
3	10	–.0259	–.0260
4	10	–.0398	–.0397
5	10	–.0179	–.0178
6	10	.0149	.0150
7	10	.0300	.0300
8	10	.0173	.0172
9	10	–.0089	–.0090
10	10	–.0246	–.0246
11	10	–.0172	–.0171
12	10	.0046	.0048
13	10	.0207	.0207
20	10	.0168	.0167
30	10	–.0087	–.0086

The *first* thing to notice here is that the table is *very* accurate, even though η and Δ *differ* from their values in the table on page 656. The *second* thing to notice is that as η ---› 0, $g_{u,v}$ ---› 1 / Δ, so that testing at $\eta = 0$ will become *exact* for *any* $r > 0$, and *any* angle $0 \le \theta \le \pi / 2$; which we call Q1. And in this case, the choice of $\Delta > 0$ *won't* matter; but from a *physics* perspective, $g_{u,v} = 1 / \Delta$ in *all* of Q1, with singularities at $(\pm\delta, 0)$, makes no sense.

When doing the testing, I noticed that at $\Delta = 10$ or 12, I could achieve *high* accuracy, if I set η to a small value, such as 1/20 (similar remarks apply to values of Δ *greater* than 12 or *less* than 10). At $\Delta = 11$, however, I could achieve this *same* degree of accuracy *without* imposing such a constraint on η. Indeed, a value of $\eta = 1/10$ was sufficient, as you can see in the tables on pages 656-7.

Therefore, we have two types of *ideal* spacelike and timelike estimators here – those that are *biased*, and those that are *unbiased*. So far, the only pair of *unbiased* ideal estimators that surface in the testing, are those for which $\Delta = 11$ and $\eta = 1/10$ …

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For the record, when testing $\mathcal{H}_x$ + R against $\mathcal{E}$, using the setup in this note, with $\delta = 1$; the pole is at $-p = -\Delta / \eta\sin(\theta) - \alpha$, so that $Res = exp(-pr) / \sqrt{p^2 + \varepsilon^2}$, and thus R is $2 \cdot Res$. Here, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, *irrespective* of the form for $g_{u,v}$, which is the *spacelike* component, in this case.

From the *definition* of $g_{u,v}$, we set $\epsilon = \eta\sin(\theta)$ and let $\mu = \alpha \cdot \epsilon + \Delta$, so that the harmonic expression $\mathcal{H}_x$ now computes to …

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(y\mathrm{r})[1/(\mu + iy\epsilon) - 1/(\mu - iy\epsilon)] + \boldsymbol{i}\sin(y\mathrm{r})[1/(\mu + iy\epsilon) + 1/(\mu - iy\epsilon)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

And finally, $\mathcal{E} = (1 / \mu) \cdot J_0(\mathrm{r}\varepsilon)$. If $\epsilon = 0$, then there is *no* pole associated with the generating function $f(s)$, so that R = 0, and the match is *exact*. That is to say, $\mathcal{H}_x = \mathcal{E}$, and again, these remarks pertain to the *spacelike* component $1 / (r \cdot \eta\sin(\theta) + \Delta)$ in $\mathcal{R}^2$. But they *also* apply to the *timelike* component $1/ (r \cdot \eta\cos(\theta) + \Delta)$ in $\mathcal{R}^2$, after making the obvious changes [see pages 635-7 for examples of how this is actually done].

Note that testing in $\mathcal{R}^3$ is really *no* different than testing in $\mathcal{R}^2$, so that it is always *sufficient* to test in $\mathcal{R}^2$ only [pp 605-6, 610]. And finally, from our *invariance* principle [pp 642-6], we only need to test at $\delta = 1$ …

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As we said on the previous page, setting $\eta = 0$ will give us an *exact* match, but it is not *realistic* physics. Therefore, if η is too small, it may be introducing *bias* into the estimator $g_{u,v}$, that shouldn't be there. Similarly, choosing a very large value for Δ may also be *unrealistic*.

Thus, we probably want η to be small, but not too small, and Δ large in comparison to η. Based on the tables in this note, a value of $\eta \approx 1/10$, and $\Delta \approx 11$ (or perhaps a little higher) might be suitable. But ultimately, the *correct* choice for η and Δ, which gives us *high* accuracy in the tables, will *only* emerge when there is agreement between theory and experiment …

A Simple Case Study, Part XLVIII

The following table compares the harmonic expression $\mathcal{H}_x$ + R with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here, $r = 2$, and η, Δ vary as per the table below. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$].

η	Δ	$\mathcal{H}_x$ + R	$\mathcal{E}$
0.1	12	.0191	.0187
0.2	11	.0213	.0204
0.3	10	.0241	.0224
0.4	9	.0277	.0249
0.5	8	.0325	.0280
0.6	7	.0391	.0320
0.7	6	.0486	.0373
0.8	5	.0632	.0448
0.9	4	.0881	.0560
1.0	3	.1362	.0746

The *first* thing to notice about this table, is that the R value is *negligible*, except for the last *two* entries. The *second* thing to notice is that as η *increases* and Δ *decreases*, testing gets *worse*. In fact, the only worthwhile entry is the *first* one.

And from the table on page 640, where $\theta = \pi / 2$ and $\delta = 1$, we *also* know that at $r = 2$, $\mathcal{H}_x \approx .022$ and $\mathcal{E} \approx .019$, if $\eta = 1$ and $\Delta = 12$. Still, this is *not* as good as the first entry above, when $\eta = 0.1$. Thus, the conclusion here is that smaller η and larger Δ, relative to η, is the best combination; but to avoid *bias*, we don't want η too small, nor do we want Δ too large.

For if we let $\eta \dashrightarrow 0$, testing becomes *exact* at $\eta = 0$; and if, for a fixed r, we let $\Delta \dashrightarrow \infty$, then each of $\mathcal{H}_x$, R and $\mathcal{E}$ will tend to *zero*. Similarly, if we let $r \dashrightarrow \infty$, then again, each of $\mathcal{H}_x$, R and $\mathcal{E}$ will tend to *zero*, provided $\varepsilon > 0$. Thus, we accomplish nothing by choosing Δ to be too large.

Note that the harmonic expression on the last page, and reproduced below,

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\mu + iy\epsilon) - 1/(\mu - iy\epsilon)] + \boldsymbol{i}\sin(yr)[1/(\mu + iy\epsilon) + 1/(\mu - iy\epsilon)]\}\, dy\,/\sqrt{y^2 - \varepsilon^2}$$

reduces to the following if $\eta = 0$ [pp 636, 327] …

$$(1 / \Delta) \cdot 2/\pi \int_{\varepsilon}^{\infty} \sin(yr)\, dy\,/\sqrt{y^2 - \varepsilon^2} \;=\; (1 / \Delta) \cdot J_0(r\varepsilon)\,.$$

A Simple Case Study, Part XLIX

The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here $\eta = 1/10$, $\Delta = 30$ and $\gamma = 1 / \eta$. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$].

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	30	.0313	.0313
1	30	.0256	.0255
2	30	.0075	.0075
3	30	–.0087	–.0086
4	30	–.0132	–.0132
5	30	–.0059	–.0059
6	30	.0050	.0050
7	30	.0100	.0100
8	30	.0057	.0057
9	30	–.0030	–.0030
10	30	–.0082	–.0082
11	30	–.0057	–.0057
12	30	.0016	.0016
13	30	.0069	.0069
20	30	.0056	.0056
30	30	–.0029	–.0029

This table can be compared to the one on page 656, where $\eta = 1/10$, $\Delta = 11$. You can see here, that with $\eta = 1/10$, $\Delta = 30$, we are virtually *exact*, out to four decimal places. And this table could be made *more* exact, were we to increase Δ further, or even reduce η, Thus, smaller η and larger Δ is the general principle here, for *high* accuracy tables; and somewhere within these ranges for η and Δ, we are probably going to find the *correct* values, where theory and experiment agree, for both the *spacelike* and *timelike* components.

And here are a few snapshots from the table above, for $\mathcal{H}_x$ and $\mathcal{E}$, when $r = 1$ …

$$\int_1^{300} -\frac{i\left(\left(-\frac{1}{30-\frac{iy}{10}}+\frac{1}{30+\frac{iy}{10}}\right)\cos(y)+i\left(\frac{1}{30-\frac{iy}{10}}+\frac{1}{30+\frac{iy}{10}}\right)\sin(y)\right)}{\pi\sqrt{-1+y^2}}\,dy = 0.0255917$$

$$\frac{J_0(1)}{30}$$

Decimal approximation

0.025506589551932

A Simple Case Study, Part L

For the record, when testing $\mathcal{H}_x$ + R against $\mathcal{E}$, using the *timelike* component, with $\delta = 1$; the pole is at $-p = -\Delta / \eta\cos(\theta) - \alpha$, so that $Res = exp(-pr) / \sqrt{p^2 + \varepsilon^2}$, and thus R is $2 \cdot Res$. Here, $\alpha = \cos(\theta)$ and $\varepsilon = \sin(\theta)$, *irrespective* of the form for $g_{u,v} = 1 / (r \cdot \eta\cos(\theta) + \Delta)$; which is the *timelike* component, in this case.

From the *definition* of $g_{u,v}$, we set $\epsilon = \eta\cos(\theta)$ and let $\mu = \alpha \cdot \epsilon + \Delta$, so that the harmonic expression $\mathcal{H}_x$ now computes to …

$$2\kappa \int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\mu + iy\epsilon) - 1/(\mu - iy\epsilon)] + \boldsymbol{i}\sin(yr)[1/(\mu + iy\epsilon) + 1/(\mu - iy\epsilon)]\}\, dy \,/ \sqrt{y^2 - \varepsilon^2}$$

And finally, $\mathcal{E} = (1 / \mu) \cdot J_0(r\varepsilon)$. If $\epsilon = 0$, then there is *no* pole associated with the generating function $f(s)$, so that R = 0, and the match is *exact*. That is to say, $\mathcal{H}_x = \mathcal{E}$, and again, these remarks pertain to the *timelike* component $1 / (r \cdot \eta\cos(\theta) + \Delta)$ in $\mathcal{R}^2$.

Note that testing in $\mathcal{R}^3$ is really *no* different than testing in $\mathcal{R}^2$, so that it is always *sufficient* to test in $\mathcal{R}^2$ only [pp 605-6, 610]. And finally, from our *invariance* principle [pp 642-6], we only need to test at $\delta = 1$.

If at $\theta = 0$, we set $\eta = 0.05$ and $\Delta = 10$, then $\mu = 10.05$; and *no matter* our choice of $r \geq ½$ (so that R will always be negligible), we find that $\mathcal{H}_x \approx 0.0995$; which exactly matches $\mathcal{E}$. Here are some snapshots from Wolfram, so you can see this effect [note that $\alpha = 1$ and $\varepsilon = 0$, so that $\mathcal{E} = 1 / \mu$, which is 0.0995, out to four decimal places]. In this case, we let $r = 2$ and 8 in $\mathcal{H}_x$ below …

$$\int_0^{300} \frac{\cos(2y)\left(\frac{1}{10.05+0.05\,i\,y} - \frac{1}{10.05-0.05\,i\,y}\right) + i\sin(2y)\left(\frac{1}{10.05+0.05\,i\,y} + \frac{1}{10.05-0.05\,i\,y}\right)}{(\pi\, i)\sqrt{y^2 - 0}}\, dy =$$

0.0995329

$$\int_0^{300} \frac{\cos(8y)\left(\frac{1}{10.05+0.05\,i\,y} - \frac{1}{10.05-0.05\,i\,y}\right) + i\sin(8y)\left(\frac{1}{10.05+0.05\,i\,y} + \frac{1}{10.05-0.05\,i\,y}\right)}{(\pi\, i)\sqrt{y^2 - 0}}\, dy =$$

0.0994966

$$\frac{1}{10.05}$$

Result

0.0995024875

A Simple Case Study, Part LI

The following table compares the harmonic expression $\mathcal{H}_x$ + R with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here $\eta = 1/100$, $\Delta = 30$ and $\gamma = 1 / \eta$. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$].

r	Δ	$\mathcal{H}_x$ + R	$\mathcal{E}$
.001	30	.0300	.0333
.002	30	.0332	.0333
.003	30	.0333	.0333
.004	30	.0333	.0333
.010	30	.0333	.0333
.020	30	.0333	.0333
.090	30	.0333	.0333
.100	30	.0333	.0333
.200	30	.0330	.0330
.300	30	.0326	.0326
.400	30	.0320	.0320
.500	30	.0313	.0313
1	30	.0255	.0255
2	30	.0075	.0075
3	30	–.0087	–.0087
9	30	–.0030	–.0030

You can see from this table, that down to about $r \approx .002$, we achieve *high* accuracy, and at $r \approx .001$, where the R value matters, accuracy is reasonable. To improve on the accuracy as r ---› 0, would mean *lowering* the value of η even further, as increasing *or* decreasing Δ doesn't seem to help much, in this case, and can actually make things *worse*.

Therefore, based on these findings, which apply to *both* the *spacelike* and *timelike* estimators, I would suggest the values of $\eta \approx 1/100$ and $\Delta \approx 30$, when doing calculations for dark energy, such as *incremental* length, area and volume; *if* we *also* want high accuracy for *very* small r. And if we don't, then as we have seen in our previous notes, *larger* choices for η and even *smaller* choices for Δ are available to us.

———

The following table compares the harmonic expression $\mathcal{H}_x$ + R with $\mathcal{E}$, for the *timelike* component in $\mathcal{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta cos(\theta) + \Delta)$. Here $\eta = 1/100$, $\Delta = 30$ and $\gamma = 1 / \eta$. The angle is $\theta = 0$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$, but see also the comments on page 662] …

r	Δ	$\mathcal{H}_x$ + R	$\mathcal{E}$
.001	30	.0300	.0333
.002	30	.0332	.0333
.003	30	.0333	.0333
.004	30	.0333	.0333
.010	30	.0333	.0333
.020	30	.0333	.0333
.090	30	.0333	.0333
.100	30	.0333	.0333
.200	30	.0333	.0333
.300	30	.0333	.0333
.400	30	.0333	.0333
.500	30	.0333	.0333
1	30	.0333	.0333
2	30	.0333	.0333
3	30	.0333	.0333
9	30	.0333	.0333

______

Even with η = 1/20 and Δ = 30, for the *spacelike* component; we can *still* get high accuracy at r = .01, as you can see in the pictures below, for $\mathcal{H}_x$ and $\mathcal{E}$, when $\theta = \pi / 2$. Thus, it becomes a *trade-off* of sorts – how much accuracy do we really need for *very* small r, versus higher values for η, in particular … ..

$$\int_1^{10000} \frac{\cos(0.01\, y)\left(\frac{1}{30+0.05\, i\, y} - \frac{1}{30-0.05\, i\, y}\right) + i \sin(0.01\, y)\left(\frac{1}{30+0.05\, i\, y} + \frac{1}{30-0.05\, i\, y}\right)}{(\pi\, i)\, \sqrt{y^2-1}}\, dy =$$

0.0331735

$$\frac{J_0(0.01)}{30}$$

Result

0.0333325…

A Simple Case Study, Part LII

We know from pages 554-5, that the following expression is *exact* for all r > 0, where $\mathbf{\Delta} > 0$ and R is equal to $2e^{-\mathbf{\Delta} r} / \mathbf{\Delta}$. Let us label this expression (†) …

$$2\kappa\int_0^\infty \{\cos(yr)[1 / (\mathbf{\Delta} + iy) - 1 / (\mathbf{\Delta} - iy)] + \boldsymbol{i}\sin(yr)\, [1 / (\mathbf{\Delta} + iy) + 1 / (\mathbf{\Delta} - iy)]\}\ dy / y \ +\ R = 1 / \mathbf{\Delta}$$

Now our harmonic expression $\mathcal{H}_x$ for the *timelike* component $1 / (r \cdot \eta\cos(\theta) + \Delta)$; where $\varepsilon = 0$, $\mu = \eta + \Delta$, $\epsilon = \eta$, is …

$$2\kappa\int_\varepsilon^\infty \{\cos(yr)[1/(\mu + iy\eta) - 1/(\mu - iy\eta)] + \boldsymbol{i}\sin(yr)[1/(\mu + iy\eta) + 1/(\mu - iy\eta)]\}\ dy / \sqrt{y^2 - \varepsilon^2}$$

Thus, if we factor out $1 / \eta$ in $\mathcal{H}_x$, we now have a form for the *integrand* which is *compatible* with the integral $\mathcal{J}$ in (†), where in $\mathcal{J}$, $\mathbf{\Delta}$ is now μ / η. Hence, from (†), it must be the case that $\mathcal{H}_x$ is *exactly* equal to $1 / \eta\mathbf{\Delta} - R$, where R is now equal to $2e^{-\mathbf{\Delta} r} / \eta\mathbf{\Delta}$.

Thus, we may write $\mathcal{H}_x = 1 / \mu - 2e^{-\mathbf{\Delta} r} / \mu$, which is equal to $(1 / \mu) \cdot (1 - 2e^{-\mathbf{\Delta} r})$; where again, $\mathbf{\Delta}$ is now equal to μ / η. From this, we see that as r ---› ∞, $\mathcal{H}_x$ becomes $1 / \mu$, and indeed, approaches this limit much *sooner* if $\mathbf{\Delta} = \mu / \eta$ is large. And that is why the integrals on page 662 converge to a value of $1 / \mu$, *even* when r = 2 or 8, as we see in the pictures there. For in that very case, $\eta = .05$ and $\Delta = 10$, so that $\mathbf{\Delta} \approx 200$.

On the other hand, as r ---› 0, $\mathcal{H}_x$ becomes, perhaps surprisingly, $-1 / \mu$, as we can see in the image below; where here $\eta = .01$, $\Delta = 30$, and r = 0.000001 … ..

$$\int_0^{10\,000\,000} \frac{1}{(\pi i)\sqrt{y^2 - 0}} \left(\cos(1\times 10^{-6}\, y)\left(\frac{1}{30.01 + 0.01\, i\, y} - \frac{1}{30.01 - 0.01\, i\, y}\right) + i\sin(1\times 10^{-6}\, y)\left(\frac{1}{30.01 + 0.01\, i\, y} + \frac{1}{30.01 - 0.01\, i\, y}\right)\right) dy = -0.0331223$$

A Simple Case Study, Part LIII

The following table compares the harmonic expression $\mathcal{H}_x$ + R with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here η varies, $\Delta = 30$, and the data indicates just how *low* we can go with *r*, and still achieve *reasonable* accuracy, for a given η. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$].

r	η	$\mathcal{H}_x$ + R	$\mathcal{E}$
.001	.01	.0300	.0333
.002	.02	.0301	.0333
.003	.03	.0301	.0333
.004	.04	.0301	.0333
.010	.10	.0303	.0333
.020	.20	.0307	.0333
.090	.90	.0330	.0333
.100	1	.0333	.0333
.200	2	.0365	.0330
.300	3	.0396	.0326
.400	4	.0426	.0320
.500	5	.0455	.0313

For example, if $r = .02$, I can achieve *reasonable* accuracy if $\eta = .20$. If I want *higher* accuracy for this choice of *r* (and there is no need to suppose we really do, when calculating the integrations for *incremental* length, say, induced by dark energy); I know this can be done by *lowering* the value of η, *without* changing the value of Δ [see the picture below, where $\eta = .05$]. Similarly, if $r = 1/10$, we achieve *high* accuracy at $\eta = 1$, $\Delta = 30$; and needn't go any lower with η in this case, if we don't want to.

But if we do, accuracy will only *improve* here, for *this* choice of *r*, if $\Delta = 30$. For example, we may decide to lower η to 1/10. Then not only would the accuracy improve for $r = 1/10$; but we *also* see from the table above, that at $r = .01$, we would now achieve *reasonable* accuracy … ..

$$\int_1^{10\,000} \frac{\cos(0.02\,y)\left(\frac{1}{30+0.05\,i\,y} - \frac{1}{30-0.05\,i\,y}\right) + i\sin(0.02\,y)\left(\frac{1}{30+0.05\,i\,y} + \frac{1}{30-0.05\,i\,y}\right)}{(\pi\,i)\,\sqrt{y^2-1}}\,dy =$$

0.0333355

A Simple Case Study, Part LIV

In this note, we wish to compare the *ideal* spacelike estimator $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$ with an alternate; namely $h_{u,v} = 1 / (r \cdot \eta(r)\sin(\theta) + \Delta)$, where $\eta(r) = \eta$ if $r \geq r_0$; otherwise $\eta(r) = \eta r / r_0$ if it is the case that $0 < r < r_0$ [we'll call this restricted range $\mathcal{R}_o$]. We suppose there exists a *small* r_0 such that for $r < r_0$, there is *no* longer high accuracy for $g_{u,v}$, when comparing $\mathcal{H}_x$ + R with $\mathcal{E}$; but that there *is* when using $h_{u,v}$ in this restricted range $\mathcal{R}_o$. Such an assumption makes sense, since we already know in $g_{u,v}$ that η must *decrease* in $\mathcal{R}_o$ as r tends to 0; if we are to achieve *high* accuracy, *especially* if Δ is fixed.

Let's begin by assuming $\theta = \pi / 2$ and $\delta = 1$, so that $g_{u,v} = 1 / (r \cdot \eta + \Delta)$. Now let us define the function $f(\epsilon) = 1 / \sqrt{\Delta + \epsilon}$, so that for *small* ϵ relative to Δ, the Taylor series expansion of $f(\epsilon)$ computes to $1 / \Delta^{1/2} - \epsilon / 2\Delta^{3/2}$. Letting $\epsilon = r \cdot \eta$, where r is in $\mathcal{R}_o$, we see that $\sqrt{g_{u,v}}$ expands as

$$1 / \Delta^{1/2} - r \cdot \eta / 2\Delta^{3/2} , \qquad (1)$$

provided $r \cdot \eta$ is *small* relative to Δ. We now integrate (1) over $\mathcal{R}_o$ and find that the result computes to

$$r_0 / \Delta^{1/2} - r_0^2 \cdot \eta / 4\Delta^{3/2} \qquad (2)$$

Now we repeat the exercise for $\sqrt{h_{u,v}}$, where $\epsilon = r \cdot \eta(r)$, and so find that the integration over $\mathcal{R}_o$ computes to

$$r_0 / \Delta^{1/2} - r_0^2 \cdot \eta / 6\Delta^{3/2} \qquad (3)$$

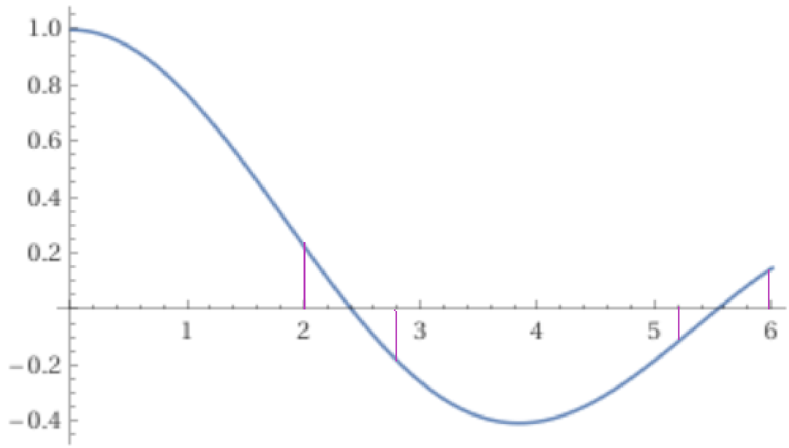
Subtracting (2) from (3) leaves us with $\boldsymbol{\Delta} = r_0^2 \cdot \eta / 12\Delta^{3/2}$, and this is the *difference* between the two integrations; one where we use our *ideal* estimator $g_{u,v}$ in $\mathcal{R}_o$, and the other being $h_{u,v}$ in $\mathcal{R}_o$. The difference [$\boldsymbol{\Delta}$] is really quite small; for if we look at the table on the previous page, where $r_0 \approx .02$ if $\eta = .20$, and $\Delta = 30$; we find that $\boldsymbol{\Delta} \approx 4 \cdot 10^{-8}$.

Thus, replacing η with $\eta(r)$ in the estimator $g_{u,v}$, where $\eta(r)$ is some *decreasing* function of r in $\mathcal{R}_o$, as r decreases [$\eta(r_0) = \eta$ and $\eta(0) = 0$]; may give us *high* accuracy when comparing $\mathcal{H}_x$ + R with $\mathcal{E}$ here, but it will make *no difference* when carrying out the integrations in this restricted range.

And so, we should feel comfortable with using the *ideal* spacelike and timelike estimators; namely $1 / (r \cdot \eta\sin(\theta) + \Delta)$ and $1 / (r \cdot \eta cos(\theta) + \Delta)$, respectively, for all $r > 0$, where η is small and Δ is large relative to η; and r_0 itself is small, if not *very* small …

A Simple Case Study, Part LV

In this note, we wish to study regions of *instability*, when comparing $\mathcal{H}_x$ with $\mathcal{E}$, which normally occur when we are evaluating these expressions *near* a *root* of $J_0(r\varepsilon)$. So we'll begin by letting the angle $\theta = \pi / 2$ for our *spacelike* component $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$, so that $\varepsilon = 1$; and here we'll set $\eta = 0.1$ and $\Delta = 30$. As always, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ and $\delta = 1$.



In the diagram above, we see a picture of $J_0(r)$, where the *first* and *second* roots, r_1 and r_2, are at roughly 2.405 and 5.520. The *unstable* regions are depicted by the vertical *magenta* lines, and are defined to be $\mathcal{U}_k = [r_k - \epsilon , r_k + \epsilon]$, for some small $\epsilon > 0$. In such a region, $\mathcal{H}_x$ and $\mathcal{E}$ will both be *very* small, but the variation can be rather extreme, as you can see in the pictures below near r_1 …

$$\int_1^{11\,000} \frac{\cos(2.4048\,y)\left(\frac{1}{30+0.1\,i y} - \frac{1}{30-0.1\,i y}\right) + i\sin(2.4048\,y)\left(\frac{1}{30+0.1\,i y} + \frac{1}{30-0.1\,i y}\right)}{(\pi\, i)\sqrt{y^2-1}}\,dy =$$

0.0000581928

$$\frac{J_0(2.4048)}{30}$$

Result

$4.42276\ldots \times 10^{-7}$

Here, the value of $\mathcal{H}_x$ is about $5.8 \cdot 10^{-5}$, whilst $\mathcal{E}$ is about $4.4 \cdot 10^{-7}$. Outside $\mathcal{U}_k$, this type of *instability* or *uncertainty* does *not* exist, so what we want to do here, is examine the integrals for $\sqrt{g_{u,v}}$ over all $\mathcal{U}_k$, and develop a *measure* which establishes a connection between ϵ and η.

For as we already know, the smaller η becomes, the *higher* the accuracy when comparing $\mathcal{H}_x$ with $\mathcal{E}$ in $\mathcal{U}_k$ [the R value can be omitted, because r is *not* small here and Δ is *large* in comparison to η].

We begin by noting that the *dark energy* function is $\xi = 2\sigma \cdot \cosh(\delta r\alpha)J_0(\delta r\varepsilon)$, so that at $\theta = \pi / 2$ with $\delta = 1$, this reduces to is $\xi = 2\sigma \cdot J_0(r)$; and thus the *sign* of ξ is simply $sgn(J_0(r))$, since $\sigma > 0$ holds true here. We now integrate $sgn(J_0(r)) \cdot \sqrt{g_{u,v}}$ over $\mathcal{U}_1$, noting that in the interval $[r_1 - \epsilon, r_1]$ the result will be *larger positive*, and in the interval $[r_1, r_1 + \epsilon]$ the result will be *smaller negative*, due to both the construction of $\sqrt{g_{u,v}}$ and $sgn(J_0(r))$.

Defining $\mu = r_1 \cdot \eta + \Delta$, and $\nu = \eta\epsilon$, we find that this integration becomes …

$$(2 / \eta)\,[2\sqrt{\mu} - \sqrt{\mu - \nu} - \sqrt{\mu + \nu}\,], \quad (*)$$

and since ν is *small* in comparison to μ, we can expand the expression above as a Taylor series to *second* order; that is to say, with

$$\sqrt{\mu + \nu} \sim \sqrt{\mu} + \nu / 2\sqrt{\mu} - \nu^2 / 8\mu^{3/2}$$

the expression (*) becomes $\mathbf{\Delta}_1 = (4 / \eta) \cdot \nu^2 / 8\mu^{3/2}$, which is equal to $\eta \cdot \epsilon^2 / 2(r_1 \cdot \eta + \Delta)^{3/2}$.

We can now go on to $\mathcal{U}_2$, noting that in the interval $[r_2 - \epsilon, r_2]$ the result will be *larger negative*, and in the interval $[r_2, r_2 + \epsilon]$ the result will be *smaller positive*, due to both the construction of $\sqrt{g_{u,v}}$ and $sgn(J_0(r))$. Hence, $\mathbf{\Delta}_2$ becomes $-\eta \cdot \epsilon^2 / 2(r_2 \cdot \eta + \Delta)^{3/2}$. We thus wind up with an *infinite* alternating series, which can be written as …

$$\mathbf{\Delta} = \epsilon^2 / 2\sqrt{\eta} \sum_{k=1}^{\infty} (-1)^{(k+1)} / (r_k + [\Delta / \eta])^{3/2}$$

And this is the amount [$\mathbf{\Delta}$] that can be attributed to *uncertainty* in the integration of $\sqrt{g_{u,v}}$ over *all* $\mathcal{U}_k$. It is a *very* small number indeed, especially if $\epsilon \sim \sqrt{\eta}$ holds true; and η is small and Δ is large relative to η. For if $\eta = 0.1$, $\Delta = 30$ and $\epsilon \sim \sqrt{\eta}$; $\mathbf{\Delta}$ is approximately 10^{-6}, summing from $k = 1$ to 6.

Outside $\mathcal{U}_k$, this type of *instability* or *uncertainty* does *not* exist, and we can see this in the pictures below, where again $\eta = 0.1$, $\Delta = 30$ and $\epsilon \sim \sqrt{\eta} \sim 0.32$. For if we choose a value of r *just outside* $\mathcal{U}_1$, $\mathcal{H}_x$ should once again agree with $\mathcal{E}$, with a fair degree of accuracy. Since $r_1 \sim 2.405$, let's move to the *left* of $r_1 - \epsilon \sim 2.08$, by setting $r = 2$. Here are the snapshots from Wolfram …

$$\int_1^{11000} -\frac{i\left(\left(-\frac{1}{30-\frac{iy}{10}}+\frac{1}{30+\frac{iy}{10}}\right)\cos(2y)+i\left(\frac{1}{30-\frac{iy}{10}}+\frac{1}{30+\frac{iy}{10}}\right)\sin(2y)\right)}{\pi\sqrt{-1+y^2}}\,dy = 0.00752712$$

$$\frac{J_0(2)}{30}$$

Decimal approximation

0.007463025971374

A Simple Case Study, Part LVI

In this note, we will show that there is actually only *one* choice for η that gives us *high* accuracy in the tables, when comparing $\mathcal{H}_x$ + R with $\mathcal{E}$, as $r \dashrightarrow 0$; and that this value is $\eta = 1$. This is true for *both* the *spacelike* and *timelike* components, as we shall see below. Here, the singularities are at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$. As always, we're working in $\mathcal{R}^2$, where θ is between 0 and $\pi / 2$.

For the *timelike* component $1 / (r \cdot \eta\cos(\theta) + \Delta)$, let us bring back our expression from page 665 for $\mathcal{H}_x$, where $\varepsilon = 0$, $\mu = \eta + \Delta$, $\epsilon = \eta$, and $\boldsymbol{\Delta} = \mu / \eta$ …

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\mu + iy\eta) - 1/(\mu - iy\eta)] + \boldsymbol{i}\sin(yr)[1/(\mu + iy\eta) + 1/(\mu - iy\eta)]\}\, dy \,/ \sqrt{y^2 - \varepsilon^2}$$

We note here [p 665] that $\mathcal{H}_x = (1 / \mu) \cdot (1 - 2e^{-\Delta r})$ and so tends to $-1 / \mu$ as $r \dashrightarrow 0$. On the other hand [p 662], for $\theta = 0$, the *pole* is at $-p = -\Delta / \eta\cos(\theta) - \alpha = -\mu / \eta$, so that the value of *Res* is now $exp(-pr) / \sqrt{p^2 + \varepsilon^2}$, which tends to $1 / p$ as $r \dashrightarrow 0$. Thus … since R = 2 · *Res*, R tends to $2\eta / \mu$ as $r \dashrightarrow 0$, and since $\mathcal{E} = (1 / \mu) \cdot J_0(r\varepsilon) = 1 / \mu$; high accuracy can *only* be achieved if $\mathcal{H}_x$ + R agrees with $\mathcal{E}$ in this case, which means $-1 / \mu + 2\eta / \mu$ *must* be equal to $1 / \mu$. In other words, there is only *one* choice for η, and that choice is $\eta = 1$.

And this is *also* true for the *spacelike* component as well; namely $1 / (r \cdot \eta\sin(\theta) + \Delta)$, as we can see in the pictures below, where $\eta = 1$, $\Delta = 30$, $r = .001$ and $\theta = \pi / 2$ …

$$\int_1^{1000000} -\frac{i\left(\left(-\frac{1}{30-iy} + \frac{1}{30+iy}\right)\cos\left(\frac{y}{1000}\right) + i\left(\frac{1}{30-iy} + \frac{1}{30+iy}\right)\sin\left(\frac{y}{1000}\right)\right)}{\pi\sqrt{-1+y^2}}\, dy =$$

$-0.0313451 + 0\,i$

value of $\mathcal{H}_x$

$$2 \times \frac{\exp(-30 \times 0.001)}{\sqrt{30^2 + 1}}$$

Result

0.064660456…

value of R

$$\frac{J_0(0.001)}{30}$$

Result

0.0333333…

value of $\mathcal{E}$

From these pictures above, we see that $\mathcal{H}_x$ + R ~ .03332, which is very close to the value of $\mathcal{E}$. And so we may conclude that *reasonable* to *high* accuracy can be achieved for all $r > 0$, if $\eta = 1$ and Δ is large, relative to η.

The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here $\eta = 1$, $\Delta = 30$ and $\gamma = 1 / \eta$. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$]. The R value is negligible, in this case …

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	30	.0315	.0313
1	30	.0260	.0255
2	30	.0081	.0075
3	30	–.0083	–.0086
4	30	–.0133	–.0132
5	30	–.0063	–.0059
6	30	.0047	.0050
7	30	.0100	.0100
8	30	.0060	.0057
9	30	–.0027	–.0030
10	30	–.0081	–.0082
11	30	–.0059	–.0057
12	30	.0013	.0016
13	30	.0068	.0069
20	30	.0056	.0056
30	30	–.0030	–.0029

This table can be compared to the one on page 661, where there, $\eta = 1/10$, all else being equal. Clearly the table there is *more* accurate than the one *here*, but the one here preserves *high* accuracy as r ---› 0, whereas the table on page 661 does not [but see also page 667]. Thus, it becomes a *trade-off* of sorts; we can leave $\eta = 1$, *if* we're satisfied with the accuracy in the table above; or we can lower the value of η to 1/10-th, say, and not worry about the loss of high accuracy as r ---› 0 [again, see page 667 for more on this issue].

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The following table compares the harmonic expression $\mathcal{H}_x$ with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here $\eta = 1$, $\Delta = 50$ and $\gamma = 1 / \eta$. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$]. The R value is negligible, in this case …

r	Δ	$\mathcal{H}_x$	$\mathcal{E}$
1/2	50	.0189	.0188
1	50	.0155	.0153
2	50	.0047	.0045
3	50	–.0051	–.0052
4	50	–.0080	–.0079
5	50	–.0037	–.0036
6	50	.0029	.0030
7	50	.0060	.0060
8	50	.0035	.0034
9	50	–.0017	–.0018
10	50	–.0049	–.0049
11	50	–.0035	–.0034
12	50	.0009	.0010
13	50	.0041	.0041
20	50	.0034	.0033
30	50	–.0018	–.0017

You can see from this table, that by *increasing* Δ to 50 and leaving $\eta = 1$, we achieve *high* accuracy, which will persist, *even* as r ---› 0. However, to get this kind of accuracy means a higher value for Δ, as it is now the only *free* variable we can change. Similar results apply to the *timelike* component; namely $g_{u,v} = 1 / (r \cdot \eta\cos(\theta) + \Delta)$, where again, $\eta = 1$ and $\Delta = 50$.

And here are some snapshots from Wolfram, for the *spacelike* component, where $\eta = 1$, $\Delta = 50$, $r = .001$ and $\theta = \pi / 2$. In this case $\mathcal{H}_x$ + R ~ .0199962, which is *extremely* close to $\mathcal{E}$ … ..

$$\int_1^{1000000} -\frac{i\left(\left(-\frac{1}{50-iy}+\frac{1}{50+iy}\right)\cos\left(\frac{y}{1000}\right)+i\left(\frac{1}{50-iy}+\frac{1}{50+iy}\right)\sin\left(\frac{y}{1000}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy =$$

$$-0.0180454+0\,i$$

value of $\mathcal{H}_x$

$$2\times\frac{\exp(-50\times 0.001)}{\sqrt{50^2+1}}$$

Result

0.03804157…

value of R

$$\frac{J_0(0.001)}{50}$$

Result

0.0200000…

value of $\mathcal{E}$

A Simple Case Study, Part LVII

The following table compares the harmonic expression $\mathcal{H}_x$ + R with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here $\eta = 1$, $\Delta = 300$ and $\gamma = 1 / \eta$. The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$, and also hold true for the *timelike* component $1 / (r \cdot \eta\cos(\theta) + \Delta)$].

r	Δ	$\mathcal{H}_x$ + R	$\mathcal{E}$
.000001	300	.00333	.00333
.00001	300	.00333	.00333
.0001	300	.00333	.00333
.001	300	.00333	.00333
.01	300	.00333	.00333
.1	300	.00332	.00332
1/2	300	.00313	.00313
1	300	.00256	.00255
2	300	.00075	.00075
3	300	–.00087	–.00086
4	300	–.00132	–.00132
5	300	–.00059	–.00059
6	300	.00050	.00050
7	300	.00100	.00100
8	300	.00057	.00057
9	300	–.00030	–.00030
10	300	–.00082	–.00082
11	300	–.00057	–.00057
12	300	.00016	.00016
13	300	.00069	.00069
20	300	.00056	.00056
30	300	–.00029	–.00029

The *first* thing to note about this table is that the R value is negligible if $r > .01$. The *second* thing to note is just how *accurate* the table is, whether r is very small or large. Now since it is the case that $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$, this means $\eta g_{u,v} = 1 / (r \cdot \sin(\theta) + \Delta / \eta)$. Thus, if we were content with values of $\eta = 1/10$ and $\Delta = 30$ for the table on page 661; then we should *also* be content with a value of $\Delta / \eta = 300$ in $\eta g_{u,v}$, where here η has been *normalized* to 1 in the $r \cdot \sin(\theta)$ term. And this is why *high* accuracy holds for all $r > 0$ here, in $\eta g_{u,v}$, as per the last research note.

To show the reader that such accuracy does *not* hold if $\eta = 1/10$ and $\Delta = 30$, we note from the table on page 666, that high accuracy starts to break down if $r \sim .01$; for in this case $\mathcal{H}_x$ + R ~ .0303 and $\mathcal{E}$ ~ .0333. Thus, *below* this threshold value of $r \sim .01$, high accuracy starts to degrade rather quickly as r decreases, if $\eta = 1/10$ and $\Delta = 30$. Only when $\eta = 1$, with Δ sufficiently large, do we obtain *high* accuracy for all $r > 0$, as we now know…

A Simple Case Study, Part LVIII

In this note, we are going to provide an *upper* bound for the expression below [pp 668-9], where for the *spacelike* component $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$, $\eta = 1$ and $\Delta = 300$.

$$\mathbf{\Delta} = \epsilon^2 / 2\sqrt{\eta} \sum_{k=1}^{\infty} (-1)^{(k+1)} / (r_k + [\Delta / \eta])^{3/2}$$

Recall this is the amount [$\mathbf{\Delta}$] that can be attributed to *uncertainty* in the integration of $\sqrt{g_{u,v}}$ over *all* $\mathcal{U}_k$. It is a *very* small number indeed, especially if $\epsilon \sim \sqrt{\eta}$ holds true; and η is small and Δ is large relative to η. For if $\eta = 0.1$, $\Delta = 30$ and $\epsilon \sim \sqrt{\eta}$; $\mathbf{\Delta}$ is approximately 10^{-6}, summing from $k = 1$ to 6.

Now from the table on the previous page, we note that for $r \geq ½$, the R value is negligible, so that the values for $\mathcal{H}_x$ and $\mathcal{E}$ are *exactly* one-tenth of the *corresponding* values in the table on page 661, where $\eta = 1/10$ and $\Delta = 30$. We'll now use the expression for $\mathbf{\Delta}$ above, with $\eta = 1/10$ and $\Delta = 30$ in $g_{u,v}$[and $\epsilon \sim \sqrt{\eta} \sim 1/3$], to *begin* our calculations; and we'll also define $\gamma = \epsilon^2 / 2\sqrt{\eta} \sim 1/6$. To *complete* the calculations, we'll simply multiply $\mathbf{\Delta}$ by $\sqrt{\eta}$, since here, this is the *equivalent* of integrating $\sqrt{\eta g_{u,v}}$, where $\eta g_{u,v}$ is now $1 / (r \cdot \sin(\theta) + \Delta)$, and $\Delta = 300$ [p 673].

Out to roughly 500,000, I can see from plots that the *zeroes* [r_k] of $J_0(r)$ are spaced about *three* apart, so we'll replace them with the sequence $\{3k - 1 \mid k = 1, 2, 3 \ldots\}$. Thus, the summation S in $\mathbf{\Delta}$ now becomes

$$\{1 / 302^{3/2} - 1 / 305^{3/2} + 1 / 308^{3/2} - 1 / 311^{3/2} + \ldots\}$$

Taking the *first* two terms, we see this can be written as $n = (1 / 302^{3/2}) \cdot (1 - 1 / (305 / 302)^{3/2})$; and since $(305 / 302)^{3/2} = (1 + 3 / 302)^{3/2} \sim 1 + 9 / 604$, we see that $n \sim (1 / 302^{3/2}) \cdot 9 / 613$, which is *less* than $(1 / 302^{3/2}) \cdot 3 / 200$. Similarly, we find by combining the *third* and *fourth* terms and calling it n, it is the case that $n < (1 / 308^{3/2}) \cdot 3 / 200$, and so on. Thus,

$$S < (3 / 200) \cdot \{1 / 302^{3/2} + 1 / 308^{3/2} + 1 / 314^{3/2} + \ldots\}$$

Now $1 / 302^{3/2}$ is *less* than the integration of $1 / x^{3/2}$ from $x = 301$ to 302; which in turn, is *less* than *one-third* of the integration of $1 / x^{3/2}$ from $x = 301$ to 307, using some *empirical* analysis here. Similarly, the number $1 / 308^{3/2}$ is *less* than the integration of $1 / x^{3/2}$ from $x = 307$ to 308; which in turn, is *less* than *one-third* of the integration of $1 / x^{3/2}$ from $x = 307$ to 313, and so on. Thus, with the constant $\kappa = \gamma / 200$ and $\varepsilon = 301$, it is the case that

$$\mathbf{\Delta} < \kappa \int_{\varepsilon}^{\infty} dx / x^{3/2}$$

Calculating the expression just above, we find $\mathbf{\Delta} < 10^{-4}$, if $\eta = 1/10$ and $\Delta = 30$. A virtually insignificant number over all $\mathcal{U}_k$, along the x-axis [again, see pages 668-9]. Thus, if $\eta = 1$ and $\Delta = 300$, we see that $\mathbf{\Delta} < 10^{-4} / 3 \sim 3 \cdot 10^{-5}$.

A Simple Case Study, Part LIX

Although *high* accuracy in the tables, for *both* the spacelike [$\mathcal{S}$] and timelike [$\mathcal{T}$] components, can be achieved as r ---› 0 with η = 1 [pp 670-3]; it was only validated if θ = π / 2 for $\mathcal{S}$ and θ = 0 for $\mathcal{T}$. In fact, further testing shows that at θ = π / 4, with η = 1, high accuracy is actually lost for $\mathcal{S}$ as the radius r ---› 0; and I suspect a similar effect holds true for $\mathcal{T}$. Here are some snapshots from Wolfram for $\mathcal{S}$, when δ = 1, θ = π / 4, r = .001, η = 1 and Δ = 300 …

$$\int_{\frac{1}{\sqrt{2}}}^{1\,000\,000} \frac{1}{(\pi\, i)\,\sqrt{y^2 - 0.5}} \left(\cos(0.001\, y)\left(\frac{1}{300.5 + 0.707\, i\, y} - \frac{1}{300.5 - 0.707\, i\, y}\right) + i \sin(0.001\, y)\left(\frac{1}{300.5 + 0.707\, i\, y} + \frac{1}{300.5 - 0.707\, i\, y}\right)\right) dy = -0.00102327$$

value of $\mathcal{H}_x$

$$\frac{2\exp(-\sqrt{2}\,(300.5 \times 0.001))}{\sqrt{2 \times 300.5^2 + 0.5}}$$

Result

0.00307686…

value of R

$$\frac{J_0\left(0.001 \times \frac{\sqrt{2}}{2}\right)}{300.5}$$

Result

0.00332779…

value of $\mathcal{E}$

In this case, $\mathcal{H}_x$ + R ~ .002, which does *not* match $\mathcal{E}$, but we *do* find that *high* accuracy returns when r = .01; for here $\mathcal{H}_x$ + R ~ .0033 and $\mathcal{E}$ ~ .0033, if θ = π / 4, η = 1 and Δ = 300. Thus, whether or not η = 1, high accuracy as r ---› 0, seems to be an issue which we can ignore for the time being; but see also page 667 on how we might deal with it …

A Simple Case Study, Part LX

Although *high* accuracy in the tables, for *both* the spacelike [$\mathcal{S}$] and timelike [$\mathcal{T}$] components, can be achieved as r ---› 0 with $\eta = 1$ [pp 670-3]; it was only validated if $\theta = \pi / 2$ for $\mathcal{S}$ and $\theta = 0$ for $\mathcal{T}$.

We note again [p 673], that *high* accuracy [$\theta = \pi / 2$] does *not* hold for $\mathcal{S}$ as r ---› 0, if $\eta = 1/10$ and $\Delta = 30$, since from the table on page 666, high accuracy starts to break down if $r \sim .01$. For in this particular case, $\mathcal{H}_x + \text{R} \sim .0303$ and $\mathcal{E} \sim .0333$; and thus, *below* this threshold value of $r_0 \sim .01$, high accuracy starts to degrade rather quickly, as r decreases, if $\eta = 1/10$ and $\Delta = 30$. Indeed, only when $\eta = 1$ with Δ sufficiently large in $\mathcal{S}$, do we obtain *high* accuracy for all $r > 0$, as we now know, *if* the angle $\theta = \pi / 2$ [p 673]. Yet at the *same* time, we have to realize that we can *also* achieve *high* accuracy for $\mathcal{S}$ if $\eta = 1/10$ and $\Delta = 30$, provided $r > r_0$ [note that $\Delta / \eta = 300$ in both cases]. And furthermore, we *also* know from page 667, that there is a way to deal with the loss of high accuracy, in this particular case.

Now as we have seen, further testing shows that at $\theta = \pi / 4$, with $\eta = 1$, high accuracy is actually lost for $\mathcal{S}$ as the radius r ---› 0 [p 675]; and a similar effect is likely true for $\mathcal{T}$ (indeed, it will be so at $\theta = \pi / 4$). Yet *even* at $\theta = \pi / 4$, with $r = .001$, $\eta = 1$ and $\Delta = 300$; we see that $\mathcal{H}_x + \text{R} \sim .002$ and $\mathcal{E} \sim .003$ [p 675].

Clearly these are fairly decent numbers for such a *small* value of r; and we should *not* expect the same or similar comparisons if $\eta = 1/10$ and $\Delta = 30$, for this particular angle. Indeed, if $r = .001$, then $\mathcal{H}_x + \text{R} \sim -.007$ and $\mathcal{E} \sim .033$ here; but accuracy returns if $r = .01$, with $\mathcal{H}_x + \text{R} \sim .032$ and $\mathcal{E} \sim .033$; when $\theta = \pi / 4$, $\eta = 1/10$, and $\Delta = 30$.

So from a *philosophical* standpoint, were we to make a decision about what value we should choose for η, based on the information above … which way should we go ? Both options [$\eta = 1/10$ and $\Delta = 30$, versus $\eta = 1$ and $\Delta = 300$], will give us *high* accuracy in the tables for all $r > r_0$, and θ between 0 and $\pi /2$; but clearly $\eta = 1$ seems to do the better job as r ---› 0. Is this enough to justify this choice of η, over all others ? It's hard to say, but in the end, the answer is likely to be decided by the *empirical* evidence, for *both* η and Δ, should we decide to use $\mathcal{S}$ and $\mathcal{T}$ as our pair of *ideal* estimators in $\mathcal{R}^3$ …

A Simple Case Study, Part LXI

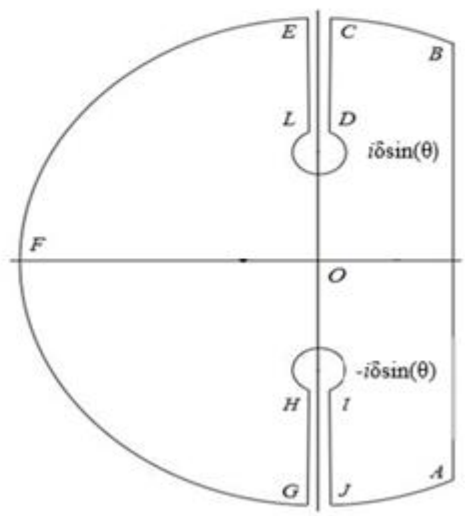
In this note, we wish to look at the *spacelike* estimator $g_{u,v} = 1 / (r \cdot \eta(r)\sin(\theta) + \Delta)$ from page 667, where $\eta(r) = \eta$ if $r \geq r_0$; otherwise $\eta(r) = \eta r / r_0$ if it is the case that $0 < r < r_0$ [we'll call this restricted range $\mathcal{R}_o$]. Now normally, we consider $\eta(r)$ to be an *unmappable* function, so that in $\mathcal{R}_o$ the argument r in $\eta(r)$ does *not* map to its *complex* equivalent in the *generating* function $f(s)$, *nor* the harmonic expression $\mathcal{H}_x$ [see also the comments on page 653]. Going forward, we'll assume that $\theta = \pi / 2$ and $\delta = 1$, with $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ …

Now if we *were* to allow r in $\eta(r)$ to map to its *complex* equivalents in $\mathcal{R}_o$, we could ask what would happen here. In this case, the generating function would become

$$f(s) = g_{u,v} \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2} = 1 / ((\eta / r_0)(s + \delta\alpha)^2 \cdot \sin(\theta) + \Delta) \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2},$$

where the *poles* associated with the *first* term in $f(s)$ are at $\pm \boldsymbol{i}\sqrt{\Delta r_0 / \eta}$. And by traversing the contour below, according to the Laplace inverse

$$\kappa \int e^{sr} f(s)\, ds,$$



we find that $Res = \sqrt{r_0 / \Delta} \cdot sin(\sqrt{\Delta r_0 / \eta} \cdot r) / \sqrt{\eta - \Delta r_0}$.

Now we see from *Res* that the term $\sqrt{\eta - \Delta r_0}$ implies $r_0 < \eta / \Delta$ $^{(*)}$; and so in particular, when it is the case that $\eta = 1/ 10$ and $\Delta = 30$ [and thus $r_0 \sim .01$ from page 666], we now find from (*) that r_0 must, in fact, be $< 1 / 300$. Such an *artificial* constraint, brought on by mapping r in $\eta(r)$ to its *complex* equivalents in $\mathcal{R}_o$, cannot really be justified; and so the recommendation here is to treat $\eta(r)$, and for that matter $\Delta(r)$, as *unmappable* functions in $g_{u,v}$, should they arise in the future.

Notice too, that since $\Delta r_0 / \eta < 1$, the poles actually lie on the y-axis *between* the branch points, which are located at $\pm \boldsymbol{i}sin(\theta)$, where in this case, $\theta = \pi / 2$ and $\delta = 1$ …

A Simple Case Study, Part LXII

In this note, we wish to study the 'specialness' of $\eta = 1$ in our spacelike [$\mathcal{S}$] and timelike [$\mathcal{T}$] estimators, when Δ / η is an *invariant*, and Δ is large, relative to η. We begin with $\mathcal{S}$, where here, we have $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$.

From page 658, the *pole* is at $-p = -\Delta / \eta\sin(\theta) - \alpha$, so that $Res = exp(-pr) / \sqrt{p^2 + \varepsilon^2}$, and thus R is $2 \cdot Res$. Furthermore, because Δ / η is an invariant, the pole is *also* an invariant; and hence *Res* and R are invariants, as well.

From the *definition* of $g_{u,v}$, we set $\epsilon = \eta\sin(\theta)$ and let $\mu = \alpha \cdot \epsilon + \Delta$, so that the harmonic expression $\mathcal{H}_x$ now computes to …

$$2\kappa\int_{\varepsilon}^{\infty} \{\cos(yr)[1/(\mu + iy\epsilon) - 1/(\mu - iy\epsilon)] + \boldsymbol{i}\sin(yr)[1/(\mu + iy\epsilon) + 1/(\mu - iy\epsilon)]\}\, dy \,/\sqrt{y^2 - \varepsilon^2}$$

Factoring out $1 / \eta$ in $\mathcal{H}_x$ thus gives us an *integrand* where $\eta = 1$ in ϵ, and Δ maps to Δ / η, where η is what we had in $\mathcal{H}_x$. And so, it must be the case that $H_x = \eta \cdot \mathcal{H}_x$, where H_x is now the harmonic expression associated with these new mappings for η and Δ. And similarly for $\mathcal{E} = (1 / \mu) \cdot J_0(r\varepsilon)$, where now $E = \eta \cdot \mathcal{E}$.

Our recent notes [p 670 *ff.*] seem to indicate that $\eta =1$ gives us better accuracy as r ---› 0 in Q1, where Q1 is defined to be $r > 0$ and θ between 0 and $\pi / 2$ [pp 670-3, 675-6]. Supposing this to be true if Δ / η is an *invariant*, and Δ is sufficiently large relative to η; we thus have $H_x + R \approx E$ in this case [p 676]. And indeed, for this to be true for any $\eta > 0$, would mean $\mathcal{H}_x + R \approx \mathcal{E}$; or equivalently, the expression $H_x / \eta + R \approx E / \eta$ must hold true, since R is an *invariant* here. Thus, we must have η equal to 1, and so we see the specialness of $\eta = 1$, in this case. Similar remarks apply to the *timelike* component $\mathcal{T}$ in Q1 … ..

The key here is to see that if Δ / η is an *invariant*, and *if* for some suitably large Δ relative to η it is the case that $H_x + R \approx E$ [where $\eta = 1$, r is small and R is *not* negligible], then $\eta = 1$ is *unique*. For if in this case $\mathcal{H}_x + R \approx \mathcal{E}$ were to hold true for some $\eta > 0$, then as we have shown above, necessarily η must be equal to 1.

The result is important because it *narrows* our search in $\mathcal{S}$ and $\mathcal{T}$, to a search for Δ, *if* we believe that reasonable to high accuracy, as r ---› 0, should be given *due* consideration when selecting a value for η [p 676].

A Simple Case Study, Part LXIII

The following table compares the harmonic expression $\mathcal{H}_x$ + R with $\mathcal{E}$, for the *spacelike* component [$\mathcal{S}$] in $\mathcal{R}^2$, which is *now* $g_{u,v}$ = 1 / ($r \cdot \eta$sin(θ) + Δ). In this table, we compare $\mathcal{H}_x$ + R with $\mathcal{E}$ by writing it as $\mathcal{H}_x$ + R | $\mathcal{E}$.

Here η = 1 and Δ = 300; and the angles are shown below, with singularities at (±δ, 0); where δ = 1, α = cos(θ), ε = sin(θ) [similar results hold in $\mathcal{R}^3$, and *also* hold true for the *timelike* component [$\mathcal{T}$], defined to be 1 / ($r \cdot \eta$cos(θ) + Δ)]. We *also* recall that for $\mathcal{S}$, testing is *exact* at θ = 0 for all r > 0, with $\mathcal{H}_x$ = $\mathcal{E}$ = 1 / Δ, since there is *no* pole here and thus R = 0.

Note that *even* at the threshold of r_0 ~ .01, we *still* achieve high accuracy, which now continues for all $r > r_0$ and *any* angle θ. Indeed, even out to *six* decimal places high accuracy persists, as it should. Note that at r = .001, we lose high accuracy, particularly in the range π / 6 < θ < π / 3; but high accuracy returns for this radius as θ ---› 0 or θ ---› π / 2. Also recall that if Δ / η is an *invariant*, better accuracy is achieved as r ---› 0, if η = 1 [p 676].

r	θ = π / 6	θ = π / 4	θ = π / 3	θ = π / 2
.001	.00150 | .00333	.00205 | .00333	.00270 | .00333	.00333 | .00333
.01	.00332 | .00333	.00330 | .00333	.00330 | .00333	.00333 | .00333
.1	.00333 | .00333	.00332 | .00332	.00332 | .00332	.00332 | .00332
1	.00312 | .00312	.00292 | .00292	.00273 | .00273	.00256 | .00255
2	.00255 | .00255	.00186 | .00186	.00126 | .00126	.00075 | .00075
3	.00171 | .00170	.00005 | .00005	–.00032 | –.00032	–.00087 | –.00086
4	.00075 | .00075	–.00006 | –.00006	–.00125 | –.00125	–.00132 | –.00132
5	–.00016 | –.00016	–.00128 | –.00128	–.00118 | –.00118	–.00059 | –.00059
6	–.00086 | –.00086	–.00123 | –.00123	–.00037 | –.00037	.00050 | .00050
7	–.00126 | –.00126	–.00065 | –.00065	.00055 | .00055	.00100 | .00100
8	–.00132 | –.00132	.00015 | .00015	.00099 | .00099	.00057 | .00057
9	–.00107 | –.00107	.00077 | .00077	.00072 | .00072	–.00030 | –.00030
10	–.00059 | –.00059	.00100 | .00100	0 | 0	–.00082 | –.00082
20	–.00082 | –.00082	.00005 | .00005	–.00043 | –.00043	.00056 | .00056
30	–.00005 | –.00005	0 | 0	.00052 | .00052	–.00029 | –.00029

A Simple Case Study, Part LXIV

In our testing so far, we have restricted ourselves to the range $0 \leq \eta \leq 1$; and indeed, the statement 'if Δ / η is an *invariant*, better accuracy is achieved as r ---› 0, if $\eta = 1$' seems to hold true here, as we can see in the table below. In this table, for the *spacelike* component; $r = .001$, $\theta = \pi / 4$ and η, Δ vary in such a way that Δ / η does *not* change. Going forward, we'll define $\boldsymbol{\Delta}$ to be $\mathcal{H}_x + \mathrm{R} - \mathcal{E}$ …

η	Δ	$\mathcal{H}_x + \mathrm{R} - \mathcal{E}$
.05	15	–.08394
.10	30	–.04043
.50	150	–.00562
.75	225	–.00272
1	300	–.00127
1.5	450	.00018
2.0	600	.00090
3.0	900	.00163

Note that up to $\eta = 1$, $\boldsymbol{\Delta}$ does, in fact, *decrease*; and were we to stop here, our statement above concerning accuracy, would be true for this value of r. But notice now, in the table, that at $\eta \sim 1.5$ the value of $\boldsymbol{\Delta}$ is fairly close to *zero*; so that for this choice of $\eta > 1$ [roughly], we achieve high accuracy.

However, from the table on page 673, we *also* know that at $\theta = \pi / 2$, high accuracy can only be achieved as r ---› 0, if $\eta = 1$ and $\Delta = 300$ [in this case $\boldsymbol{\Delta} \sim 0$ if $r = .001$]. And indeed, were we to let $\eta = 1.5$ and $\Delta = 450$ in that table [p 673], we would find that $\boldsymbol{\Delta} \sim .00165$, if $r = .001$.

Thus, the statement 'if Δ / η is an *invariant*, better accuracy is achieved as r ---› 0, if $\eta = 1$' is really only *consistent* with the range $0 \leq \eta \leq 1$. As to whether or not we want to include $\eta > 1$ in our study of ideal *spacelike* and *timelike* estimators, where Δ / η is an *invariant*; that is another matter. For here, the consistency we just spoke of when $0 \leq \eta \leq 1$, no longer applies …

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Finally, if $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$ is our *spacelike* component, and Δ / η is an *invariant*; then if $\mathcal{N}$ is the *normalized* variant of $g_{u,v}$; that is to say, $\mathcal{N} = 1 / (r \cdot \sin(\theta) + \Delta / \eta)$; then $g_{u,v} > \mathcal{N}$ if $\eta < 1$ and $g_{u,v} < \mathcal{N}$ if $\eta > 1$. Only when $\eta = 1$ do these inequalities vanish, if $0 \leq \theta \leq \pi / 2$. Similar remarks apply to the *timelike* component $1 / (r \cdot \eta\cos(\theta) + \Delta)$, if $0 \leq \theta \leq \pi / 2$. Recall again, that for the *spacelike* component, testing is *exact* at $\theta = 0$, and for the *timelike* component, testing is *exact* at $\theta = \pi / 2$; since in both cases there is no *pole*, and thus the R value is *zero*.

A Simple Case Study, Part LXV

We have learned from our more recent studies, that as r ---› 0, *high* accuracy holds for the *spacelike* component [$\mathcal{S}$] at θ = π / 2, if η = 1. Similarly, as r ---› 0, *high* accuracy holds for the *timelike* component [$\mathcal{T}$] at θ = 0, if η = 1. Going forward, we shall suppose Δ / η is an *invariant*.

Since testing is *exact* for $\mathcal{S}$ at θ = 0, and *exact* for $\mathcal{T}$ at θ = π / 2, we'll simply define η to be 1 here, since the choice of η really doesn't matter in this case. Now we could ask if for *any* θ between 0 and π / 2, there exists a *fixed* η and a *fixed* Δ, such that Δ / η is an *invariant* and as r ---› 0, *high* accuracy holds, for both $\mathcal{S}$ and $\mathcal{T}$.

The answer is roughly *yes*, as we can see in the table below, where for $\mathcal{S}$; r = .0001, θ = π / 4 and η, Δ vary in such a way that Δ / η does *not* change [going forward, we'll define **Δ** to be $\mathcal{H}_x$ + R – $\mathcal{E}$].

η	Δ	$\mathcal{H}_x$ + R – $\mathcal{E}$
.05	15	–.12306
.10	30	–.05927
.50	150	–.00825
.75	225	–.00399
1	300	–.00187
1.5	450	.00026
2.0	600	.00132
3.0	900	.00238

For here we see that if r = .0001, **Δ** ~ 0 if $\eta \sim \sqrt{2}$; and similarly on the previous page – if r = .001, **Δ** ~ 0 if $\eta \sim \sqrt{2}$. In fact, for η = 1 / *sin*(θ), high accuracy will hold for *any* 0 < θ ≤ π / 2, as r ---› 0, where again, the choice of η is arbitrary if θ = 0 [and this is the *approximate* generalization of page 670]. And in this specific case; **Δ** ~ $2 \cdot 10^{-7}$ if r = .001 and **Δ** ~ $3 \cdot 10^{-8}$ if r = .0001, when $\eta \sim \sqrt{2}$.

But the problem with this approach is that η now varies with θ; and indeed, as θ ---› 0, η(θ) ---› ∞. And this means Δ ---› ∞, since Δ / η is an *invariant*. So while we could choose η to be $\sqrt{2}$, for $\mathcal{S}$ in Q1, where r > 0 and 0 ≤ θ ≤ π / 2; we are now *also equally* justified in choosing η = 1 / *sin*(θ) for *any* 0 < θ ≤ π / 2, within reason. And once we make this choice, η is *fixed* for $\mathcal{S}$ in *all* of Q1, where $\mathcal{S}$ is defined to be 1 / ($r \cdot \eta$sin(θ) + Δ). And a similar argument applies to $\mathcal{T}$ = 1 / ($r \cdot \eta$cos (θ) + Δ).

So is there a 'best choice' for the constant η, given what we now know ? It does *not* appear there is, unless we appeal to the *theoretical* argument on page 670, which is *exact* for the *timelike* component as the angle θ ---› 0. And in this case, η = 1 as r ---› 0 …

A Simple Case Study, Part LXVI

To see the power of $\eta = 1$ in action, for the *spacelike* component [$\mathcal{S}$] when θ = π / 2, and the *timelike* component [$\mathcal{T}$] when θ = 0; some snapshots have been included below, when r = .000001, $\eta = 1$ and Δ = 300. Only here, at these angles, is it true that *high* accuracy is achieved as *r* ---› 0; and indeed, $\eta = 1$ is *unique* if Δ / η is an *invariant* [p 678].

$$\int_1^{10\,000\,000} -\frac{i\left(\left(-\frac{1}{300-iy}+\frac{1}{300+iy}\right)\cos\left(\frac{y}{1\,000\,000}\right)+i\left(\frac{1}{300-iy}+\frac{1}{300+iy}\right)\sin\left(\frac{y}{1\,000\,000}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy =$$

$$-0.00333131 + 0\,i$$

$$2\times\frac{\exp(-300\times 1\times 10^{-6})}{\sqrt{300^2+1}}$$

Result

0.006664629941…

$$\frac{J_0(1\times 10^{-6})}{300}$$

Result

0.00333333…

For $\mathcal{S}$, just above, $\mathcal{H}_x$ + R computes to .00333332, whilst $\mathcal{E}$ = .00333333 [below $\mathcal{H}_x$ and to the *left* is R, and to the *right* is $\mathcal{E}$].

$$\int_0^{10\,000\,000} \frac{\cos(1\times 10^{-6}\,y)\left(\frac{1}{301+iy}-\frac{1}{301-iy}\right)+i\sin(1\times 10^{-6}\,y)\left(\frac{1}{301+iy}+\frac{1}{301-iy}\right)}{(\pi\,i)\sqrt{y^2-0}}\,dy =$$

$$-0.00332026$$

Input

$$2\left(\frac{1}{301}\exp(-301\times 1\times 10^{-6})\right)$$

Result

0.006642518573…

$$\frac{1}{301} \quad \text{(irreducible)}$$

Decimal approximation

0.003322259136212

For $\mathcal{T}$, just above, $\mathcal{H}_x$ + R computes to .003322259, whilst $\mathcal{E}$ = .003322259 [below $\mathcal{H}_x$ and to the *left* is R, and to the *right* is $\mathcal{E}$] … ..

A Simple Case Study, Part LXVII

A mistake was made in the calculations for the table on page 681, which has been corrected. I would encourage the reader to reread this page; for there we asked the question ‘if for *any* θ between 0 and π / 2, there exists a *fixed* η and a *fixed* Δ, such that Δ / η is an *invariant* and as the radius *r* ---› 0, *high* accuracy holds, for both $\mathcal{S}$ and $\mathcal{T}$’.

The original calculations led me to believe the answer was *no*, but the revised calculations indicate that the answer is roughly *yes*. In turn, this means that within reason, there is *no preferred* choice of η when considering high accuracy as *r* ---› 0; unless we defer *strictly* to the theoretical proof on page 670, where it is shown that $\eta = 1$ when $\theta = 0$, for the *timelike* component.

The revision on page 681, in particular, gives consideration to the *spacelike* component [$\mathcal{S}$], where now $\eta = 1 / \mathit{sin}(\theta)$ is some *fixed* value in $\mathcal{S} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$, for all of Q1.

Both the tables and the calculations on pages 680 and 681 have been checked and rechecked for accuracy, and I am satisfied at this point that they are correct …

Corrections

When dealing with a function $f(z)$ that has a *pole* at z_0, the *residue* is actually defined to be

$$\lim_{z \to z_0} (z - z_0) f(z)$$

Thus, $f(z)$ must be *properly* factored to compute this limit; and what this means for us when looking at the *generating* function for the *spacetime* component [$\boldsymbol{S}$]; namely

$$f(s) = g_{u,v} \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2} = 1 / ((s + \delta\alpha) \cdot \eta\sin(\theta) + \Delta) \cdot 1 / \sqrt{s^2 + (\delta\varepsilon)^2},$$

is that $Res = exp(-pr) / \eta \sin(\theta) \cdot \sqrt{p^2 + \varepsilon^2}$, if $\delta = 1$. Similarly for the *timelike* component [$\mathcal{T}$], where $\sin(\theta)$ is replaced with $\cos(\theta)$ in *Res*. Here $\boldsymbol{S}$ is defined to be $1 / (r \cdot \eta\sin(\theta) + \Delta)$ and $\mathcal{T}$ is defined to be $1 / (r \cdot \eta\cos(\theta) + \Delta)$.

Now beginning on page 559 and going *forward* from there, the term $\eta\sin(\theta)$ was omitted in *Res* for $\boldsymbol{S}$, as was the term $\eta\cos(\theta)$ in $\mathcal{T}$, regrettably. But including this term in *Res* actually doesn't affect the calculations very much, if at all, for pages 559-655; and has *no effect* on residue calculations *prior* to page 559.

Indeed, most of our computations [pp 559-655] set $\theta = \pi / 2$ for $\boldsymbol{S}$ and $\theta = 0$ for $\mathcal{T}$; where in *all* cases η was equal to 1 [the variable η only took on other values, starting at page 656]. Now from pages 640-655, *all* of the tables involved testing where the R value was *negligible* [R = 2 · *Res*]; and this won't change under the *new* definition for *Res* above. And so, our starting point for any fixes is page 656, which we shall now look at.

The *first* table of interest to us is on page 663, and within that table, the first *two* lines, as shown below ...

r	Δ	$\mathcal{H}_x$ + R	$\mathcal{E}$
.001	30	.0300	.0333
.002	30	.0332	.0333

Under the *new* definition for *Res*, and thus R, $\mathcal{H}_x$ + R now computes to .0333, which *exactly* matches $\mathcal{E}$, for *both* lines above. All other lines remain the same, because R is now negligible. In fact, even at $r = .000001$, the match is *exact* here, thus showing us that high accuracy as r ---› 0 does *not* depend on any particular value of η [in this case $\eta = .01$ and $\Delta = 30$]. And similarly for the table on page 664.

———

The following table compares the harmonic expression $\mathcal{H}_x$ + R with $\mathcal{E}$, for the *spacelike* component in $\mathcal{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here η varies, $\Delta = 30$, and the data indicates *at least* how *low* we can go with *r*, and still achieve *reasonable* accuracy, for a given η.

The angle is $\theta = \pi / 2$, with singularities at $(\pm\delta, 0)$; and $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$]. This table was copied from page 666, but has now been updated for the *new* version of *Res*, and thus R. You can see just how accurate the data is, *even* down to *r* = .000001 …

r	η	$\mathcal{H}_x$ + R	$\mathcal{E}$
.000001	.01	.0333	.0333
.000001	.02	.0333	.0333
.000001	.03	.0333	.0333
.000001	.04	.0333	.0333
.000001	.10	.0333	.0333
.000001	.20	.0333	.0333
.000001	.90	.0333	.0333
.000001	1	.0333	.0333
.000001	2	.0333	.0333
.000001	3	.0332	.0333
.000001	4	.0330	.0333
.000001	5	.0329	.0333

———

The proof on page 670, while interesting, must now change. And here, using the syntax on that page, the new *Res* tends to $1 / \eta p = 1 / \mu$, so that R tends to $2 / \mu$ as $r \dashrightarrow 0$. Thus, we are simply left with the identity $-1 / \mu + 2 / \mu = 1 / \mu$; and hence there is *no* preference for η, when considering high accuracy as the radius *r* shrinks to *zero*. In this case, we are dealing with the *timelike* component, where the angle $\theta = 0$, but you can also see the effect in the table above for the *spacelike* component.

Notice too, that in the tables on pages 670-673, $\eta = 1$ and $\theta = \pi / 2$, for the *spacelike* element, so that the old and new *Res* are one and the same thing. The comments at the bottom of page 673 no longer apply, using the new *Res*.

———

On page 675, we show some snapshots from Wolfram for $\mathcal{S}$, when $\delta = 1$, $\theta = \pi / 4$, $r = .001$, $\eta = 1$ and $\Delta = 300$. Here $\mathcal{S}$ is the *spacelike* component; and the old *Res* is being used there. Using the new *Res*, the R value now becomes ~ .00435133, since we now multiply the old *Res* by $\sqrt{2}$. In so doing, $\mathcal{H}_x$ + R ~ .003328 and $\mathcal{E}$ ~ .003328. The two are now identical for this angle and a relatively small *r*, and it should be pointed out that there is *no* change to the way we calculate $\mathcal{H}_x$ and $\mathcal{E}$.

———

The comments on page 676 are no longer valid using the new *Res*; and speaking of residues, the calculation on page 677 was done correctly. Page 678 speaks about the specialness of $\eta = 1$, when considering high accuracy as r ---› 0; but these comments are also no longer valid.

———

On page 679, we prepared a rather detailed table comparing $\mathcal{H}_x$ + R with $\mathcal{E}$, by writing this using the expression $\mathcal{H}_x$ + R | $\mathcal{E}$. A portion of the table is shown below for the *spacelike* component [S] in $\mathcal{R}^2$, which is *now* $g_{u,v} = 1 / (r \cdot \eta\sin(\theta) + \Delta)$. Here $\eta = 1$ and $\Delta = 300$, and we note that this data was compiled using the old *Res*. But the *second* table shows what happens when we use the new *Res* ...

r	$\theta = \pi / 6$	$\theta = \pi / 4$	$\theta = \pi / 3$	$\theta = \pi / 2$
.001	.00150 | .00333	.00205 | .00333	.00270 | .00333	.00333 | .00333

r	$\theta = \pi / 6$	$\theta = \pi / 4$	$\theta = \pi / 3$	$\theta = \pi / 2$
.001	.00333 | .00333	.00333 | .00333	.00333 | .00333	.00333 | .00333

Here are some snapshots from Wolfram showing us what happens at $\theta = \pi / 3$ in the *second* table, just above, where the R value is below $\mathcal{H}_x$ and to the *left*; whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. In this case $\mathcal{H}_x$ + R ~ .00332839 and $\mathcal{E}$ ~ .00332853. Note that the R value has now been divided by $\sqrt{3}$ / 2.

$$\int_{\frac{\sqrt{3}}{2}}^{1\,000\,000} \frac{1}{(\pi i)\sqrt{y^2 - 0.75}} \left(\cos(0.001 y)\left(\frac{1}{300.433 + 0.866 i y} - \frac{1}{300.433 - 0.866 i y}\right) + i \sin(0.001 y)\left(\frac{1}{300.433 + 0.866 i y} + \frac{1}{300.433 - 0.866 i y}\right)\right) dy = -0.00137708$$

$$\left(2 \times \frac{2}{\sqrt{3}}\right) \times \frac{\exp\left(-\frac{1}{0.866}\,(300.433 \times 0.001)\right)}{\sqrt{\left(\frac{1}{0.866} \times 300.433\right)^2 + 0.75}}$$

Result

0.00470547…

$$\frac{J_0\left(0.001 \times \frac{\sqrt{3}}{2}\right)}{300.433}$$

Result

0.00332853…

Thus, our s*pacelike* and *timelike* estimators now demonstrate high accuracy as r ---› 0, and *also* demonstrate high accuracy as the radius r increases, away from the origin. And furthermore, the choice of η is arbitrary, so long as Δ is sufficiently large, relative to η.

———

The tables on pages 680-681 are no longer useful to us, as they were calculated using the old *Res*. But they did inspire me to go back and check my calculations on pages 559-62, where it all started ; for in these tables I saw patterns in the numbers, that indicated to me something was amiss with the way *Res* was being calculated. The calculations on page 682 are acceptable, because there the old and new *Res* agree. Page 683 is simply some commentary on where I was with my thoughts, on that day …

———

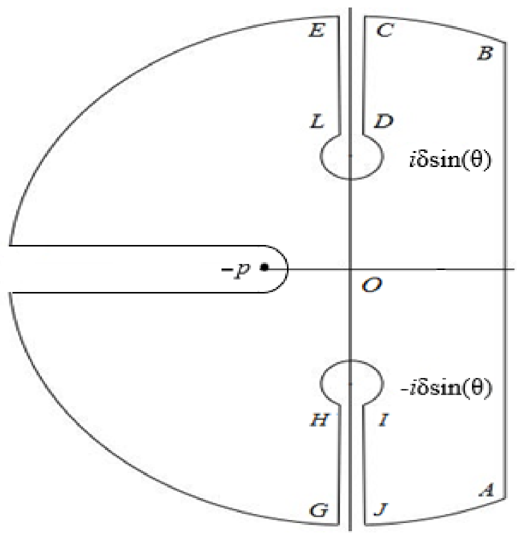
Note that the *invariance* principle for the harmonic equation $\boldsymbol{\mathcal{H}_x}$ + R ≈ $\boldsymbol{\mathcal{E}}$ on pages 642-6, still holds for the new *Res* = *exp*(–*pr*) / ηsin(θ) · $\sqrt{p^2 + \varepsilon^2}$, in the case of the *spacelike* component [$\boldsymbol{\mathcal{S}}$], if δ = 1. If η or θ ---› 0, the *dominant* term in the pole –p = –Δ / ηsin(θ) – α is –Δ / ηsin(θ), so that the denominator in *Res* is now ~ Δ. The numerator in *Res* now tends to *zero*, and thus, so does the R value.

Thus, there are no convergence issues with *Res* here, and similarly for the *timelike* component [$\mathcal{T}$], where *Res* = *exp*(–*pr*) / ηcos(θ) · $\sqrt{p^2 + \varepsilon^2}$, and η ---› 0 or θ ---› π / 2. In this particular case, the pole –p = –Δ / ηcos(θ) – α, if δ = 1. We presume, of course, that Δ > 0.

Are $1 / (r \cdot \eta\sin(\theta) + \Delta)^{\beta}$ and $1 / (r \cdot \eta\cos(\theta) + \Delta)^{\beta}$ Ideal Estimators For All $\beta > 0$

In this note, we wish to investigate the possibility that $1 / (r \cdot \eta\sin(\theta) + \Delta)^{\beta}$ and $1 / (r \cdot \eta\cos(\theta) + \Delta)^{\beta}$ are *ideal* estimators, for *any* real number $\beta > 0$, provided Δ is sufficiently large, relative to η. We now know this is so if $\beta = 1$, in so much as high accuracy holds true for $r > 0$ and $0 \le \theta \le \pi / 2$; where the choice of η and Δ are *arbitrary*, but *greater* than *zero* [pp 684-7].

Let us begin with $\beta = ½$, where $\eta = 1$, $\Delta = 300$, $\theta = \pi / 2$, and $r = .001$, for the *spacelike* estimator [$\mathcal{S}$]. In this case, we use the example on pages 556-7, according to the contour below, where the *branch* point $-p = -\Delta$, $\delta = 1$, and $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$. Thus, the R value is the integration over the *horizontal* branch lines, *doubled*. Again, $\mathcal{S} = 1 / (r \cdot \eta\sin(\theta) + \Delta)^{\beta}$ and $\mathcal{T} = 1 / (r \cdot \eta\cos(\theta) + \Delta)^{\beta}$.



$\boxed{\varepsilon = \sin(\theta)}$

$\boxed{p = \delta\alpha + \Delta}$

Here are the snapshots from Wolfram, where R is below $\mathcal{H}_x$ and to the *left*, whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. Now $\mathcal{H}_x$ + R ~ .0577343 and $\mathcal{E}$ ~ .0577350, and the two are remarkably close; so this is our *first* hint that we have a *family* of *ideal* estimators for $\mathcal{S}$ and $\mathcal{T}$, for all $\beta > 0$.

$$\int_1^{1\,000\,000} -\frac{i\left(\left(-\frac{1}{\sqrt{300-iy}}+\frac{1}{\sqrt{300+iy}}\right)\cos\left(\frac{y}{1000}\right)+i\left(\frac{1}{\sqrt{300-iy}}+\frac{1}{\sqrt{300+iy}}\right)\sin\left(\frac{y}{1000}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy =$$

0.00709185

$$\frac{J_0(0.001)}{\sqrt{300}}$$

$$\int_0^{10\,000} \frac{2\,e^{(-300-x)/1000}}{\pi\sqrt{x}\,\sqrt{1+(300+x)^2}}\,dx = 0.0506424$$

Result

0.0577350…

Now we'll set $r = 2$, and repeat the testing. In this case the R value is negligible, and here are the snapshots for $\mathcal{H}_x$ and $\mathcal{E}$, which also agree rather well.

$$\int_1^{16000} -\frac{i\left(\left(-\frac{1}{\sqrt{300-iy}}+\frac{1}{\sqrt{300+iy}}\right)\cos(2y)+i\left(\frac{1}{\sqrt{300-iy}}+\frac{1}{\sqrt{300+iy}}\right)\sin(2y)\right)}{\pi\sqrt{-1+y^2}}\,dy =$$

0.0129818

$$\frac{J_0(2)}{10\sqrt{3}}$$

Decimal approximation

0.012926340160626

Now we'll set $\beta = 2$ and $r = .001$, with all other parameters the same as above. In this case the generating function is …

$$f(s) \;=\; g_{u,v} \cdot 1/\sqrt{s^2+(\delta\varepsilon)^2} \;=\; 1/(s+\Delta)^2 \cdot 1/\sqrt{s^2+1}\,,$$

and the pole p at $-\Delta$ is of order 2; so *Res* is calculated as lim $s \dashrightarrow p$ $\{(s-p)^2 f(s)\,exp(sr)\}'$, where the tick means differentiate with respect to s. And this computes to

$$Res \;=\; [r+\Delta/(\Delta^2+1)]\cdot[1/\sqrt{\Delta^2+1}\,]\cdot exp(-\Delta r)\,.$$

Here are the snapshots from Wolfram, where R is below $\mathcal{H}_x$ and to the *left*, whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. Now $\mathcal{H}_x$ + R ~ .0000111109, which is remarkably close to $\mathcal{E}$ …

$$\int_1^{1000000} -\frac{i\left(\left(-\frac{1}{(300-iy)^2}+\frac{1}{(300+iy)^2}\right)\cos\left(\frac{y}{1000}\right)+i\left(\frac{1}{(300-iy)^2}+\frac{1}{(300+iy)^2}\right)\sin\left(\frac{y}{1000}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy =$$

$-0.0000102902 + 0\,i$

$$2\left(\left(0.001+\frac{300}{300^2+1}\right)\times\frac{1}{\sqrt{300^2+1}}\exp(-300\times 0.001)\right)$$

Result

0.0000214011…

$$\frac{J_0(0.001)}{300^2}$$

Result

0.0000111111…

And here are the images when β = 2 and *r* = 2, with all other parameters the same. In this case, the R value is *negligible*. Note that the agreement is fairly decent for the integration range, which is as high as I could go on Wolfram for this value of *r* …

$$\int_1^{16000} -\frac{i\left(\left(-\frac{1}{(300-i y)^2}+\frac{1}{(300+i y)^2}\right)\cos(2 y)+i\left(\frac{1}{(300-i y)^2}+\frac{1}{(300+i y)^2}\right)\sin(2 y)\right)}{\pi\sqrt{-1+y^2}}\,dy =$$

$$2.53042\times 10^{-6}$$

$$\frac{J_0(2)}{90000}$$

Decimal approximation

$$2.4876753237915074567615\ldots\times 10^{-6}$$

Thus, at this point, from the empirical evidence above, we can say with reasonable confidence that $\boldsymbol{S} = 1 / (r \cdot \eta\sin(\theta) + \Delta)^{\beta}$ and $\mathcal{T} = 1 / (r \cdot \eta\cos(\theta) + \Delta)^{\beta}$ are very likely *ideal* estimators for all $\beta > 0$, if Δ is sufficiently large relative to η …

A Few More Things

Here is the *complete* table from page 679, using the new *Res* [pp 684-7]; which compares the harmonic expression $\mathcal{H}_x$ + R with $\mathcal{E}$, for the *spacelike* component [S] in $\mathcal{R}^2$. Note that S is defined to be $1 / (r \cdot \eta\sin(\theta) + \Delta)$. In this table, we compare $\mathcal{H}_x$ + R with $\mathcal{E}$ by writing it as $\mathcal{H}_x$ + R | $\mathcal{E}$.

Here $\eta = 1$ and $\Delta = 300$; and the angles are shown below, with singularities at $(\pm\delta, 0)$; where $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$ [similar results hold in $\mathcal{R}^3$, and *also* hold true for the *timelike* component [$\mathcal{T}$], defined to be $1 / (r \cdot \eta\cos(\theta) + \Delta)$]. We *also* recall that for S, testing is *exact* at $\theta = 0$ for all $r > 0$, with $\mathcal{H}_x = \mathcal{E} = 1 / \Delta$, since there is *no* pole here and thus R = 0.

Note that high accuracy will persist, even down to $r \sim .000001$ [and likely lower], using the new *Res* [pp 684-7]; and that the choice of $\eta > 0$ is *arbitrary*, so long as $\Delta > 0$ is sufficiently large, relative to η. Also note that at $r \sim .03$ [and higher], the R value is *negligible*.

r	$\theta = \pi / 6$	$\theta = \pi / 4$	$\theta = \pi / 3$	$\theta = \pi / 2$
.001	.00033 | .00333	.00033 | .00333	.00033 | .00333	.00333 | .00333
.01	.00333 | .00333	.00333 | .00333	.00333 | .00333	.00333 | .00333
.1	.00333 | .00333	.00332 | .00332	.00332 | .00332	.00332 | .00332
1	.00312 | .00312	.00292 | .00292	.00273 | .00273	.00256 | .00255
2	.00255 | .00255	.00186 | .00186	.00126 | .00126	.00075 | .00075
3	.00171 | .00170	.00005 | .00005	–.00032 | –.00032	–.00087 | –.00086
4	.00075 | .00075	–.00006 | –.00006	–.00125 | –.00125	–.00132 | –.00132
5	–.00016 | –.00016	–.00128 | –.00128	–.00118 | –.00118	–.00059 | –.00059
6	–.00086 | –.00086	–.00123 | –.00123	–.00037 | –.00037	.00050 | .00050
7	–.00126 | –.00126	–.00065 | –.00065	.00055 | .00055	.00100 | .00100
8	–.00132 | –.00132	.00015 | .00015	.00099 | .00099	.00057 | .00057
9	–.00107 | –.00107	.00077 | .00077	.00072 | .00072	–.00030 | –.00030
10	–.00059 | –.00059	.00100 | .00100	0 | 0	–.00082 | –.00082
20	–.00082 | –.00082	.00005 | .00005	–.00043 | –.00043	.00056 | .00056
30	–.00005 | –.00005	0 | 0	.00052 | .00052	–.00029 | –.00029

Some additional testing was done on the *spacelike* component $S = 1 / (r^v \cdot \eta\sin(\theta) + \Delta)$, and *also* on the component $S = 1 / (r \cdot \eta\sin^v(\theta) + \Delta)$, where $v > 0$. In the *first* case, testing was *not* acceptable [or free from artificial constraints (p 677) when calculating *Res*] if $v \neq 1$; but in the *second* case, high accuracy was achievable on the testing that was done, when $\theta = \pi / 4$.

Here are some snapshots from Wolfram, for the *second* case, when $\eta = 1$, $\Delta = 300$, and $\nu = 2$; with singularities at $(\pm\delta, 0)$, where $\delta = 1$, $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$. Note that the pole $-p$ is $-\Delta / \sin^{\nu}(\theta) - \alpha$, and that *Res* computes to $exp(-pr) / \eta\sin^{\nu}(\theta) \cdot \sqrt{p^2 + \varepsilon^2}$, if $\delta = 1$ [pp 684-7]. In these snapshots, just below; $r = .01$, and R is below $\mathcal{H}_x$ and to the *left*; whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. In this case $\mathcal{H}_x$ + R ~ .00332939, which is very close to $\mathcal{E}$ ~ .00332937.

$$\int_{\frac{1}{\sqrt{2}}}^{1\,000\,000} \frac{\cos(0.01\,y)\left(\frac{1}{300.353+0.5\,i\,y} - \frac{1}{300.353-0.5\,i\,y}\right) + i\sin(0.01\,y)\left(\frac{1}{300.353+0.5\,i\,y} + \frac{1}{300.353-0.5\,i\,y}\right)}{(\pi\,i)\sqrt{y^2 - 0.5}}\,dy = 0.003313$$

$$\frac{2\exp(-600.707 \times 0.01)}{\frac{1}{2}\sqrt{600.707^2 + 0.5}}$$

Result

0.0000163893…

$$\frac{J_0\left(0.01 \times \frac{\sqrt{2}}{2}\right)}{300.353}$$

Result

0.00332937…

When $r = 2$, the R value is *negligible*, and here are the snapshots for $\mathcal{H}_x$ and $\mathcal{E}$, where we see that the agreement is fairly decent …

$$\int_{\frac{1}{\sqrt{2}}}^{16\,000} \frac{\cos(2\,y)\left(\frac{1}{300.353+0.5\,i\,y} - \frac{1}{300.353-0.5\,i\,y}\right) + i\sin(2\,y)\left(\frac{1}{300.353+0.5\,i\,y} + \frac{1}{300.353-0.5\,i\,y}\right)}{(\pi\,i)\sqrt{y^2 - 0.5}}\,dy = 0.00186372$$

$$\frac{J_0\left(2 \times \frac{\sqrt{2}}{2}\right)}{300.353}$$

Result

0.00186159…

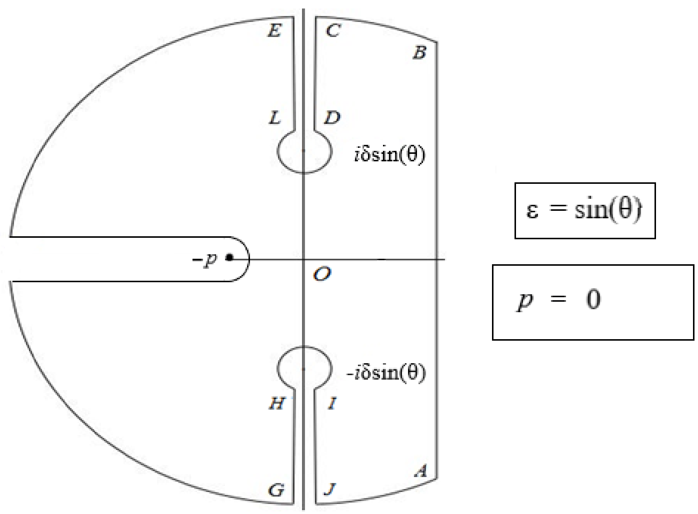
A Few More Things, Part II

In the case of the *spacelike* component, $\boldsymbol{S} = 1 / (r^{v} \cdot \eta\sin(\theta) + \Delta)$, some progress has been made when $v = ½$; and we'll also discuss the case when $v = 2$. The generating function for $\boldsymbol{S}$ is

$$f(s) \ = \ 1 / \{((s + \delta\alpha)^{v} \cdot \eta\sin(\theta) + \Delta) \sqrt{s^2 + (\delta\varepsilon)^2}\},$$

and if $\eta = 1$, $\theta = \pi / 2$, $v = ½$ and $\delta = 1$; this function reduces to

$$f(s) \ = \ 1 / \{(s^{v} + \Delta) \sqrt{s^2 + 1}\}.$$



If we now agree that in the contour γ above, s^{v} is *positive* along the *positive* x-axis, where $v = ½$; then we can use γ to find the R value [a *doubling* of the expression below, along the *horizontal* branch lines], by setting the branch point p to *zero*. We do this because of the s^{v} term in the denominator of $f(s)$.

$$\kappa \int e^{sr} f(s)\, ds$$

In so doing, we find that R computes to

$$(2 / \pi) \int_{0}^{\infty} \sqrt{x}\, exp(-rx) \,/\, ((x + \Delta^2) \sqrt{x^2 + 1}\,)\, dx$$

and here are some snapshots from Wolfram, when $r = .001$ and $\Delta = 300$ …

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In the snapshots that follow; r = .001, and R is below $\mathcal{H}_x$ and to the *left*; whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. In this case $\mathcal{H}_x$ + R ~ .003327, which is close to $\mathcal{E}$ ~ .003333.

$$\int_1^{10\,000\,000} -\frac{i\left(\left(-\frac{1}{300+\sqrt{-i\,y}}+\frac{1}{300+\sqrt{i\,y}}\right)\cos\left(\frac{y}{1000}\right)+i\left(\frac{1}{300+\sqrt{-i\,y}}+\frac{1}{300+\sqrt{i\,y}}\right)\sin\left(\frac{y}{1000}\right)\right)}{\pi\sqrt{-1+y^2}}\,dy = 0.00294503$$

$$\int_0^{10\,000} \frac{2\,e^{-x/1000}\sqrt{x}}{\pi\,(90\,000+x)\sqrt{1+x^2}}\,dx = 0.000382325$$

$$\frac{J_0(0.001)}{300}$$

Result

0.00333333…

In the snapshots that follow; r = 1, and R is below $\mathcal{H}_x$ and to the *left*; whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. In this case $\mathcal{H}_x$ + R ~ .0025502, which is very close to $\mathcal{E}$ ~ .0025506.

$$\int_1^{30\,000} -\frac{i\left(\left(-\frac{1}{300+\sqrt{-i\,y}}+\frac{1}{300+\sqrt{i\,y}}\right)\cos(y)+i\left(\frac{1}{300+\sqrt{-i\,y}}+\frac{1}{300+\sqrt{i\,y}}\right)\sin(y)\right)}{\pi\sqrt{-1+y^2}}\,dy = 0.00254625$$

$$\int_0^{12} \frac{2\,e^{-x}\sqrt{x}}{\pi\,(90\,000+x)\sqrt{1+x^2}}\,dx = 3.99946\times 10^{-6}$$

$$\frac{J_0(1)}{300}$$

Decimal approximation

0.002550658955193

Thus, the estimator $\mathcal{S} = 1 / (r^{\nu} \cdot \eta\sin(\theta) + \Delta)$ looks promising if $\nu = ½$; but unfortunately if $\nu = 2$, then with $\theta = \pi / 2$, and $\delta = 1$, *Res* computes to

$$\sqrt{1/\Delta}\cdot sin(\sqrt{\Delta/\eta}\cdot r)\,/\sqrt{\eta-\Delta}\,,$$

thereby placing an *artificial* constraint on Δ, in so much as Δ must now be *less* than η [p 677].

A Few More Things, Part III

On pages 563-5, using a rather *intuitive* approach, we showed that the R value over the branch lines $\mathcal{U}$ and $\mathcal{L}$ (before doubling), computed to the the following, after replacing .707 with $\sqrt{2}/2$ in the *original* integrals on page 564. Here, we wanted to know if $1 / (\sqrt{r^4 + \Delta^4})$ was a suitable template for $C_{u,v} \approx \sigma g_{u,v}(0)$, where $\Delta > 0$; and in this case, $\delta = 0$, $\Delta = 1$, $r = 1$.

$$\int_0^{12} -\frac{i\, e^{-(1-i)/\sqrt{2}-x}}{\pi\left(\frac{1-i}{\sqrt{2}}+x\right)\sqrt{1+\left(\frac{1-i}{\sqrt{2}}+x\right)^4}}\,dx = 0.0581038 + 0.0702193\,i$$

$$\int_0^{12} \frac{i\, e^{-(1+i)/\sqrt{2}-x}}{\pi\left(\frac{1+i}{\sqrt{2}}+x\right)\sqrt{1+\left(\frac{1+i}{\sqrt{2}}+x\right)^4}}\,dx = 0.0581038 - 0.0702193\,i$$

We then went on to say in Part II [pp 570-2], that by using *phases*, the same should hold true; and indeed, on page 571 produced the integrals for $\mathcal{U}$ and $\mathcal{L}$ below, that very closely match the results above in Part I …

$$\int_0^{12} \frac{\exp(-(x+0.707-0.707\,i))}{\pi\left((x+0.707-0.707\,i)\sqrt{(x\,(0.707+0.707\,i)+1)^4-1}\right)}\,dx =$$
$$0.0581305 + 0.0702366\,i$$

$$\int_0^{12} \frac{\exp(-(x+0.707+0.707\,i))}{\pi\left((x+0.707+0.707\,i)\sqrt{(x\,(0.707-0.707\,i)+1)^4-1}\right)}\,dx =$$
$$0.0581305 - 0.0702366\,i$$

And when we replace .707 with $\sqrt{2}/2$, in the integrals just above, we obtain an *exact* match with the *first* set of integrals at the top of the page, in keeping with our comments on page 572 …

$$\int_0^{12} \frac{e^{-(1-i)/\sqrt{2}-x}}{\pi\left(\frac{1-i}{\sqrt{2}}+x\right)\sqrt{-1+\left(1+\frac{(1+i)x}{\sqrt{2}}\right)^4}}\,dx = 0.0581038 + 0.0702193\,i$$

$$\int_0^{12} \frac{e^{-(1+i)/\sqrt{2}-x}}{\pi\left(\frac{1+i}{\sqrt{2}}+x\right)\sqrt{-1+\left(1+\frac{(1-i)x}{\sqrt{2}}\right)^4}}\,dx = 0.0581038 - 0.0702193\,i$$

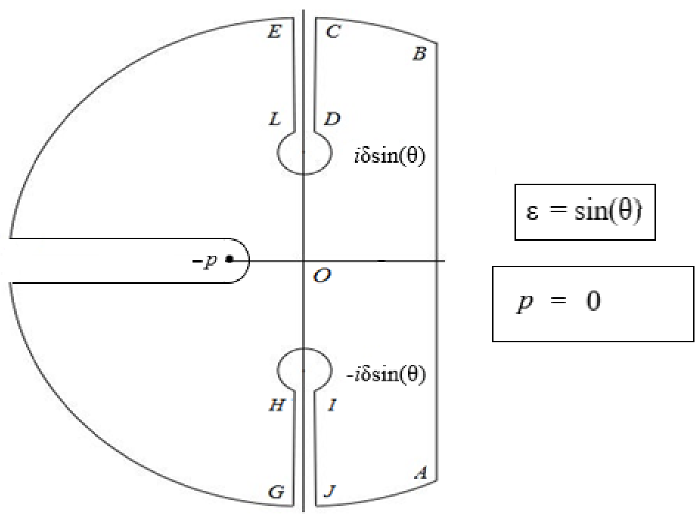
A Few More Things, Part IV

In the case of the *spacelike* component, $\boldsymbol{S} = 1 / (r^{v} \cdot \eta\sin(\theta) + \Delta)$, some progress has been made when $v = ⅓$; and we'll also briefly discuss the case when $v = 3$. The generating function for $\boldsymbol{S}$ is

$$f(s) = 1 / \{((s + \delta\alpha)^{v} \cdot \eta\sin(\theta) + \Delta) \sqrt{s^2 + (\delta\varepsilon)^2}\},$$

and if $\eta = 1$, $\theta = \pi / 2$, $v = ⅓$ and $\delta = 1$; this function reduces to

$$f(s) = 1 / \{(s^{v} + \Delta) \sqrt{s^2 + 1}\}.$$



Since the phases along the *upper* and *lower* horizontal branch lines are π and $-\pi$, respectively; the term $s^{v} + \Delta$ is *not* zero here if $s = -\Delta^3$, when $v = ⅓$. Thus, we can use the contour γ above to find the R value [a *doubling* of the expression below, along the *horizontal* branch lines], by setting the branch point p to *zero*. We do this because of the s^{v} term in the denominator of $f(s)$.

$$\kappa \int e^{sr} f(s)\, ds$$

In so doing, we find that R computes to

$$(2 / \pi) \int_{0}^{\infty} x^{1/3}\, exp(-rx) \sin(\pi/3) \,/\, \left((x^{1/3}\, exp(\boldsymbol{i}\pi/3) + \Delta)\, (x^{1/3}\, exp(-\boldsymbol{i}\pi/3) + \Delta) \sqrt{x^2 + 1}\right) dx$$

and here are some snapshots from Wolfram, when $r = .001$ and $\Delta = 300$ …

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In the snapshots that follow; r = .001, and R is below $\mathcal{H}_x$ and to the *left*; whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. In this case $\mathcal{H}_x$ + R ~ .003326, which is close to $\mathcal{E}$ ~ .003333.

$$\int_1^{10\,000\,000} -\frac{i\left(\left(-\frac{1}{300+\sqrt[3]{-i y}}+\frac{1}{300+\sqrt[3]{i y}}\right)\cos\left(\frac{y}{1000}\right)+i\left(\frac{1}{300+\sqrt[3]{-i y}}+\frac{1}{300+\sqrt[3]{i y}}\right)\sin\left(\frac{y}{1000}\right)\right)}{\pi\sqrt{-1+y^2}}\, dy = 0.00318002$$

$$\frac{J_0(0.001)}{300}$$

$$\int_0^{30\,000} \frac{\sqrt{3}\, e^{-x/1000}\, \sqrt[3]{x}}{\pi\left(300+e^{-(i\pi)/3}\sqrt[3]{x}\right)\left(300+e^{(i\pi)/3}\sqrt[3]{x}\right)\sqrt{1+x^2}}\, dx = 0.00014553$$

Result

0.00333333…

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In the snapshots that follow; r = 1, and R is below $\mathcal{H}_x$ and to the *left*; whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. In this case $\mathcal{H}_x$ + R ~ .0025477, which is close to $\mathcal{E}$ ~ .0025506.

$$\int_1^{30\,000} -\frac{i\left(\left(-\frac{1}{300+\sqrt[3]{-i y}}+\frac{1}{300+\sqrt[3]{i y}}\right)\cos(y)+i\left(\frac{1}{300+\sqrt[3]{-i y}}+\frac{1}{300+\sqrt[3]{i y}}\right)\sin(y)\right)}{\pi\sqrt{-1+y^2}}\, dy = 0.00254407$$

$$\frac{J_0(1)}{300}$$

$$\int_0^{30\,000} \frac{\sqrt{3}\, e^{-x}\, \sqrt[3]{x}}{\pi\left(300+e^{-(i\pi)/3}\sqrt[3]{x}\right)\left(300+e^{(i\pi)/3}\sqrt[3]{x}\right)\sqrt{1+x^2}}\, dx = 3.68181\times 10^{-6}$$

Decimal approximation

0.002550658955193

Thus, the estimator $\mathcal{S} = 1 / (r^{\nu} \cdot \eta\sin(\theta) + \Delta)$ looks promising if ν = ⅓; but if ν = 3, *simple* poles exist when $s^{\nu} + \Delta = 0$. These poles are at $-\Delta^{1/3}$ and $\Delta^{1/3} \cdot exp(\pm \boldsymbol{i}\pi/3)$; and so the first issue we face is whether to include the *complex* poles in the contour γ, by setting Re(AB) > $\Delta^{1/3} \cdot \cos(\pi/3)$, since Δ is always greater than 0. We'd have to test both ways – with and without the *complex* poles – and see if either approach gives us a match for $\mathcal{H}_x$ + R ≈ $\mathcal{E}$. Personally, I'm not optimistic here, and while our testing, for the most part, has been successful for these variants [p 688 *ff*.]; I still believe that the *best* choice for our ideal estimators is the *simplest* choice; namely $1 / (r \cdot \eta\sin(\theta) + \Delta)^{\beta}$ and $1 / (r \cdot \eta\cos(\theta) + \Delta)^{\beta}$, where β is equal to 1 …

Are $1 / (r \cdot \eta\sin(\theta) + \Delta)^{\beta}$ and $1 / (r \cdot \eta\cos(\theta) + \Delta)^{\beta}$ Ideal Estimators For All $\beta > 0$, Part II

In this note, we'll look at the *time* component $\mathcal{T} = 1 / (r \cdot \eta\cos(\theta) + \Delta)^{\beta}$, when $\beta = ½$ and Δ is sufficiently large, relative to η. We now know that if $\beta = 1$, high accuracy holds true here, for $r > 0$ and $0 \leq \theta \leq \pi / 2$; where the choice of η and Δ are *arbitrary*, but *greater* than *zero* [pp 684-7].

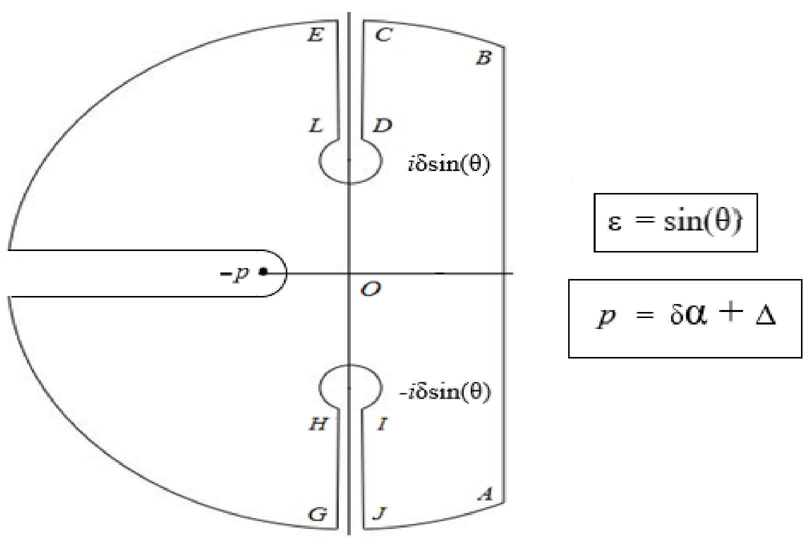
The *generating* function for this case is

$$f(s) \; = \; 1 / \{((s + \delta\alpha) \cdot \eta\cos(\theta) + \Delta)^{\beta} \sqrt{s^2 + (\delta\varepsilon)^2}\},$$

where $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$; and if $\eta = 1$, $\theta = 0$, $\beta = ½$ and $\delta = 1$, this function reduces to

$$f(s) \; = \; 1 / \{(s + \Delta + 1)^{\beta} \sqrt{s^2 + 0}\}.$$

Let us set $\Delta = 300$ and $r = .001$, for the *timelike* estimator [$\mathcal{T}$]. In this case, then, the *branch* point is at $-p = -(\Delta + 1)$; so that the R value is the integration over the *horizontal* branch lines, *doubled*, in the contour γ as shown below …



In this case, the R value computes to

$$(2 / \pi) \int_{0}^{\infty} exp(-r(x + p)) / \sqrt{x}\,(x + p)\; dx$$

and here are the snapshots from Wolfram, where R is below $\mathcal{H}_x$ and to the *left*, whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. Now $\mathcal{H}_x$ + R ~ .0576384 and $\mathcal{E}$ ~ .0576390, and the two are remarkably close, as we can see in the calculations …

$$\int_0^{1000000} -\frac{i\left(\left(-\frac{1}{\sqrt{301-iy}}+\frac{1}{\sqrt{301+iy}}\right)\cos\left(\frac{y}{1000}\right)+i\left(\frac{1}{\sqrt{301-iy}}+\frac{1}{\sqrt{301+iy}}\right)\sin\left(\frac{y}{1000}\right)\right)}{\pi\sqrt{y^2}}\,dy =$$

0.00716783

$$\int_0^{10000} \frac{2\,e^{(-301-x)/1000}}{\pi\sqrt{x}\,(301+x)}\,dx = 0.0504706$$

$$\frac{1}{\sqrt{301}}$$

Decimal approximation

0.057639041770423

Now we'll set $r = 2$, and repeat the testing. In this case the R value is negligible, and here is the snapshot for $\mathcal{H}_x$, which agrees closely with $\mathcal{E}$ above.

$$\int_0^{16000} -\frac{i\left(\left(-\frac{1}{\sqrt{301-iy}}+\frac{1}{\sqrt{301+iy}}\right)\cos(2y)+i\left(\frac{1}{\sqrt{301-iy}}+\frac{1}{\sqrt{301+iy}}\right)\sin(2y)\right)}{\pi\sqrt{y^2}}\,dy =$$

0.057639

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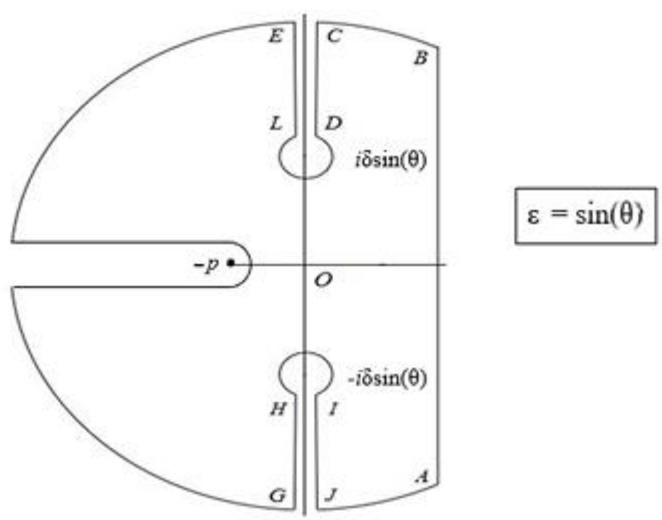
In the case of the *timelike* component, $\mathcal{T} = 1 / (r^v \cdot \eta\cos(\theta) + \Delta)$, some progress has been made when $v = ½$. The generating function for $\mathcal{T}$ is

$$f(s) \;=\; 1 / \{((s + \delta\alpha)^v \cdot \eta\cos(\theta) + \Delta)\,\sqrt{s^2 + (\delta\varepsilon)^2}\},$$

where $\alpha = \cos(\theta)$, $\varepsilon = \sin(\theta)$; and if $\eta = 1$, $\theta = 0$, $v = ½$ and $\delta = 1$, this function reduces to

$$f(s) \;=\; 1 / \{((s + 1)^v + \Delta)\,\sqrt{s^2 + 0}\}.$$



If we now agree that in the contour γ above, $(s + 1)^v$ is *positive* along the *x*-axis if $s > -1$, where $v = ½$; then we can use γ to find the R value [a *doubling* of the expression below, along the *horizontal* branch lines], by setting the branch point $-p$ to -1. We do this because of the $(s + 1)^v$ term in the denominator of $f(s)$.

$$\kappa \int e^{sr} f(s)\, ds$$

In so doing, we find that R computes to

$$(2 / \pi) \int_0^{\infty} \sqrt{x}\, exp(-r(x + 1)) \,/\, ((x + \Delta^2)\,(x + 1))\, dx$$

In the snapshots that follow; $r = .001$, and R is below $\mathcal{H}_x$ and to the *left*; whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. In this case $\mathcal{H}_x$ + R ~ .00332227, which is very close to $\mathcal{E}$ ~ .00332226.

$$\int_0^{10\,000\,000} -\frac{i\left(\left(-\frac{1}{300+\sqrt{1-iy}}+\frac{1}{300+\sqrt{1+iy}}\right)\cos\left(\frac{y}{1000}\right)+i\left(\frac{1}{300+\sqrt{1-iy}}+\frac{1}{300+\sqrt{1+iy}}\right)\sin\left(\frac{y}{1000}\right)\right)}{\pi\sqrt{y^2}}\,dy = 0.00294979$$

$$\int_0^{10\,000} \frac{2\, e^{(-1-x)/1000}\sqrt{x}}{\pi\,(1+x)\,(90\,000+x)}\,dx = 0.000372483$$

$$\frac{1}{301} \quad \text{(irreducible)}$$

Decimal approximation

0.003322259136212

In the snapshots that follow; $r = 1$, and R is below $\mathcal{H}_x$ and to the *left*; whilst $\mathcal{E}$ is below $\mathcal{H}_x$ and to the *right*. In this case $\mathcal{H}_x$ + R ~ .00332229, which is very close to $\mathcal{E}$ ~ .00332226.

$$\int_0^{30\,000} -\frac{i\left(\left(-\frac{1}{300+\sqrt{1-iy}}+\frac{1}{300+\sqrt{1+iy}}\right)\cos(y)+i\left(\frac{1}{300+\sqrt{1-iy}}+\frac{1}{300+\sqrt{1+iy}}\right)\sin(y)\right)}{\pi\sqrt{y^2}}\,dy = 0.00332118$$

$$\frac{1}{301} \quad \text{(irreducible)}$$

$$\int_0^{10\,000} \frac{2\, e^{-1-x}\sqrt{x}}{\pi\,(1+x)\,(90\,000+x)}\,dx = 1.11675\times 10^{-6}$$

Decimal approximation

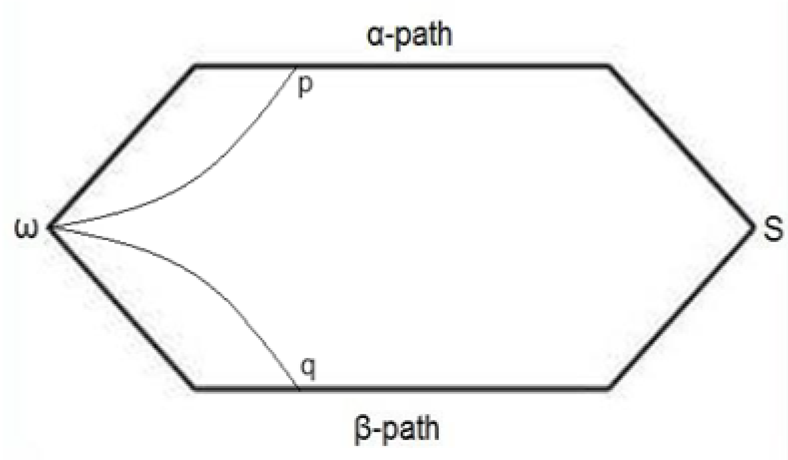
0.003322259136212

Let us recall too, that $\mathcal{E} = g_{u,v}(\delta\alpha) \cdot J_0(\delta r \varepsilon)$, where $\delta\alpha$ is a *radial* measure, and that testing for the *timelike* component is *exact* if $\theta = \pi / 2$ …

A Necessary Condition For Pseudo-Roots

In previous research, we discussed the *undecidable* nature of *pseudo*-roots, when looking at the Riemann Hypothesis [pp 302-4, 315-16, 342-3, 359, 376-83].

Initially, we showed by the use of Fourier transforms, that we could generate a signal ***S*** from *either* the α-line or the β-line in $\mathcal{C}$* – the critical strip *minus* the critical line [pp 294-6]; where here, it is the case that ω is a root or *pseudo*-root of ζ(s), lying on *both* the α-line and the β-line. Thus, it is the case that $\zeta(p) = \zeta(q)$, where $p = \alpha \pm i\varepsilon$, and $q = \beta \pm i\varepsilon$, for some $\varepsilon \geq 0$ and $\beta = 1 - \alpha$.



From Riemann's Functional Equation, as shown below and labelling as (*) …

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

we see immediately that by setting $s = p$ in (*), and calling this equation (1); and by setting $s = q$ in (*), and calling this equation (2); we arrive at the following result, *after* dividing (1) by (2) …

$$(2\pi)^{(\beta - \alpha)} \sin(\pi q / 2)\, \Gamma(1 - q) \,/\, \sin(\pi p / 2)\, \Gamma(1 - p) = |\, \zeta(q)\, |^2 \,/\, |\, \zeta(p)\, |^2 \qquad (\dagger)$$

Defining the *left*-hand side of (†) to be ψ, the first thing we note is that along the *critical* line, where $\alpha = \beta = ½$, ψ is equal to 1, which agrees with the *right*-hand side of (†). And this is because $p = q$ on this line.

Now in $\mathcal{C}$* – the critical strip *minus* the critical line – if p and q are *pseudo*-roots, so that $\zeta(p) = \zeta(q) \neq 0$; then ψ must *also* be equal to 1 in (†). But since *pseudo*-roots are *not* verifiable, the statement ψ = 1 cannot be verified in this case. We can simply say that ψ = 1, but nothing more; and this is the necessary condition for *pseudo*-roots in $\mathcal{C}$*.

On the other hand, if ψ = 1, then from (†) we see that $|\zeta(p)| = |\zeta(q)|^{(\ddagger)}$, in which case p and q may or may not be *pseudo*-roots. If they are, ψ = 1 *cannot* be verified; but if they aren't, our methodology cannot say if (‡) is or isn't an *undecidable* state, since the condition $\zeta(p) \neq \zeta(q)$ excludes p and q from the analysis [pp 382-3] in this case.

Now let's consider the state $\zeta(p) + \zeta(q) = \zeta(p) + \zeta(q)^{(\S)}$, so that at p, we are comparing the *left*-hand side of (§) with the *right*-hand side of (§), at q. If $\zeta(p) = \zeta(q)$, then p and q are *pseudo*-roots and (§) is an *undecidable* state. And this is because the *left*-hand side of (§) is now $2\zeta(p)$, and the *right*-hand side of (§) is now $2\zeta(q)$.

On the other hand, if $\zeta(p) \neq \zeta(q)$, then for an observer $\mathcal{O}$, in the *signal's* frame of reference, that is situated at p or q, (§) is most certainly *not* undecidable. $\mathcal{O}$ is 'blind to the line' and so, won't know if it's really at p or at q [pp 382-3], but *will* know that (§) is *always* true, in this case. And thus, the points (p, q) are *excluded* by $\mathcal{O}$, based *solely* on *algebraic* considerations.

But since $\mathcal{O}$ knows $\zeta(p) \neq \zeta(q)^{(3)}$, then $\mathcal{O}$ will know that being situated at p is *not* the same as being situated at q. $\mathcal{O}$ is still 'blind to the line' but now knows there is a *difference* between $\boldsymbol{\ell}_\alpha$ and $\boldsymbol{\ell}_\beta$, at p versus at q, because of (3). However, relative to the signal $\boldsymbol{S}$, it should *never* be possible to detect such a difference between $\boldsymbol{\ell}_\alpha$ and $\boldsymbol{\ell}_\beta$, at these two points$^{(4)}$ [pp 382-3].

And so, the condition $\zeta(p) \neq \zeta(q)$ excludes p and q from the analysis, in this case, as it should. And this is based *solely* on (4), which is the *exclusion* principle [$\mathscr{P}$]. Thus, the *algebraic* approach supports $\mathscr{P}$, and conversely, $\mathscr{P}$ supports the *algebraic* approach, in this instance. But we can go further now and conclude that our analysis on pp 382-3 is *also* correct, in so much as (4) is indeed true, at the *point* level.

Thus, we see the importance of the *exclusion* principle here. Were we to disregard this principle, in general; then on the previous page, we might have concluded that $|\,\zeta(p)\,| = |\,\zeta(q)\,|^{(\ddagger)}$ was an *undecidable* state *for all* (p, q) on the lines $\boldsymbol{\ell}_\alpha$ and $\boldsymbol{\ell}_\beta$, respectively, where (‡) was true. But this is not necessarily so …

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The notes above have been rewritten somewhat, to clarify the *exclusion* principle, and justify its use, particularly on pages 382-3. There, we came to the conclusion that roots and *pseudo*-roots cannot be verified in $\mathcal{C}^*$ …

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To show that *pseudo*-roots *may* exist, consider the case where $p = \frac{1}{4} + 6.288\boldsymbol{i}$, and $q = \frac{3}{4} + 6.288\boldsymbol{i}$. The moduli *squared* and values for $\{\zeta(p), \zeta(q)\}$ are shown below; and while they are not *pseudo*-roots in this case, it is quite possible that *pseudo*-roots might exist somewhere in $\mathcal{C}^*$, even though they are *not* verifiable. Notice here how close the moduli *squared* are to one another.

$$\left\{\left|\zeta(0.25 + 6.288\, i)\right|^2, \left|\zeta(0.75 + 6.288\, i)\right|^2\right\} \qquad \left\{\zeta(0.25 + 6.288\, i), \zeta(0.75 + 6.288\, i)\right\}$$

$$\{0.916571, 0.916583\} \qquad \{0.86573 + 0.408758\, i, 0.90174 + 0.321635\, i\}$$

Now consider the case where $s = ¼ + 6.289\boldsymbol{i}$, and $t = ¾ + 6.289\boldsymbol{i}$. The moduli *squared* and values for $\{\zeta(s), \zeta(t)\}$ are shown below; and while they are not *pseudo*- roots in this case, we note here that $| \zeta(s) |^2 > | \zeta(t) |^2$, whereas in the previous case just above, $| \zeta(p) |^2 < | \zeta(q) |^2$.

$\{|\zeta(0.25 + 6.289\,i)|^2, |\zeta(0.75 + 6.289\,i)|^2\}$ $\{\zeta(0.25 + 6.289\,i), \zeta(0.75 + 6.289\,i)\}$

$\{0.916969, 0.916909\}$ $\{0.865919 + 0.408843\,i, 0.901899 + 0.321695\,i\}$

Thus, as we move from (p, q) to (s, t), we see from a *continuity* argument, that there must be some ε such that $6.288 < \varepsilon < 6.289$; and $u = ¼ + \boldsymbol{i}\varepsilon$ and $v = ¾ + \boldsymbol{i}\varepsilon$, for which $| \zeta(u) |^2 = | \zeta(v) |^2$. Clearly u and v won't be *pseudo*-roots, but the argument does tell us that $| \zeta(p) | = | \zeta(q) |^{(‡)}$ is *not* an undecidable state if $\zeta(p) \neq \zeta(q)$, more generally. And in this particular case, $\varepsilon \approx 6.28817$, as we can see in the images below …

$\{|\zeta(0.25 + 6.28817\,i)|^2, |\zeta(0.75 + 6.28817\,i)|^2\}$ $\{\zeta(0.25 + 6.28817\,i), \zeta(0.75 + 6.28817\,i)\}$

$\{0.916638, 0.916638\}$ $\{0.865762 + 0.408772\,i, 0.901767 + 0.321645\,i\}$

In fact, there are probably *infinitely* many (p, q) in $\mathcal{C}^*$ – the critical strip *minus* the critical line, for which (‡) is true; where $p = \alpha \pm \boldsymbol{i}\varepsilon$ and $q = \beta \pm \boldsymbol{i}\varepsilon$, for some $\varepsilon \geq 0$; and $\beta = 1 - \alpha$ with $\zeta(p) \neq \zeta(q)$. Still the mystery remains if p and q are *pseudo*-roots, so that $\zeta(p) = \zeta(q)$. For here our methodology implies they cannot be verified in $\mathcal{C}^*$ …

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Some additional commentary was added to the notes above. In particular, an estimate for ε is shown in the *second* set of images on this page …

On The Implicit and Explicit Nature of The Fourier Transforms Concerning The Riemann Hypothesis

From page 294, let us bring back *one* of the Fourier transforms that is *directly* involved in our analysis of the Riemann Hypothesis [$\mathcal{R}$].

$$\int_0^\infty \{(A + C)\sin(yr) - (B + D)\cos(yr)\}dy / y =$$

$$\pi e^{-r(1-\alpha)}\{2(1-\alpha)\cos(\varepsilon r) - 2\varepsilon\sin(\varepsilon r)\} / \{(1-\alpha)^2 + \varepsilon^2\} \qquad \dots (3)$$

Now we said there, that the following coefficients *implicitly* reference the α-line,

$$A = \{\zeta(\alpha + \boldsymbol{i}y - \boldsymbol{i}\varepsilon) + \zeta(\alpha - \boldsymbol{i}y + \boldsymbol{i}\varepsilon)\}/2$$
$$B = \{\zeta(\alpha + \boldsymbol{i}y - \boldsymbol{i}\varepsilon) - \zeta(\alpha - \boldsymbol{i}y + \boldsymbol{i}\varepsilon)\}/2\boldsymbol{i}$$
$$C = \{\zeta(\alpha + \boldsymbol{i}y + \boldsymbol{i}\varepsilon) + \zeta(\alpha - \boldsymbol{i}y - \boldsymbol{i}\varepsilon)\}/2$$
$$D = \{\zeta(\alpha + \boldsymbol{i}y + \boldsymbol{i}\varepsilon) - \zeta(\alpha - \boldsymbol{i}y - \boldsymbol{i}\varepsilon)\}/2\boldsymbol{i},$$

but we can go further now, and note that in the coefficients; both y and ε can be thought of as *real* variables that *attach* to the α-line [$\boldsymbol{\ell}_\alpha$] as imaginary parts. Seen in this light, (3) becomes a Fourier transform over $\boldsymbol{\ell}_\alpha$, and the reference is no longer just *implicit*; but rather *explicit*.

And similarly for the Fourier transform on page 295 over the β-line [$\boldsymbol{\ell}_\beta$]; namely

$$\int_0^\infty f(y)\cos(yr)dy, \qquad (*)$$

where

$$f(y) = (\varepsilon + y) / (\beta^2 + (\varepsilon + y)^2) + (\varepsilon - y) / (\beta^2 + (\varepsilon - y)^2),$$

and the denominators can be written as

$$(\beta^2 + (\varepsilon + y)^2) = (\beta + (\varepsilon + y)\boldsymbol{i})(\beta - (\varepsilon + y)\boldsymbol{i})$$

$$(\beta^2 + (\varepsilon - y)^2) = (\beta + (\varepsilon - y)\boldsymbol{i})(\beta - (\varepsilon - y)\boldsymbol{i})$$

The Fourier transform over $\boldsymbol{\ell}_\beta$ is now no longer just *implicit*; but rather *explicit*, where the variables y and ε *attach* to the β-line as imaginary parts.

Notice too, that for this transform over $\boldsymbol{\ell}_\beta$, it is to be *multiplied* by the constant $-2\varepsilon/\{(1-\alpha)^2 + \varepsilon^2\}$, which is equal to $-2\varepsilon/\{\beta^2 + \varepsilon^2\}$; and thus the constant *attaches* to the β-line quite naturally [p 295].

The *offset* to (3) is $(\pi / 2)\, g(0)$, where $g(s) = \zeta(\alpha - s + i\varepsilon) + \zeta(\alpha - s - i\varepsilon)$; and for *pseudo*-roots, computes to the *same* value, whether we are on ℓ_α or ℓ_β [pp 302-3]. So we needn't worry about which line we are calculating on here …

y, ε $\qquad\qquad$ y, ε

ℓ_α $\qquad\qquad$ ℓ_β

In the diagram above, the Fourier transform in (3) occurs *explicitly* along ℓ_α, because y and ε attach to the α-line. However, the Fourier transform in (*) [previous page] occurs *explicitly* along ℓ_β, because y and ε now attach to the β-line. Using the methods on pp 294-5, the *entire* signal S in (3), and shown below, can be generated *either* from ℓ_α or ℓ_β, where here *two* Fourier transforms are actually used on ℓ_β, along with their respective constants.

$$\pi e^{-r(1-\alpha)}\{2(1-\alpha)\cos(\varepsilon r) - 2\varepsilon\sin(\varepsilon r)\} \;/\; \{(1-\alpha)^2 + \varepsilon^2\}$$

Now when we look at S, just above, we notice the parameter ε appears; and indeed, if S was generated from the Fourier transform on ℓ_α, we could say *in* S that ε attaches to ℓ_α. On the other hand, if S was generated from the Fourier transforms on ℓ_β, then we could say *in* S that ε attaches to ℓ_β.

Thus, asking 'which line' the signal S came from, is *equivalent* to asking which line ε actually attaches to *in* S. We could *also* ask 'which line' the signal S came from, relative to the Fourier transforms themselves, as we do in previous notes; and hence the *equivalency* between the two approaches here. Either way, it is *not* possible to decide; and this is why it is *not* possible to decide which line [ℓ_α or ℓ_β] the root or *pseudo*-root, itself, actually comes from.

Said a little differently, there is *no* way to distinguish the root or *pseudo*-root on ℓ_α from the root or *pseudo*-root on ℓ_β, and hence there is *no* mathematical method $\mathcal{M}$ which can *prove* their existence in $\mathcal{C}^*$ – the critical strip *minus* the critical line – or *disprove* their existence in $\mathcal{C}^*$ *without* ambiguity [pp 376-83]. In short, roots and *pseudo*-roots cannot be verified in $\mathcal{C}^*$ …

Table Of Contents, To Date

Pages 383-460

Largely devoted to a study of the α-γ equations, which are connected to the *time* component of the Schwarzschild solution, for the *interior* of a star. A long and somewhat detailed section, but we eventually arrive at our final destination. Lots of pictures and tables included along the way.

Pages 461-513

Understanding and interpreting the *coupled* field equations within the context of general relativity and dark energy singularities. The *associative* principle under addition, relative to σ, is one of the more important highlights. At times, this section reminds me of taking a *long* and *winding* road to some final destination, but eventually we reach that point. I would encourage the reader to be patient with this piece. Some good diagrams as well, to help clarify things.

Pages 514-705

A *mixed* bag covering a lot of different topics, like *tangible* and *intangible* spaces and vacuums; *normalized* versus *general* solutions for the coupled equations in the *intangible* space; the dark energy density function and Laplace compliancy; *inner* blocks and solving the equations as a *sum* over *normalized* component solutions; interpreting the *harmonic* expression within the context of the generalized *equivalency* theorem; and more recently, a tutorial on how to use the harmonic expression here, as well as a *Simple Case Study* series. Lots of good diagrams and snapshots to clarify the text, as well as some recent material on the Riemann Hypothesis.

A Message From The Light

You save, redeem and heal yourself. You always have ... you always will. You were born with the power to do so from before the beginning of the world ... so says the Light ...

Of all the things I have come across within the world of NDE research, the words above, brought back to us from one near-death experiencer, have always stood out.

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The last *eighty-three* updates precede the Table of Contents, and can be found on pages 566-705.

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'Why even the stones know more than we do … indeed, even a grain of sand.
If I could talk to the stones or the grain of sand, what deep secrets would they tell me ?'